Regularity of the Boltzmann equation in bounded domains

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The dynamics of rarefied gases are governed by Boltzmann equation (1872)

$$\underbrace{\partial_t F + v \cdot \nabla_x F}_{\textit{Free Transport}} = \underbrace{Q(F,F)}_{\textit{Collisions}},$$

where $\forall t \geq 0, \forall x, v \in \mathbb{R}^3$, $F(t,x,v)$ denotes the particles distribution

and $Q(F,F)$ is the collision operator which takes the form

$$Q(F_1,F_2) := Q_{\text{gain}}(F_1,F_2) - Q_{\text{loss}}(F_1,F_2)$$

Mesoscopic description: statistic description aiming at describing particles behavior

Free transport

We suppose that the considered gas is made up of monoatomic identical particles

In the absence of external forces, if the interactions between the particles are not considered, they move along straight lines with constant speed

$$\forall t \geq 0, \forall x, v \in \mathbb{R}^3 \quad F(t, x + vt, v) = const = F(0, x, v)$$

Hence, their distribution is given by

$$\forall t \geq 0, \forall x, v \in \mathbb{R}^3 \quad \partial_t F(t, x, v) + v \cdot \nabla_x F(t, x, v) = 0$$

Collision operator

- Binary collisions
- Instantaneous collisions
- Elastic collisions : conservation of momentum and kinetic energy

$$\underbrace{v + v_*}_{\text{pre-collisional}} = \underbrace{v' + v'_*}_{\text{post-collisional}} \qquad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

This is equivalent to the existence of a unitary vector $\omega \in \mathbb{S}^2$ such that

$$\begin{cases} v' = v + [(v_* - v) \cdot \omega]\omega, \\ v'_* = v_* - [(v_* - v) \cdot \omega]\omega \end{cases}$$

$\Rightarrow |v - v_*| = |v' - v'_*|$ and $\left| \frac{v - v_*}{|v - v_*|} \cdot \omega \right| = \left| \frac{v' - v'_*}{|v' - v'_*|} \cdot \omega \right|$

- Microreversible collisions
- Molecular chaos

Under the previous hypotheses Boltzmann proved that the general equation becomes

$$\partial_t F + v \cdot \nabla_x F = Q(F,F),$$

$$Q_{\text{loss}}(F_1,F_2)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(|v-v_*|, \left|\frac{v-v_*}{|v-v_*|}\cdot\omega\right|\right) F_1(v_*)F_2(v)\,\mathrm{d}\omega\,\mathrm{d}v_*$$

$$Q_{\text{gain}}(F_1,F_2)(t,x,v') = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(|v-v_*|, \left|\frac{v-v_*}{|v-v_*|}\cdot\omega\right|\right) F_1(v_*)F_2(v)\,\mathrm{d}\omega\,\mathrm{d}v_*$$

$$Q(F_1,F_2)(t,x,v) = Q_{\text{gain}}(F_1,F_2) - Q_{\text{loss}}(F_1,F_2)$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(|v-v_*|, \left|\frac{v-v_*}{|v-v_*|}\cdot\omega\right|\right)\Big[F_1(v'_*)F_2(v') - F_1(v_*)F_2(v)\Big]\mathrm{d}\omega\,\mathrm{d}v_*$$

In our case, the collision operator takes the form

$$Q(F_1, F_2) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |v - v_*|^\kappa q_0(\theta)\Big[F_1(v'_*)F_2(v') - F_1(v_*)F_2(v)\Big]\mathrm{d}\omega\mathrm{d}v_*,$$

where θ is the deviation angle and the collision rule is

$$\begin{cases} v' = v + [(v_* - v)\cdot\omega]\omega, \\ v'_* = v_* - [(v_* - v)\cdot\omega]\omega \end{cases}$$

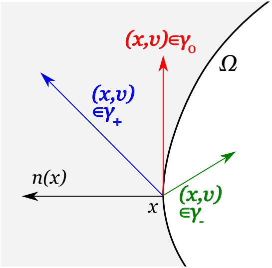
- Hard potential $0 \leq \kappa \leq 1$
- Angular cutoff $0 \leq q_0(\theta) \leq C|\cos\theta|$ with $\cos\theta = \frac{v_* - v}{|v* - v|}\cdot\omega$

Bounded domain $\Omega \subset \mathbb{R}^3$

The boundary of the phase space is

$$\gamma := \{(x, v) \in \partial\Omega \times \mathbb{R}^3\},$$

where $n = n(x)$ the outward normal direction at $x \in \partial\Omega$



We decompose γ as

$$\gamma_- = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, \quad \textit{the incoming set}$$
$$\gamma_+ = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, \quad \textit{the outcoming set}$$
$$\gamma_0 = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, \quad \textit{the grazing set}$$

Boundary conditions on γ_-

In-flow boundary condition:

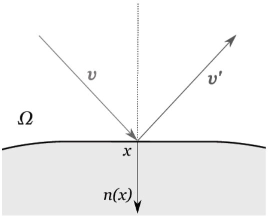
$$\forall t \geq 0,\ \forall (x,v) \in \gamma_- \quad F(t,x,v) = g(t,x,v)$$

where g precribes the density of the incoming particles.

Specular reflection boundary condition: $\forall t \geq 0,\ \forall (x,v) \in \gamma_-$

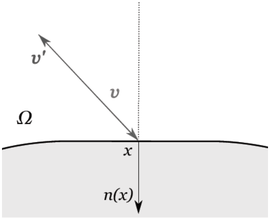
$$F(t,x,v) = F(t,x,R_x v),$$

where $R_x v := v - 2n(x)(n(x) \cdot v)$



Bounce-back reflection boundary condition: $\forall t \geq 0,\ \forall (x,v) \in \gamma_-$

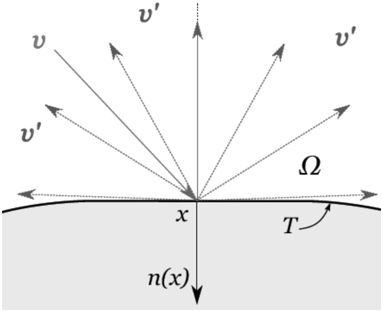
$$F(t,x,v) = F(t,x,-v)$$



Diffuse boundary condition: $\forall t \geq 0,\ \forall (x,v) \in \gamma_-$

$$F(t,x,v) = c_{\mu_T} \mu_T(v) \int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\}\mathrm{d}u$$

Where $c_{\mu_T} \int_{n(x)\cdot u>0} \mu_T(u)\{n(x)\cdot u\}\mathrm{d}u = 1$ and $\mu_T = \frac{1}{2\pi T} e^{-\frac{|v|^2}{2T}}$ is a global Maxwellian distribution with constant temperature $T > 0$



Known results in a general bounded domain

Existence and uniqueness of solutions

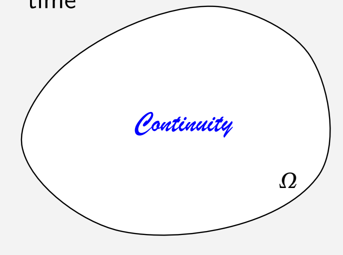
- Existence of renormalized DiPerna-Lions solutions (weak regularity) : Hamdache, Arkeryd, Cercignani, Maslova, Mischler,...
- Perturbative framework (stronger solutions) : Domains with a particular geometry: Ukai, Asano, Guiraud, General Domains: Guo

Time-decay towards an absolute Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$

- Desvillettes-Villani, Villani : If $F(t)$ exists in H^k with uniform in t bound, $k >> 1$ then $F(t) \to \mu$ with some polynomial rate
- Guo : $F(t) \to \mu$ in L^∞ with $e^{-\lambda t}$ rate

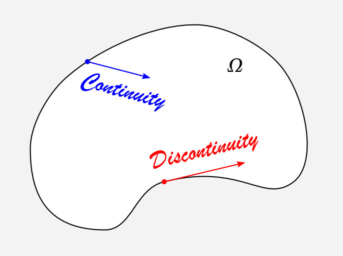
[Guo 2010]

- Existence and uniqueness of a strong global solution in a weighted in speed $L^\infty_{x,v}$ space
- Time-decay towards an absolute Maxwellian with an exponential rate
- In the case of a strictly convex domain, for general boundary conditions, $C^0_{x,v}$ regularity away from γ_0 for all positive time



[Kim 2011]

- In the case of a non convex domain, for diffuse, in-flow, bounce-back boundary conditions, a discontinuity may appear in non convexity points and propagates inside the domain through a linear trajectory



Regularity Estimates

- BV, Sobolev, Hölder regularity results for the Vlasov equation in a half space with various boundary conditions [Guo 1995]
- Hölder regularity results for the Vlasov equation in convex domains with Specular BC [Hwang-Velazquez 2010]

In the case of Boltzmann equation very rare results exist when the domain is non-trivial and in the presence of boundary conditions

Perturbative framework

Let $\mu = e^{-\frac{|v|^2}{2}}$ be a global normalized Maxwellian

IDEA : look for solutions of the form $F = \sqrt{\mu} f$

Then f satisfies

$$\partial_t f + v \cdot \nabla_x f = \Gamma_{\text{gain}}(f, f) - \nu(\sqrt{\mu} f) f$$

where

$$\begin{aligned}
\nu(\sqrt{\mu} f)(v) = \nu(F)(v) &:= \frac{1}{\sqrt{\mu} f} Q_{\text{loss}}(\sqrt{\mu} f, \sqrt{\mu} f)(v) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*|^\kappa q_0(\theta) \sqrt{\mu(v_*)} f(v_*) \mathrm{d}\omega \mathrm{d}v_*
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\text{gain}}(f_1, f_2)(v) &:= \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} f_1, \sqrt{\mu} f_2)(v) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*|^\kappa q_0(\theta) \sqrt{\mu(v_*)} f_1(v'_*) f_2(v') \mathrm{d}\omega \mathrm{d}v_*
\end{aligned}$$

The corresponding boundary conditions for f are followings :

In-flow boundary condition :

$$f(t,x,v) = \frac{g(t,x,v)}{\sqrt{\mu(v)}}, \quad \text{on } \gamma_-$$

Diffuse boundary condition :

$$f(t,x,v) = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \quad \text{on } \gamma_-$$

Specular reflection boundary condition :

$$f(t,x,v) = f(t,x,R_x v), \quad \text{on } \gamma_-$$

Bounce-back reflection boundary condition :

$$f(t,x,v) = f(t,x,-v), \quad \text{on } \gamma_-$$

For the initial datum f_0, compatibility conditions are necessary

In-flow boundary compatibility condition:

$$f_0(x,v) = \frac{1}{\sqrt{\mu(v)}} g(0,x,v) \quad \text{on } \gamma_-$$

Diffuse boundary compatibility condition:

$$f_0(x,v) = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f_0(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \quad \text{on } \gamma_-$$

Specular reflection boundary compatibility condition:

$$f_0(x,v) = f_0(x, R_x v), \quad \text{on } \gamma_-$$

Bounce-back reflection boundary compatibility condition:

$$f_0(x,v) = f_0(x,-v), \quad \text{on } \gamma_-$$

Analysis of the characteristics

From now on we consider a domain with a smooth boundary

Let Ω be a bounded open subset of $\mathbb{R}^3$, i.e.

$$\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}, \quad \text{and} \quad \partial\Omega = \{x \in \mathbb{R}^3 : \xi(x) = 0\}$$

for a smooth $\xi : \mathbb{R}^3 \to \mathbb{R}$

For all $x \in \bar{\Omega} = \Omega \cup \partial\Omega$ we say that the domain is strictly convex if :

$$\sum_{i,j} \partial_{ij}\xi(x)\zeta_i\zeta_j \geq C_\xi|\zeta|^2 \quad \text{for all} \quad \zeta \in \mathbb{R}^3$$

We assume that $\nabla\xi(x) \neq 0$ when $|\xi(x)| \ll 1$ and we define the unit outward normal as $n(x) = \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$

Analysis of the characteristics

For $(x, v) \in \bar{\Omega} \times \mathbb{R}^3$ we define $t_{\mathbf{b}}(x, v)$ be the backward exit time as

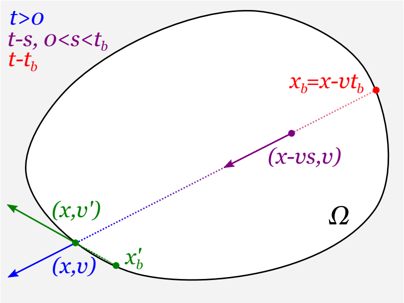
$$t_{\mathbf{b}}(x, v) = \inf\{\tau > 0 : x - sv \notin \Omega\},$$

and $x_{\mathbf{b}}(x, v) = x - t_{\mathbf{b}}(x, v)v$

Note: the particle hits the boundary at time $t - t_{\mathbf{b}}(x, v)$

The characteristics ODE of the Boltzmann equation is

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s), \quad \frac{\mathrm{d}V(s)}{\mathrm{d}s} = 0$$



$t_{\mathbf{b}}(x,v)$ and $x_{\mathbf{b}}(x,v)$ may have a singular behavior when

$$n(x_{\mathbf{b}}(x,v)) \cdot v = 0$$

Define the grazing singular set as:

$$\mathfrak{S}_{\mathbf{b}} := \{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = 0\}$$

Role of $\mathfrak{S}_{\mathbf{b}}$: stationary transport equation

$$v \cdot \nabla_x f(x,v) = 0$$

$$f|_{\gamma_-} = g,$$

where g is a smooth function, then the solution is

$$f(x,v) = g(x_{\mathbf{b}}(x,v), v) = g(x - t_{\mathbf{b}}(x,v)v, v)$$

$\Longrightarrow f(x,v)$ might be singular on the singular grazing set $\mathfrak{S}_{\mathbf{b}}$.

In the case the characteristic touches the boundary we need to define generalized characteristics :

Let $(x, v) \notin \gamma_0$ and $(t^0, x^0, v^0) = (t, x, v)$

the stochastic diffuse cycles are defined as:

$$(t^1, x^1, v^1) = (t - t_{\mathbf{b}}(x, v), x - t_{\mathbf{b}}(x, v)v, v^1) \quad \text{with } n(x^1) \cdot v^1 > 0$$

and for $\ell \geq 1$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), v^{\ell+1}) \quad \text{with } n(x^\ell) \cdot v^\ell > 0$$

the specular cycles, are defined for all $\ell \geq 1$:

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), v^\ell - 2n(x^\ell)(v^\ell \cdot n(x^\ell)))$$

the bounce-back cycles are defined for all $\ell \geq 1$:

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^{\ell} - t_{\mathbf{b}}(x^{\ell}, v^{\ell}), x_{\mathbf{b}}(x^{\ell}, v^{\ell}), -v^{\ell})$$

Then for $\ell \geq 1$

$$\begin{aligned} t^{\ell} &= t^1 - (\ell - 1) t_{\mathbf{b}}(x^1, v^1), \\ x^{\ell} &= \frac{1 - (-1)^{\ell}}{2} x^1 + \frac{1 + (-1)^{\ell}}{2} x^2, \\ v^{\ell+1} &= (-1)^{\ell+1} v \end{aligned}$$

In all cases we define the backward trajectory as

$$\begin{aligned} X_{\mathbf{cl}}(s; t, x, v) &= \sum_{\ell} \mathbf{1}_{[t^{\ell+1}, t^{\ell})}(s) \{ x^{\ell} - (t^{\ell} - s) v^{\ell} \}, \\ V_{\mathbf{cl}}(s; t, x, v) &= \sum_{\ell} \mathbf{1}_{[t^{\ell+1}, t^{\ell})}(s) v^{\ell} \end{aligned}$$

$$\partial_t G + v \cdot \nabla_x G = 0$$
$$G(0,x,v) = G_0(x,v)$$

- In the case of in-flow BC $G(t,x,v) = g(t,x,v)\ \forall (x,v)\gamma_-$ then for $0 \leq s \leq t \leq t_{\mathbf{b}}(x,v)$

$$G(t,x,v) = G(s, x-(t-s)v, v) = G_0(x - tv, v).$$

while for for $t_{\mathbf{b}}(x,v) \leq s \leq t$

$$G(t,x,v) = G(s, x-(t-s)v, v) = g(t - t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v)$$

- In the case of specular or bounce-back reflection BC, then for $0 \leq s \leq t$

$$\begin{aligned} G(t,x,v) &= G(s, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) \\ &= G_0(X_{\mathbf{cl}}(0;t,x,v), V_{\mathbf{cl}}(0;t,x,v)) \end{aligned}$$

- In the case of diffuse reflection BC, it is more difficult to explicit the solution

We define: the concave (singular) grazing boundary as

$$\gamma_0^S := \{(x,v) \in \gamma_0 \ : t_{\mathbf{b}}(x,v) \neq 0 \text{ and } t_{\mathbf{b}}(x,-v) \neq 0\},$$

the outward inflection grazing boundary as

$$\begin{aligned}\gamma_0^{I_+} := \{(x,v) \in \gamma_0 \ : t_{\mathbf{b}}(x,v) \neq 0,\ t_{\mathbf{b}}(x,-v) = 0 \\ \text{and } \exists \delta > 0 \text{ s.t. } x + \tau v \in \bar{\Omega}^c,\ \forall \tau \in (0,\delta)\},\end{aligned}$$

the inward inflection grazing boundary as

$$\begin{aligned}\gamma_0^{I_-} := \{(x,v) \in \gamma_0 \ : t_{\mathbf{b}}(x,v) = 0,\ t_{\mathbf{b}}(x,-v) \neq 0 \\ \text{and } \exists \delta > 0 \text{ s.t. } x - \tau v \in \bar{\Omega}^c,\ \forall \tau \in (0,\delta)\},\end{aligned}$$

and the convex grazing boundary as

$$\gamma_0^V := \{(x,v) \in \gamma_0 \ : t_{\mathbf{b}}(x,v) = 0 \text{ and } t_{\mathbf{b}}(x,-v) = 0\}$$

$$\mathfrak{S}_{\mathbf{b}} := \{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = 0\}$$

we have

$$\mathfrak{S}_{\mathbf{b}} = \gamma_0^V \cup \mathfrak{S}_{\mathbf{b}}^S \cup \mathfrak{S}_{\mathbf{b}}^{I_-},$$

where

$$\mathfrak{S}_{\mathbf{b}}^S := \left\{(x,v) \in \mathfrak{S}_{\mathbf{b}} : (x_{\mathbf{b}}(x,v), v) \in \gamma_0^S\right\} \not\supseteq \gamma_0^S,$$

and

$$\mathfrak{S}_{\mathbf{b}}^{I_-} := \left\{(x,v) \in \mathfrak{S}_{\mathbf{b}} : (x_{\mathbf{b}}(x,v), v) \in \gamma_0^{I_-}\right\} \supseteq \gamma_0^{I_-},$$

while

$$\left\{(x,v) \in \mathfrak{S}_{\mathbf{b}} : (x_{\mathbf{b}}(x,v), v) \in \gamma_0^V\right\} = \gamma_0^V$$

and

$$\left\{(x,v) \in \mathfrak{S}_{\mathbf{b}} : (x_{\mathbf{b}}(x,v), v) \in \gamma_0^{I_+}\right\} = \emptyset,$$

Lemma (Guo10, Kim11)

- $t_{\mathbf{b}}(x,v)$ *is lower semicontinuous;*
- *if* $v \cdot n(x_{\mathbf{b}}(x,v)) < 0$ *then* $(t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v))$ *are smooth functions of* (x,v)

Assume $(x_0, v_0) \in \mathfrak{S}_{\mathbf{b}}$, *with* $v_0 \neq 0$ *and* $0 < t_{\mathbf{b}}(x_0, v_0) < +\infty$

- *If* $(x_0, v_0) \in \mathfrak{S}_{\mathbf{b}}^{I-}$ *then* $t_{\mathbf{b}}(x,v)$ *is continuous around* (x_0, v_0)
- *If* $(x_0, v_0) \in \mathfrak{S}_{\mathbf{b}}^{S}$ *then* $t_{\mathbf{b}}(x,v)$ *is not continuous around* (x_0, v_0)

Define the discontinuity set

$$\mathfrak{D} := \mathfrak{D}_0 \cup \mathfrak{D}_i$$

where

$$\mathfrak{D}_0 := \left\{(0,+\infty) \times [\gamma_0^S \cup \gamma_0^V \cup \gamma_0^{I_+}]\right\},$$

$$\mathfrak{D}_i := \Big\{(t,x,v) \in (0,+\infty) \times \{\Omega \times \mathbb{R}^3 \cup \gamma_+\} \;\; : t \geq t_{\mathbf{b}}(x,v) \text{ and } (x_{\mathbf{b}}(x,v),v) \in \gamma_0^S\Big\}$$

Define the continuity set

$$\mathfrak{C} = \mathfrak{C}_0 \cup \mathfrak{C}_- \cup \mathfrak{C}_{0,-}$$

where

$$\mathfrak{C}_0 := \left\{ \{0\} \times \bar{\Omega} \times \mathbb{R}^3 \right\},$$

$$\mathfrak{C}_- := \left\{ (0,+\infty) \times [\gamma_- \cup \gamma_0^{I_-}] \right\},$$

$$\mathfrak{C}_{0,-} := \Big\{ (t,x,v) \in (0,+\infty) \times \{\Omega \times \mathbb{R}^3 \cup \gamma_+\} \ : t < t_{\mathbf{b}}(x,v) \quad \text{or } (x_{\mathbf{b}}(x,v),v) \in \gamma_- \cup \gamma_0^{I_-} \Big\}$$

Reference case: Linear transport equation with in-flow BC

$$\{\partial_t + v\cdot\nabla_x + \nu\}f = H,$$

$$f(0,x,v) = f_0(x,v), \quad f(t,x,v)|_{\gamma_-} = g(t,x,v),$$

where $\nu(t,x,v) \geq 0$

compatibility conditions:

$$f_0(x,v) = g(0,x,v) \quad \text{for} \quad (x,v) \in \gamma_-$$

By Duhamel formula, denoting $\nu(s) = \nu(s, x-(t-s)v, v)$, we have

$$\begin{aligned} f(t,x,v) =& \mathbf{1}_{\{t\leq t_{\mathbf{b}}\}} e^{-\int_{\mathbf{0}}^{t}\nu(s)ds} f_0(x-tv,v) + \mathbf{1}_{\{t>t_{\mathbf{b}}\}} e^{-\int_{\mathbf{0}}^{t_{\mathbf{b}}}\nu(s)ds} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\ &+ \int_0^{\min(t,t_{\mathbf{b}})} e^{-\int_{\mathbf{0}}^{s}\nu(\tau)d\tau} H(t-s, x-sv, v)\mathrm{d}s \end{aligned}$$

Assume Ω is a smooth bounded domain.

Theorem (Guo10)

Let $\omega(v)$ be a weight function, and suppose that $f_0 \geq 0$ is s.t.

$$||\omega f_0||_\infty + \sup_{0 \leq t < +\infty} e^{\lambda t} ||\omega g(t)||_\infty < \delta$$

for a $\lambda, \delta > 0$,
then $\exists!$ a solution f s.t.

$$\sup_{0 \leq t < +\infty} ||e^{\lambda' t} f(t)||_\infty \lesssim ||\omega f_0||_\infty + \sup_{0 \leq t < +\infty} e^{\lambda t} ||\omega g(t)||_\infty,$$

for $0 < \lambda' < \lambda$

Suppose Ω is strictly convex and f_0, g continuous then

$$f \text{ is continuous on } [0, +\infty) \times \left[\{\bar{\Omega} \times \mathbb{R}^3\} \setminus \gamma_0\right]$$

Let now Ω be strictly convex

$$\{\partial_t + v\cdot\nabla_x + \nu\}f = H, \quad f(0,x,v) = f_0(x,v), \quad f(t,x,v)|_{\gamma_-} = g(t,x,v),$$

where $\nu(t,x,v) \geq 0$ and

$$f_0(x,v) = g(0,x,v) \quad \text{for} \quad (x,v) \in \gamma_-$$

$\nabla_x f_0, \nabla_v f_0, \partial_t g, \partial_{\tau_i} g, \nabla_v g$ can be obtained directly

$$\begin{aligned} \partial_t f_0 &:= -v\cdot\nabla_x f_0 - \nu(0,x,v)f_0 + H(0,x,v), \\ \partial_n g &:= \frac{1}{n\cdot v}\Big\{-\partial_t g - \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} g - \nu g + H\Big\} \end{aligned}$$

Note

$$\nabla_x g := \frac{n}{n\cdot v}\Big\{-\partial_t g - \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} g - \nu g + H\Big\} + \sum_{i=1}^{2}\tau_i\partial_{\tau_i} g$$

Theorem (GKTT17)

For any fixed $p \in [1, \infty)$, assume

$$\begin{aligned} \partial_t f_0, \nabla_x f_0, \nabla_v f_0, \quad &\in \quad L^p(\Omega \times \mathbb{R}^3), \\ \langle v \rangle g, \partial_t g, \nabla_v g, \partial_{\tau_i} g, \partial_n g \quad &\in \quad L^p([0, T] \times \gamma_-), \end{aligned}$$

and some conditions on the integrability of H.

Then for sufficiently small $T > 0$ $\exists ! f$ s.t.

$$f, \partial_t f, \nabla_x f, \nabla_v f \in C^0([0, T]; L^p(\Omega \times \mathbb{R}^3))$$

and the traces are compatible with initial datum and boundary conditions

$$\partial_t f|_{\gamma_-} = \partial_t g, \ \nabla_v f|_{\gamma_-} = \nabla_v g, \ \nabla_x f|_{\gamma_-} = \nabla_x g, \ \ on \ \ \gamma_-,$$
$$\nabla_x f(0, x, v) = \nabla_x f_0, \ \nabla_v f(0, x, v) = \nabla_v f_0, \ \partial_t f(0, x, v) = \partial_t f_0, \ \ in \ \Omega \times \mathbb{R}^3$$

Here $\langle v \rangle = \sqrt{1 + |v|^2}$

By direct computation for $t \neq t_{\mathbf{b}}$, we can compute

$$\partial_t f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \quad \nabla_x f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} \quad \text{and} \quad \nabla_v f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}$$

using

$$\nabla_x t_{\mathbf{b}} = \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \qquad \nabla_v t_{\mathbf{b}} = -\frac{t_{\mathbf{b}}\, n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})},$$

$$\nabla_x x_{\mathbf{b}} = I - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\otimes v, \qquad \nabla_v x_{\mathbf{b}} = -t_{\mathbf{b}} I + \frac{t_{\mathbf{b}}\, n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\otimes v$$

Through estimates we can prove that

$$\partial f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}} \equiv \left[\partial_t f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}, \nabla_x f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}, \nabla_v f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}\right] \in L^\infty([0,T]; L^p(\Omega \times \mathbb{R}^3))$$

On the other hand, thanks to the compatibility condition, we need to show f has the same trace on the set

$$\mathcal{M} \equiv \{(t_{\mathbf{b}}(x,v), x, v) \in [0,T] \times \Omega \times \mathbb{R}^3\}$$

Main fact: Let $\phi(t,x,v) \in C_c^\infty((0,T) \times \Omega \times \mathbb{R}^3)$ then

$$\int_0^T \iint_{\Omega \times \mathbb{R}^3} f \partial \phi = -\int_0^T \iint_{\Omega \times \mathbb{R}^3} \partial f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}} \phi,$$

so that $f \in W^{1,p}$ with weak derivatives given by $\partial f \mathbf{1}_{\{t \neq t_{\mathbf{b}}\}}$

Note that for $\partial = [\partial_t, \partial_{x_i}, \partial_{v_i}]$ $i = 1, 2, 3$ the derivative ∂f satisfies

$$\{\partial_t + v\cdot\nabla_x + \nu\}\partial f = \mathcal{H}, \quad \partial f(0,x,v) = \partial f_0(x,v), \quad \partial f(t,x,v)|_{\gamma_-} = \partial g(t,x,v),$$

where

$$\mathcal{H} = -[\partial v]\cdot\nabla_x f - \partial\nu f + \partial H$$

and $\partial_{x_i} g$ are given by

$$\nabla_x g = \frac{n}{n\cdot v}\Big\{-\partial_t g - \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} g - \nu g + H\Big\} + \sum_{i=1}^{2}\tau_i\partial_{\tau_i} g$$

Idea: Previous estimates can be applied to this case

$$||\partial f(t)||_p^p + \int_0^t |\partial f|_{\gamma_+,p}^p \lesssim ||\partial f_0||_p^p + \int_0^t |\partial g|_{\gamma_-,p}^p + p\int_0^t \iint_{\Omega\times\mathbb{R}^3} |\partial\mathcal{H}||\partial f|^{p-1}$$

Existence theorem for diffuse, specular reflection, or bounce-back reflection BC

Assume $\Omega = \{\xi < 0\}$ is a smooth bounded domain.
Suppose $f_0 \geq 0$ satisfies the compatibility conditions and for $0 < \theta < 1/4$

$$||e^{\theta|v|^2} f_0||_\infty < +\infty$$

Theorem (Existence/Uniqueness (Guo10), (GKTT16))

There exists a unique solution $F = \sqrt{\mu} f \geq 0$ of the Boltzmann equation on $[0, T^*]$ with $T^* = T^*(||e^{\theta|v|^2} f_0||_\infty)$. Furthermore

$$\sup_{0 \leq t \leq T^*} ||e^{\theta'|v|^2} f(t)||_\infty \lesssim P(||e^{\theta|v|^2} f_0||_\infty),$$

for $0 < \theta' < \theta < 1/4$ and some polynomial P.

If $\|e^{\theta|v|^2}\{f_0 - \sqrt{\mu}\}\|_\infty << 1$ then $T^* = +\infty$.
(ξ has to be real analytic in the specular reflection case).

Idea of the proof: use a positive preserving iteration

$F^m := \sqrt{\mu} f^m$, for all $m \in \mathbb{N}$

$$\partial_t f^{m+1} + v \cdot \nabla_x f^{m+1} + \nu(\sqrt{\mu} f^m) f^{m+1} = \Gamma_{\mathrm{gain}}(f^m, f^m),$$
$$f^{m+1}|_{t=0} = f_0 \geq 0, \quad f^0 \equiv f_0 \geq 0$$

NOTE: for every step m we are considering a linear transport equation

$$\nu = \nu(\sqrt{\mu} f^m), \quad H = \Gamma_{\mathrm{gain}}(f^m, f^m)$$

with in-flow BC

$$\forall (x, v) \in \gamma_- \quad f^{m+1}(t, x, v) = g^m(t, x, v)$$

- Diffuse reflection boundary condition, on $(x, v) \in \gamma_-$,

$$f^{m+1}(t,x,v) = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f^m(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u$$
$$=: g^m(t,x,v),$$

- Specular reflection boundary condition, on $(x, v) \in \gamma_-$,

$$f^{m+1}(t,x,v) \ = \ f^m(t,x,R_x v) =: g^m(t,x,v),$$

where $R_x v = v - 2n(x)(n(x)\cdot v)$.

- Bounce-back reflection boundary condition, on $(x, v) \in \gamma_-$,

$$f^{m+1}(t,x,v) \ = \ f^m(t,x,-v) =: g^m(t,x,v)$$

For all $m \in \mathbb{N}$

$$\partial_t f^{m+1} + v \cdot \nabla_x f^{m+1} + \nu(\sqrt{\mu} f^m) f^{m+1} = \Gamma_{\mathrm{gain}}(f^m, f^m),$$
$$f^{m+1}|_{t=0} = f_0 \geq 0, \quad f^0 \equiv f_0 \geq 0,$$
$$\forall (x,v) \in \gamma_- \quad f^{m+1}(t,x,v) = g^m(t,x,v)$$

By Duhamel formula

$$\begin{aligned} f^{m+1}(t,x,v) =& \mathbf{1}_{\{t \leq t_{\mathbf{b}}\}} e^{-\int_0^t \nu(\sqrt{\mu} f^m)} f_0(x - tv, v) \\ &+ \mathbf{1}_{\{t > t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu(\sqrt{\mu} f^m)} g^m(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\ &+ \int_0^{\min(t, t_{\mathbf{b}})} e^{-\int_0^s \nu(\sqrt{\mu} f^m)} \Gamma_{\mathrm{gain}}(f^m, f^m)(t - s, x - vs, v) \mathrm{d}s \end{aligned}$$

Note: by Grad estimates for $0 < \theta < 1/4$, $p \in [1, +\infty)$

$$||\Gamma_{\mathrm{gain}}(g_1, g_2)||_p \lesssim_{\theta,p} ||e^{\theta|v|^2} g_1||_\infty ||g_2||_p$$

and

$$||\nu(\sqrt{\mu} g_1) g_2||_p \lesssim_{\theta,p} ||e^{\theta|v|^2} g_2||_\infty ||g_1||_p$$

Use an $L^2 - L^\infty$ interpolation argument to find estimates

$$\sup_{0 \le t \le T} ||e^{\theta'|v|^2} f^{m+1}(t)||_\infty \lesssim ||e^{\theta|v|^2} f_0||_\infty,$$

for some $0 < \theta' < \theta$

Convex case

In this part we concentrate on a strictly convex domain

Let Ω be a bounded open subset of $\mathbb{R}^3$, i.e.

$$\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}, \quad \text{and} \quad \partial\Omega = \{x \in \mathbb{R}^3 : \xi(x) = 0\}$$

for a smooth $\xi : \mathbb{R}^3 \to \mathbb{R}$

For all $x \in \bar{\Omega} = \Omega \cup \partial\Omega$ we assume the domain is strictly convex :

$$\sum_{i,j} \partial_{ij}\xi(x)\zeta_i\zeta_j \geq C_\xi |\zeta|^2 \quad \text{for all} \quad \zeta \in \mathbb{R}^3$$

We assume that $\nabla\xi(x) \neq 0$ when $|\xi(x)| \ll 1$ and we define the unit outward normal as $n(x) = \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$

Time derivative for diffuse, specular reflection, or bounce-back reflection BC

Suppose $f_0 \geq 0$ satisfies the compatibility conditions and for $0 < \bar{\theta}, \theta < 1/4$

$$||e^{\theta|v|^2} f_0||_\infty + ||e^{\bar{\theta}|v|^2} \partial_t f_0||_\infty < +\infty$$

Recall $\partial_t f_0 := -v \cdot \nabla_x f_0 - \nu(\sqrt{\mu} f_0) f_0 + \Gamma_{\mathrm{gain}}(f_0, f_0)$

Theorem (Time derivative regularity (GKTT16))

$$\sup_{0 \leq t \leq T^*} ||e^{\bar{\theta}|v|^2} \partial_t f(t)||_\infty \lesssim P(||e^{\bar{\theta}|v|^2} \partial_t f_0||_\infty) + P(||e^{\theta|v|^2} f_0||_\infty),$$

for $0 < \theta' < \min\{\bar{\theta}, \theta\} < 1/4$ *and some polynomial* P.

IDEA: Use again the positive preserving iteration

$$\partial_t f^{m+1}(t,x,v)\mathbf{1}_{\{t\neq t_\mathbf{b}\}} =$$

$$- \mathbf{1}_{\{t<t_\mathbf{b}\}} e^{-\int_0^t \nu(\sqrt{\mu}f^m)}[\nu(\sqrt{\mu}f^m)f_0 + \int_0^t \partial_t \nu(\sqrt{\mu}f^m)\ f_0 + v\cdot\nabla_x f_0](x-tv,v)$$

$$+ \mathbf{1}_{\{t>t_\mathbf{b}\}} e^{-\int_0^{t_\mathbf{b}} \nu(\sqrt{\mu}f^m)}[\partial_t g^m - \int_0^{t_\mathbf{b}} \partial_t \nu(\sqrt{\mu}f^m)](t-t_\mathbf{b}, x_\mathbf{b}, v)$$

$$- \int_0^{\min(t,t_\mathbf{b})} e^{-\int_0^s \nu(\sqrt{\mu}f^m)} \int_0^s \partial_t \nu(\sqrt{\mu}f^m)\ \Gamma_{\mathrm{gain}}(f^m,f^m)(t-s,x-vs,v)\mathrm{d}s$$

$$+ \int_0^{\min(t,t_\mathbf{b})} e^{-\int_0^s \nu(\sqrt{\mu}f^m)} \partial_t \Gamma_{\mathrm{gain}}(f^m,f^m)(t-s,x-vs,v)\mathrm{d}s$$

$$+ \mathbf{1}_{\{t<t_\mathbf{b}\}} e^{-\int_0^t \nu(\sqrt{\mu}f^m)} \Gamma_{\mathrm{gain}}(f^m,f^m)_{|t=0}(x-tv,v)$$

How to deal with space and velocity derivatives ?

One of the crucial ingredient for our results is the construction of a distance function towards the grazing set γ_0

Definition (Kinetic Distance)

For $(x, v) \in \bar{\Omega} \times \mathbb{R}^3$,

$$\alpha(x, v) := |v \cdot \nabla\xi(x)|^2 - 2\{v \cdot \nabla^2\xi(x) \cdot v\}\xi(x).$$

Properties:

- vanishes exactly on the grazing boundary
- invariant along the characteristics (up to some quantity in $|v|$)

Velocity Lemma

Lemma (Velocity Lemma (Guo10))

Along the backward trajectory $X_{\mathbf{cl}}, V_{\mathbf{cl}}$ we define

$$\alpha(s;t,x,v) := \alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)).$$

Then there exists $\mathcal{C} = \mathcal{C}(\xi) > 0$ such that, for all $0 \leq s_1, s_2 \leq t$,

$$e^{-\mathcal{C}|v||s_1-s_2|}\alpha(s_1;t,x,v) \ \leq \ \alpha(s_2;t,x,v) \ \leq \ e^{\mathcal{C}|v||s_1-s_2|}\alpha(s_1;t,x,v).$$

This Lemma implies that in a strictly convex domain, the singular set γ_0 cannot be reached via the trajectories starting from interior points inside the domain, and hence γ_0 does not really participate in or interfere with the interior dynamics

Main IDEA : for $(x, v) \in \gamma_-$,

$$\nabla_v f(t,x,v) \quad = \quad c_\mu \nabla_v \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u$$

IDEM for tangential derivatives $\partial_{\tau_i} f(t,x,v)$

PROBLEM : How to control the normal spatial derivative close to γ_-?

$$\partial_n f(t,x,v) = -\frac{1}{n(x)\cdot v}\left\{\partial_t f + \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} f - \Gamma_{\text{gain}}(f,f) + \nu(\sqrt{\mu}f)f\right\},$$

and

$$\int_{\gamma_-} |\partial_n f|^p |v\cdot n| \mathrm{d}v \lesssim \underbrace{\int_{\mathbb{R}^3} |v\cdot n|^{1-p}}_{<\infty \quad \text{when} \quad p<2}$$

Diffuse boundary conditions, convex case

Suppose $f_0 \geq 0$ satisfies the diffuse BC compatibility conditions and for $0 < \theta < 1/4$, $1 < p < 2$,

$$||\nabla_x f_0||_p + ||\nabla_v f_0||_p + ||e^{\theta|v|^2} f_0||_\infty < +\infty$$

Theorem ($W^{1,p}$ propagation (GKTT16))

$$f \in L^\infty_{loc}([0, T^*]; W^{1,p}(\Omega \times \mathbb{R}^3))$$

and for all $0 \leq t \leq T$

$$||\nabla_x f(t)||_p^p + ||\nabla_v f(t)||_p^p + \int_0^t \left[|\nabla_x f(s)|_{\gamma,p}^p + |\nabla_v f(s)|_{\gamma,p}^p \right] \mathrm{d}s$$
$$\lesssim_t \ ||\nabla_x f_0||_p^p + ||\nabla_v f_0||_p^p + P(||e^{\theta|v|^2} f_0||_\infty),$$

where P is some polynomial.

Here $|f(s)|_{\gamma,p}^p = \int_{\partial\Omega\times\mathbb{R}^3} |f(s,x,v)|^p |n(x)\cdot v| \mathrm{d}S_x \, \mathrm{d}v$

Idea of the proof: For $\partial_{\mathbf{e}} = [\partial_x, \partial_v]$, ∂f^m satisfies

$$\{\partial_t + v \cdot \nabla_x + \nu(\sqrt{\mu} f^m)\}\partial f^{m+1} = \mathcal{H}^m, \quad \partial f^{m+1}(0,x,v) = \partial f_0(x,v),$$

$$\partial f^{m+1}(t,x,v)|_{\gamma_-} = \partial g^m(t,x,v)$$

where

$$\mathcal{H}^m \ = \ -[\partial v] \cdot \nabla_x f^{m+1} - \partial[\nu(\sqrt{\mu} f^m)] f^{m+1} + \partial[\Gamma_{\text{gain}}(f^m, f^m)],$$

$$||\partial f^{m+1}(t)||_p^p + \int_0^t |\partial f^{m+1}|_{\gamma_+,p}^p$$

$$\lesssim ||\partial f_0||_p^p + \int_0^t |\partial g^m|_{\gamma_-,p}^p + p \int_0^t \iint_{\Omega \times \mathbb{R}^3} |\partial \mathcal{H}^m| |\partial f^{m+1}|^{p-1}$$

$p < 2$ to bound the red term

Idea of the proof: For $\partial_{\mathbf{e}} = [\partial_x, \partial_v]$, ∂f^m satisfies

$$\{\partial_t + v \cdot \nabla_x + \nu(\sqrt{\mu} f^m)\}\partial f^{m+1} = \mathcal{H}^m, \quad \partial f^{m+1}(0,x,v) = \partial f_0(x,v),$$

$$\partial f^{m+1}(t,x,v)|_{\gamma_-} = \partial g^m(t,x,v)$$

where

$$\mathcal{H}^m \;=\; -[\partial v]\cdot \nabla_x f^{m+1} - \partial[\nu(\sqrt{\mu} f^m)] f^{m+1} + \partial[\Gamma_{\mathrm{gain}}(f^m, f^m)],$$

$$\sup_{0\leq t\leq T_*} ||\partial f^m||_p^p + \int_0^{T_*} |\partial f^m|_{\gamma,p}^p \lesssim_{\Omega, T_*} ||\partial f_0||_p^p + P(||e^{\theta|v|^2} f_0||_\infty),$$

for some polynomial $P \Longrightarrow$ weak convergence for $p > 1$

Suppose $f_0 \geq 0$ satisfies the diffuse BC compatibility conditions and for $0 < \theta < 1/4$, $2 \leq p < +\infty$, and $\frac{p-2}{2p} < \beta < \frac{p-1}{2p}$,

$$||\alpha^\beta \nabla_x f_0||_p + ||\alpha^\beta \nabla_v f_0||_p + ||e^{\theta|v|^2} f_0||_\infty < \infty$$

Theorem ($W^{1,p}$ propagation (GKTT16))

There exists $\varpi > 0$ *s.t.* $e^{-\varpi\langle v\rangle t}\alpha^\beta \nabla_{x,v} f \in L^\infty_{loc}([0, T^*]; L^p(\Omega \times \mathbb{R}^3))$ *and for all* $0 \leq t \leq T$

$$||e^{-\varpi\langle v\rangle t}\alpha^\beta \nabla_{x,v} f(t)||_p^p + \int_0^t |e^{-\varpi\langle v\rangle t}\alpha^\beta \nabla_{x,v} f(s)|_{\gamma,p}^p \mathrm{d}s$$
$$\lesssim_t \ ||\alpha^\beta \nabla_{x,v} f_0||_p^p + P(||e^{\theta|v|^2} f_0||_\infty),$$

where P *is some polynomial.*

Note: $\partial f(t) \sim e^{\varpi\langle v\rangle t}$, for a $\varpi > 0$ that is determined by the geometry of $\partial\Omega$, for example if ξ is quadratic we can set $\varpi = 0$

Suppose $f_0 \geq 0$ satisfies the compatibility conditions and for $0 < \theta < 1/4$

$$||\alpha^{1/2}\nabla_{x,v} f_0||_\infty + ||e^{\theta|v|^2} f_0||_\infty < +\infty$$

Theorem ($W^{1,\infty}$ and C^1 propagation (GKTT16))

There exists $\varpi > 0$ s.t.

$$e^{-\varpi\langle v\rangle t}\alpha^{1/2}\nabla_{x,v} f \in L^\infty([0, T^*]; L^\infty(\Omega \times \mathbb{R}^3))$$

and for all $0 \leq t \leq T^*$,

$$||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\nabla_{x,v} f(t)||_\infty \ \lesssim_t \ ||\alpha^{1/2}\nabla_{x,v} f_0||_\infty + P(||e^{\theta|v|^2} f_0||_\infty)$$

where P is some polynomial.

If $\alpha^{1/2}\nabla_{x,v} f_0 \in C^0(\bar{\Omega} \times \mathbb{R}^3)$ and

$$\partial_t f_0 = c_\mu \sqrt{\mu} \int_{n\cdot u>0} \{u \cdot \nabla_x f_0 + \nu(\sqrt{\mu(u)} f_0) f_0 - \Gamma(f_0, f_0)\} \sqrt{\mu} \{n \cdot u\} \mathrm{d}u,$$

is valid for $\gamma_- \cup \gamma_0$, then $f \in C^1$ away from the grazing set γ_0.

Diffuse boundary conditions, non convex case

The singular set

$$\mathfrak{S}_{\mathbf{b}} := \{(x, v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x, v)) \cdot v = 0\},$$

is a set of co-dimension 1 in $\Omega \times \mathbb{R}^3$: We look for BV regularity

IDEA : remove a tubular neighborhood of $\mathfrak{S}_{\mathbf{b}}$ using cut off functions in order to obtain $W^{1,1}$ estimates

Notation:

$$||f||_{BV} := ||f||_{L^1(\Omega)} + ||f||_{\tilde{BV}},$$

where

$$||f||_{\tilde{BV}} := \sup \left\{ \iint_{\Omega\times\mathbb{R}^3} f \,\mathrm{div}\varphi \,\mathrm{d}x\mathrm{d}v : \varphi \in C_c^1(\Omega\times\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3),\ |\varphi| \leq 1 \right\} < \infty$$

Suppose $f_0 \geq 0$ satisfies the compatibility conditions and for $0 < \theta < 1/4$

$$||f_0||_{BV} + ||e^{\theta|v|^2} f_0||_\infty < +\infty$$

Theorem (BV propagation (GKTT15))

$$f \in L^\infty([0, T^*]; BV(\Omega \times \mathbb{R}^3))$$

and $\nabla_{x,v} f \mathrm{d}\gamma$ *is a Radon measure on* $\partial\Omega \times \mathbb{R}^3$*. Moreover, for all* $0 \leq t \leq T^*$

$$||f(t)||_{BV} \lesssim_{T^*,\Omega} ||f_0||_{BV} + P(||e^{\theta|v|^2} f_0||_\infty),$$

for some polynomial P *and* $\nabla_{x,v} f_\gamma(t)$ *is a Radon measure* σ_t *on* $\partial\Omega \times \mathbb{R}^3$ *such that*

$$\int_0^{T^*} |\sigma_t(\partial\Omega \times \mathbb{R}^3)| \mathrm{d}t \lesssim_{T^*,\Omega} ||f_0||_{BV} + P(||e^{\theta|v|^2} f_0||_\infty).$$

Here f_γ is the trace of f on γ

Idea of the proof: we reduce, as usual to a simpler linear problem

$$\partial_t f + v \cdot \nabla_x f + \nu f = H, \qquad f|_{t=0} = f_0,$$

where $\nu = \nu(t, x, v) \geq 0$, H, are smooth enough, with the in-flow boundary condition

$$f(t, x, v) = g(t, x, v) \quad (x, v) \in \gamma_-$$

Then as usual we could apply the positive preserving iteration scheme to produce $W^{1,1}$ estimates for the sequence

PROBLEM: Solutions of such a transport equation are discontinuous on $\mathfrak{S}_{\mathbf{b}}$

$\Longrightarrow$ use a smooth cut-off function $\chi_\varepsilon(x, v)$ vanishing on an open neighborhood of $\mathfrak{S}_\mathbf{b}$:

$$\begin{aligned} \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \nu f^\varepsilon &= \chi_\varepsilon H \quad \text{in } (x, v) \in \Omega \times \mathbb{R}^3, \\ f^\varepsilon|_{t=0} &= \chi_\varepsilon f_0 \quad \text{in } (x, v) \in \Omega \times \mathbb{R}^3, \\ f^\varepsilon(t, x, v) &= \chi_\varepsilon g(t, x, v) \quad (x, v) \in \gamma_-. \end{aligned}$$

A uniform-in-ε bound of $\partial f^\varepsilon = [\nabla_x f^\varepsilon, \nabla_v f^\varepsilon]$ in $L^1(\Omega \times \mathbb{R}^3)$ will be enough to prove the thesis

$$\sup_{0 \le s \le t} ||\partial f^\varepsilon(s)||_1 + \int_0^t |\partial f^\varepsilon(s)|_{\gamma,1} \lesssim ||f_0||_{BV} + P(||e^{\theta|v|^2} f_0||_\infty)$$

where P is a polynomial

Recap and optimality of results

In convex domains	In non-convex domains
C^0 away from γ_0	*NO* C^0 Discontinuity created on γ_0 and propagated along the grazing trajectories [Kim 11]
$W^{1,p}$ for $1 \leq p < 2$	
NO H^1 : c-ex (transport eq.)	BV regularity
Weighted $W^{1,p}$ for $p \in [2, +\infty]$	
C^1 away from γ_0	
NO $W^{2,1}$: c-ex	

Further results

Other boundary conditions

- Specular boundary conditions : in convex domains, propagation of C^1 regularity away from γ_0 (with the help of the kinetic distance) [GKTT16]
- Bounce-back boundary conditions : same [GKTT16], in non-convex domains: propagation of discontinuity [Kim 11]
- Maxwell boundary conditions: continuity away from the grazing trajectories [Briant Guo 15]

Non-isothermal boundary Results of existence of strong solutions, uniqueness and stability (exponential convergence towards the solution of the stationnary problem) for a (not too much) varying boundary temperature. Continuity propagation in convex domains [Esposito Guo Kim Marra 13].