

# Second order conformally invariant elliptic equations

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• Theorem 3-1 (Luc Nguyen, L.) (Blow up analysis)

Assume  $\{u_k\} \in C^2(B_2)$ ,

$$f(\lambda(A^{u_k})) = 1, \quad u_k > 0, \quad \text{in } B_2, \quad \sup_{B_1} u_k \rightarrow \infty.$$

Then  $\forall \epsilon > 0$ , after passing to a subsequence,

$\exists \{x_k^1, \dots, x_k^m\} \subset B_2(0)$ ,  $1 \leq m \leq \bar{m}$ ,

$$|x_k^i - x_k^j| \geq K^{-1} > 0, \quad \forall k, i \neq j,$$

$$u_k(x_k^i) = \sup_{B_\delta(x_k^i)} u_k.$$

$$K^{-1} \leq \frac{u_k(x_k^i)}{u_k(x_k^j)} \leq K, \quad \forall i, j, k,$$

$$|u_k(x) - U^{x_k^i, u_k(x_k^i)}(x)| \leq \epsilon U^{x_k^i, u_k(x_k^i)}(x), \quad \forall x \in B_\delta(x_k^i).$$

$$\frac{1}{K\delta^{n-2}u_k(x_k^1)} \leq u_k(x) \leq \frac{K}{\delta^{n-2}u_k(x_k^1)}, \quad \text{in } B_{\frac{3}{2}}(0) \setminus \cup_{i=1}^m B_\delta(x_k^i), \quad \forall k.$$

—  $U^{\bar{x}, \mu}(x) = \mu U(\mu^{\frac{2}{n-2}}(x - \bar{x}))$ ,

—  $U(x) = (1 + |x|^2)^{\frac{2-n}{2}}$  satisfies  $f(\lambda(A^U)) = 1$ ,

—  $\bar{m}, K$  depend only on  $(f, \Gamma)$ ,  $\delta$  depends on  $(f, \Gamma)$  and  $\epsilon$ .

**Proposition 3-1.** (Strengthened Liouville type theorem) Assume  $0 < v \in C^0(\mathbb{R}^n)$ ,  $0 < v_k \in C^2(B_{R_k})$ ,  $R_k \rightarrow \infty$ ,

$$f(\lambda(A^{v_k})) = 1 \text{ in } B_{R_k}, \quad v_k \rightarrow v \text{ in } C_{loc}^0(\mathbb{R}^n).$$

Then 
$$v(x) = \left( \frac{a}{1 + a^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \quad a > 0, \bar{x} \in \mathbb{R}^n.$$

- $v$  satisfies  $f(\lambda(A^v)) = 1$ , in  $\mathbb{R}^n$  in viscosity sense.

**Open Problem.** Let  $0 < v \in C_{loc}^0(\mathbb{R}^n)$  satisfy

$$f(\lambda(A^v)) = 1, \text{ in } \mathbb{R}^n \text{ in viscosity sense.}$$

Is it true that

$$v(x) = \left( \frac{a}{1 + a^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \quad a > 0, \bar{x} \in \mathbb{R}^n?$$

**Proof of Proposition 3-1.**  $0 < v$  superharmonic, so

$$|y|^{n-2}v(y) \geq 2c_0 > 0, \quad \forall |y| \geq 1.$$

Passing to subsequence, shrinking  $R_k$ , shrinking  $c_0$ , may assume

$$|v_k(y) - v(y)| \leq (R_k)^{-n}, \quad v_k(y) \geq c_0(R_k)^{2-n}, \quad \forall |y| \leq R_k.$$

- **Define** for  $x \in \mathbb{R}^n$ ,  $|x| + 1 \leq R_k/4$ ,

$$\bar{\lambda}_k(x) = \sup\{0 < \mu \leq \frac{R_k}{4} \mid (v_k)_{x,\lambda} \leq v_k \text{ in } B_{R_k}(0) \setminus B_\lambda(x), \forall 0 < \lambda < \mu\},$$

where  $(v_k)_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} v_k\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)$ , the Kelvin transformation.

- $\bar{\lambda}_k(x)$  well defined and  $\exists C(x) > 0$  such that

$$0 < \frac{1}{C(x)} \leq \bar{\lambda}_k(x) \leq \frac{R_k}{4}, \quad \forall k.$$

— **Proof based on :**

- Local gradient estimates:

$$u \in C^2(B_2), \quad f(\lambda(A^u)) = 1, \quad 0 < u \leq b, \quad \text{in } B_2$$

implies

$$|\nabla \log u| \leq C \quad \text{in } B_1$$

—  $C$  depends only on  $(f, \Gamma)$  and  $b$ .

• Set  $\bar{\lambda}(x) = \liminf_{k \rightarrow \infty} \bar{\lambda}_k(x) \in (0, \infty]$ .

• Can prove (maximum principle, Hopf Lemma): either  $\bar{\lambda}(x) \equiv \infty \forall x$  or  $\bar{\lambda}(x) < \infty \forall x$ .

$\bar{\lambda}(x) \equiv \infty$  leads to:  $v \equiv \text{Constant}$ , which can be ruled out.

$\bar{\lambda}(x) < \infty \forall x$  leads to:

$$\lim_{|y| \rightarrow \infty} |y|^{n-2} v_{x, \bar{\lambda}(x)}(y) = \alpha := \liminf_{|y| \rightarrow \infty} |y|^{n-2} v(y) < \infty, \quad \forall x.$$

- **We have arrived at:**  $0 < v \in C_{loc}^{0,1}(\mathbb{R}^n)$ ,  $\Delta v \leq 0$  in  $\mathbb{R}^n$ , for every  $x \in \mathbb{R}^n$ , there exists  $0 < \bar{\lambda}(x) < \infty$  such that

$$v_{x, \bar{\lambda}(x)}(y) \leq v(y), \quad \forall |y - x| \geq \bar{\lambda}(x),$$

$$\lim_{|y| \rightarrow \infty} |y|^{n-2} v_{x, \bar{\lambda}(x)}(y) = \alpha := \liminf_{|y| \rightarrow \infty} |y|^{n-2} v(y), \quad \forall x.$$

- **Claim.** We can deduce from the above that

$$v(x) = b \left( \frac{a}{1 + a^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \quad a, b > 0, \bar{x} \in \mathbb{R}^n.$$

Since  $v_k \rightarrow v$  in  $C_{loc}^0(\mathbb{R}^n)$  and  $f(\lambda(A^{v_k})) = 1$ , can prove that  $b = 1$ .

- **A Lemma.** For  $n \geq 2$ ,  $B_1 \subset \mathbb{R}^n$ ,  $w_1, w_2 \in C^0(B_1)$ ,  $w_1, w_2$  differentiable at 0,  $u \in L^1_{loc}(B_1 \setminus \{0\})$ ,  $\Delta u \leq 0$  in  $B_1 \setminus \{0\}$ ,

$$u(y) \geq \max\{w_1(y), w_2(y)\}, \quad y \in B_1 \setminus \{0\},$$

$$w_1(0) = w_2(0) = \liminf_{y \rightarrow 0} u(y).$$

Then

$$\nabla w_1(0) = \nabla w_2(0).$$

- Apply the lemma with  $w_1 = w^{(x)}$ ,  $w_2 = w^{(\tilde{x})}$ ,  $u = v_\psi$ ,  $x, \tilde{x} \in D$ .



• **Proof of Claim.** Let  $\psi(y) = \frac{y}{|y|^2}$ , and

$$w^{(x)} := (v_x, \bar{\lambda}(x))_\psi, \quad x \in \mathbb{R}^n.$$

• Then, for some  $\delta(x) > 0$ ,

$$v_\psi \geq w^{(x)} \text{ in } B_{\delta(x)} \setminus \{0\},$$

$$w^{(x)}(0) = \alpha = \liminf_{y \rightarrow 0} v_\psi(y),$$

$$\Delta v_\psi \leq 0, \text{ in } B_{\delta(x)} \setminus \{0\},$$

• Let  $D = \{x \in \mathbb{R}^n \mid v \text{ is differentiable at } x\}$ . Since  $v \in C_{loc}^{0,1}(\mathbb{R}^n)$ , Lebesgue measure  $|\mathbb{R}^n \setminus D| = 0$ .

• By **the Lemma**,

$$\nabla w^{(x)}(0) = \nabla w^{(\tilde{x})}(0), \quad \forall x, \tilde{x} \in D.$$

Namely, for some  $V \in \mathbb{R}^n$ ,

$$\nabla w^{(x)}(0) = V, \quad \forall x \in D.$$

- A calculation yields

$$\nabla w^{(x)}(0) = (n-2)\alpha x + \alpha^{\frac{n}{n-2}} v(x)^{\frac{n}{n-2}} \nabla v(x).$$

- Thus

$$\nabla_x \left( \frac{n-2}{2} \alpha^{\frac{n}{n-2}} v(x)^{-\frac{2}{n-2}} - \frac{n-2}{2} \alpha |x|^2 + V \cdot x \right) = 0, \quad \forall x \in D.$$

- Consequently, for some  $\bar{x} \in \mathbb{R}^n$  and  $d \in \mathbb{R}$ ,

$$v(x)^{-\frac{2}{n-2}} \equiv \alpha^{-\frac{2}{n-2}} |x - \bar{x}|^2 + d \alpha^{-\frac{2}{n-2}}.$$

- Since  $v > 0$ , we must have  $d > 0$ , so

$$v(x) \equiv \left( \frac{\alpha^{\frac{2}{n-2}}}{d + |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}.$$

Claim proved.

**Proposition 3-2.** Assume  $\{v_k\} \in C^2(B_{R_k})$ ,  $R_k \rightarrow \infty$ ,

$$f(\lambda(A^{v_k}))(y) = 1, \quad 0 < v_k(y) \leq v_k(0) = 1, \quad |y| \leq R_k. \quad (1)$$

Then  $\forall \epsilon > 0$ ,  $\exists k'_0 = k'_0(\epsilon)$  and  $\delta' = \delta'(\epsilon)$  such that  $\forall k > k'_0$ ,

$$|v_k(y) - U(y)| \leq 2\epsilon U(y), \quad \forall |y| \leq \delta' R_k. \quad (2)$$

Recall:

$$\begin{aligned} \text{--- } U(x) &:= \left( \frac{1}{1+|x|^2} \right)^{\frac{n-2}{2}} \\ \text{--- } A^U &\equiv 2I, \quad f(\lambda(A^U)) \equiv 1 \end{aligned}$$

By the local gradient estimates and by Proposition 3-1, after passing to subsequence,

$$v_k \rightarrow U, \text{ in } C_{loc}^0(\mathbb{R}^n).$$

**Lemma 1.**  $\forall \epsilon > 0, \exists k_0$ , such that  $\forall k \geq k_0$ ,

$$\min_{|y|=r} v_k(y) \leq (1 + \epsilon)U(y), \quad \forall 0 < r < R_k.$$

**Proof.**

• Facts:

$$U_{0,\lambda}(y) < U(y), \quad \forall 0 < \lambda < 1, |y| > \lambda,$$

$$U_{0,1} \equiv U.$$

$$U_{0,\lambda}(y) > U(y), \quad \forall \lambda > 1, |y| > \lambda.$$

- Contradiction argument: If for some  $\epsilon > 0$ ,  $\exists r_k$

$$\min_{|y|=r_k} v_k(y) > (1 + \epsilon)U(y).$$

- Then, using the above facts of  $U$ ,  $r_k \rightarrow \infty$ , and

$$(v_k)_\lambda(y) \leq v_k(y), \quad \forall 0 < \lambda < 1 + \epsilon^2, |y| = r_k.$$

- Sending  $k \rightarrow \infty$ .

$$U_\lambda(y) \leq U(y), \quad \forall 0 < \lambda < 1 + \epsilon^2, \lambda < |y| < \infty.$$

Violating the above facts of  $U$ . Lemma 1 proved.

**Lemma 2.**  $\forall \epsilon > 0, \exists$  small  $\delta_1 > 0$ , large  $r_1 > 0$ , such that for large  $k$ ,

$$v_k(y) \geq (1 - \epsilon)U(y), \quad \forall |y| \leq \delta_1 R_k,$$

$$\int_{r_1 \leq |y| \leq \delta_1 R_k} v_k^{\frac{n+2}{n-2}} \leq \epsilon.$$

**Proof.** Since  $v_k \rightarrow U$  in  $C_{loc}^0(\mathbb{R}^n)$ ,  $\exists r_1$  such that for large  $k$

$$v_k(y) \geq (1 - \epsilon^2)U(y), \quad \forall |y| \leq r_1,$$

$$v_k(y) \geq (1 - \epsilon^2)(r_1)^{2-n}, \quad \forall |y| = r_1,$$

Superharmonicity of  $v_k$ , maximum principle, we have

$$v_k(y) \geq (1 - \epsilon^2) (|y|^{2-n} - (R_k)^{2-n}), \quad r_1 \leq |y| \leq R_k.$$

Thus, for any  $\delta_1 \in (0, \epsilon^{\frac{2}{n-2}})$ ,

$$v_k(y) \geq (1 - 2\epsilon^2)|y|^{2-n}, \quad r_1 \leq |y| \leq \delta_1 R_k.$$

The equation of  $v_k$  implies that  $\exists \delta > 0$ ,

$$-\Delta v_k(y) \geq \frac{n-2}{2} \delta v_k(y)^{\frac{n+2}{n-2}} \quad \text{in } r_1 \leq |y| \leq \delta_1 R_k.$$

This implies

$$v_k(y) \geq (1 - 2\epsilon^2)|y|^{2-n} + \frac{1}{C}|y|^{2-n} \int_{2r_1 \leq |x| \leq \delta_1 R_k/8} \delta v_k(x)^{\frac{n+2}{n-2}} dx, \quad \forall |y| = \frac{\delta_1 R_k}{2}.$$

By Lemma 1,

$$(1 + 2\epsilon^2)|y|^{2-n} \geq v_k(y), \quad \forall |y| = \frac{\delta_1 R_k}{2}.$$

Lemma 2 follows from the above.

Since  $v_k \leq 1$ , by Lemma 2, for any  $\epsilon > 0$ , we have, for large  $k$ ,

$$\int_{r_1 \leq |y| \leq \delta_1 R_k} v_k^{\frac{2n}{n-2}} \leq \epsilon.$$

• Small energy implies  $L^\infty$  bound — consequence of Liouville, as showed before.

**Lemma 3.**  $\exists \delta_0 > 0$  and  $C_0 > 1$  such that if  $0 < u \in C^2(B_2)$ ,

$$f(\lambda(A^u)) = 1, \text{ in } B_2, \quad \int_{B_2} u^{\frac{2n}{n-2}} \leq \delta_0,$$

then

$$u \leq C_0 \text{ in } B_1.$$



**Lemma 4.**  $\exists C, \delta_4 > 0$ , independent of  $k$ , such that

$$v_k(y) \leq CU(y), \quad \forall |y| \leq \delta_4 R_k.$$

**Proof.**  $\forall 4r_1 < r < \delta_1 R_k/4$ , consider

$$\tilde{v}_k(z) = r^{\frac{n-2}{2}} v_k(rz), \quad \frac{1}{4} < |z| < 4.$$

For large  $k$ ,

$$\int_{\frac{1}{4} < |z| < 4} \tilde{v}_k(z)^{\frac{2n}{n-2}} = \int_{\frac{r}{4} < |\eta| < 4r} v_k(\eta)^{\frac{2n}{n-2}} \leq \epsilon := \delta_0,$$

where  $\delta_0 > 0$  is the number in Lemma 3.

- By Lemma 3,

$$\tilde{v}_k(z) \leq C, \quad \frac{1}{3} < |z| < 3,$$

for some universal constant  $C$ .

- By local gradient estimates,

$$|\nabla \log \tilde{v}_k(z)| \leq C, \quad \frac{1}{2} < |z| < 2.$$

- Thus

$$\max_{|z|=1} \tilde{v}_k(z) \leq \min_{|z|=1} \tilde{v}_k(z).$$

i.e.

$$\max_{|x|=r} v_k(x) \leq C \min_{|x|=r} v_k(x) \leq CU(r).$$

— used Lemma 1 for last inequality. Lemma 4 follows immediately.

**Proof of Proposition 3-2.** Only need to prove that there exists  $\delta'$  and  $k'_0$  such that for any  $k \geq k'_0$ ,

$$v_k(y) \leq (1 + 2\epsilon)U(y), \quad \forall |y| \leq \delta' R_k.$$

Suppose the contrary, passing to subsequence,  $\exists |y_k| = \delta_k R_k$ ,

$\delta_k \rightarrow 0^+$ , but

$$v_k(y_k) = \max_{|y|=\delta_k R_k} v_k(y) \geq (1 + 2\epsilon)U(y_k).$$

Since  $v_k \rightarrow v$  in  $C_{loc}^0(\mathbb{R}^n)$ ,  $|y_k| \rightarrow \infty$ .

Consider rescaling of  $v_k$ :

$$\hat{v}_k(z) := |y_k|^{n-2} v_k(|y_k|z), \quad |z| < \frac{\delta_4 R_k}{|y_k|} \rightarrow \infty.$$

We have

$$f_k(\lambda(A^{\hat{v}_k}))(z) := |y_k|^{-2} f(\lambda(A^{v_k}))(z) = |y_k|^{-2}, \quad |z| < \frac{\delta_4 R_k}{|y_k|}.$$

Since  $\hat{v}_k \leq C$ , we can apply gradient estimates to  $f_k$  to obtain:

$\forall 0 < \alpha < \beta < \infty, \exists C(\alpha, \beta)$  such that for large  $k$ ,

$$|\nabla \log \hat{v}_k(z)| \leq C(\alpha, \beta), \quad \forall \alpha < |z| < \beta.$$

We know from Lemma 1 and the above

$$\min_{|z|=1} \hat{v}_k(z) \leq 1 + \frac{5\epsilon}{4},$$

and

$$\max_{|z|=1} \hat{v}_k(z) \geq 1 + \frac{3\epsilon}{2}.$$

Passing to subsequence, for some  $0 < v^* \in C_{loc}^{0,1}(\mathbb{R}^n \setminus \{0\})$ ,

$$\hat{v}_k \rightarrow \hat{v}^* \quad \text{in } C_{loc}^{1,\alpha}(\mathbb{R}^n \setminus \{0\}), \quad \forall 0 < \alpha < 1,$$

and  $v^*$  satisfies in viscosity sense

$$\lambda(A^{\hat{v}^*}) \in \partial\Gamma, \quad \mathbb{R}^n \setminus \{0\}.$$

## Theorem

$u \in C_{loc}^{0,1}(\mathbb{R}^n \setminus \{0\})$ ,  $\lambda(A^u) \in \partial\Gamma$  in  $\mathbb{R}^n \setminus \{0\}$ , viscosity sense

implies

$u$  radially symmetric about the origin 0.

So  $\hat{v}^*$  radially symmetric.

**Remark.** If  $f$  is not assumed to be homogeneous,  $\hat{v}^*$  does not necessarily satisfy  $\lambda(A^{\hat{v}^*}) \in \partial\Gamma$ ,  $\mathbb{R}^n \setminus \{0\}$ .

Passing to subsequence,

$$\min_{|z|=1} \hat{v}^*(z) \leq 1 + \frac{5\epsilon}{4},$$

$$\max_{|z|=1} \hat{v}^*(z) \geq 1 + \frac{3\epsilon}{2}.$$

Contradiction. Proposition 3-2 proved.

## Proof of Theorem 1.

- By a previously known energy estimate of,

$$\int_{B_{1.9}} u_k^{\frac{2n}{n-2}} \leq C.$$

- $\exists 1.8 < r_1 < r_2 < 1.9$ ,

$$\int_{B_{r_2} \setminus B_{r_1}} u_k^{\frac{2n}{n-2}} \leq \delta_0.$$

- $\exists r_1 < r_3 < r_4 < r_2$  such that

$$u_k \leq C, \text{ in } B_{r_4} \setminus B_{r_3},$$

- Go to a maximum point of  $u_k$  in  $B_{r_4}$ , and apply Proposition 3-2, ..., then apply Proposition 3-2 again in the region ... Since each time, it takes away a fixed amount of energy, it stops in finite times (the total energy is bounded by  $C$ ).

Theorem 1 is proved.

Recall

$\Gamma \subset \mathbb{R}^n$  open, convex, symmetric cone, vertex at origin

$$\Gamma_n \subset \Gamma \subset \Gamma_1$$

$$\Gamma_n := \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \forall i\}, \quad \Gamma_1 := \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0\}$$

$f \in C^1(\Gamma) \cap C^0(\bar{\Gamma})$  symmetric function

$f_{\lambda_i} > 0$  in  $\Gamma \forall i$ ,  $f > 0$  in  $\Gamma$ ,  $f = 0$  on  $\Gamma$