Non linear problems in non local diffusion processes

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The Laplacian

The Laplacian has, at first sight, a somewhat uninspiring formula:

$$\text{Laplacian}(u) = u_{xx} + u_{yy} + u_{zz} + \ldots,$$

It is not clear from looking at it, why this combination is so particular: Why is the Laplacian rotational invariant, independent of the system of coordinates, and it represents diffusion ?

This can be seen, though, if we write the Laplacean as an infinitesimal limit of a gain-loss of density at the point x_0

The Laplacian is in fact:

$$\Delta u(x) = \lim_{r\to 0} \; r^{-(n+2)} \int_{B(0)} u(x+y) - u(x) \, dy$$

A "gain –loss" of particles jumping in the position x minus those leaving x.

It is the infinitesimal limit of integral operators

In that sense, the heat equation

$$u_t = \Delta(u)$$

reflects the fact that the density u at the point x has compared itself with its infinitesimal neighborhood and it is trying to revert to its surrounding average.

The other extreme:The master equation

The equation gives a general setting for the idea of particles that jump randomly at various speeds

:

$$u_t\ (x,t) = \iint [u(y,s) - u\ (x,t)]\ K(x,y,t,s)\ dyds$$

The kernel k is positive, supported in the past (s<t) and symmetric in x and y (divergence related to enegry considerations) or right and left ,and keeps Count of those particles reaching and leaving (x,t)

If the transport is instantaneous, K is supported in the present, i.e. in t=s (K(x,y) symmetric in x and y)

a) $u_t\ (x,t) = \int [u(y,t) - u(x,t)]\ K(x,y)\ dy$

or

b) $u_t\ (x,t) = \int [u(x+(y,t) - u(x,t)]\ K(x,y)\ dy$

If K becomes concentrates around (x,t), properly scaled, gives rise to a second order equation

From a) $u_t\ (x,t) = \mathrm{div}\ a_{i,j}\ \mathrm{grad}\ u$ *where x gives y the same weight that y gives x*

or from b) $u_t\ (x,t) = a_{i,j}\ D_{i,j}\ u$ *where x gives same weoght to the right than the left*

a) Is associated to energy condiderations, while b) to random processes

Classical infinitesimal examples :
Viscous flows like Navier Stokes

The heat equation (heat flow)
Irreversibility , smoothing of
the flow, and decay to equilibrium

"Infinitesimal" equations cover the bulk of the theory since classical mechanics model phenomena assuming that the global evolution is determined by the interaction of adjacent particles

(Heat ,waves, elasticity,magnetism…)

The Laplacian plays a central role, being linear and invariant under dilations, rotations, variational and non variatonal, that lends to the use of energy methods and Banach space approach

The nonlinear theory expanded considerably in the '50 and '60, new tools were develop

Harmonic analysis (singular integrals, pseudo differential operators, Sobolev spaces, H^1, BMO)
The Di Giorgi, Nash, Moser theory , equations with degenerate coefficients
Boundary behavior for non smooth boundaries (Calderon commutators, work of Dahlberg, Widman, Kenig on Harmonic measure , boundary Harnack, trace theory from numerical analysis (Lions, Magenes)….

There are roughly two (interconnected) branches of the non linear theory for diffusive processes:
Functional analytical:: harmonic maps, Navier Stokes,, critical Sobolev…
In these equations the highest order is linear but there is criticality in the interaction with non linear lower order terms

The second one , more related to geometric and measure theoretic issues corresponds to:

Quasi linear variational problems like minimal surfaces, p Laplacians.

Generalized minimal surfaces theory

Equations from geometry, like the symmetric functions of the Hessian (Monge Ampere and related equations) or fully non linear equations, like the Extremal Operators

Domain optimization , phase transition probelms, *Non local theory (non local diffuson processes)*

An important family is given by the (instantaneous) kernels

a) $\int [u(y)- u(x)]\, K(x,y)\, dy$

or

b) $\int [u(x+y)- u(x)]\, K(x,y)\, dy$

Natural in the stationary (elliptic) theory as well as in the evolutionary

Of particular interest are the "fractional Laplacians" :

The s- Laplacian:" Convolving" with the function

$V_s = |x|^{-n-2s}$ *dicreases differentiability by 2s for 0< s<1*

The inverse operation, convolving with:

: $V_{-s}(y) = |y|^{2s-n}$ $(s<1)$

Is the potential that, by convolution, inverts the s Laplacian , i.e. it is the fundamental solution of the s-Laplacian,

It improves by " 2s" derivatives the original function (invariant under dilations, rotations,"divergence" and "no divergence"

They also set the order of differentiability

(We consider kernels satisfying

$a|x-y|^{-n-2s} < K(x,y) < A|x-y|^{-n-2s}$

Fractional time derivatives are " directed in time

$$\int^t (u(t)-u(s)) / (t-s)^{1+\alpha} ds$$

Here too, u(t) compares itlelf to the past, in a non infinitesimal way

Memory equations appear for instance in reservoir modeling where the properties of the media are modified by the past flow and in many models from physiscs (see,for instance, G Zaslavski)

Non linear evolution equations

Energy methods, the quasigeostrophic and other quasilinear equations

A fundamental technique :De Giorgi’s geome trical approach

Apply DG ideas to

:The critical quasigeostrophic equation.

$$u_t(x,t)=\Delta^{1/2}+R^{\perp}u\ \mathrm{grad}\ (u)$$

(C-Vasseur) (related results by Kisselev et al)

:Quasi linear, non local equations:

(C-Chang Vasseur)

Porous media with potential pressure

C Soria Vazquez

Equations with memory:

The time derivative is replaced by a fractional time derivative.

The ***Caputo derivative*** *is particularly appropriate :*

$$D^{\alpha}_{t}\, u\,(x,t) = \int^{t} (u(x,t)-u(x,s))\,(t-s)^{-(1+\alpha)}\, ds$$

since it naturally quantifies the influence of the past "densities" into the current one.
It is natural, in this context to prescribe initial data from -∞.

Uniformly elliptic case with rough kernels :

$$D^{\alpha}_{t}\, u\,(x,t) = \int [u(y,t) - u(x,t)]\, K(x,y)\, dx$$

where the only requirement on K is to be symmetric and

$$m\ \ |x-y|^{-(n+s)}\ < K(x,y) < M\ \ |x-y|^{-(n+s)}$$

and we showed existence and regularity of solutions.
(Allen, C, Vasseur)

The main issue is of course , how does the time derivative contribution to an energy formula looks like?

How does the time integral behaves under multiplication by a monotone h(u), to attain an energy integral?

For the uniformly elliptic case the quadratic energy case suffices:

multiply by h(u)=u, to obtain

$$\int^T u(t) \int^t (u(t)-u(s)) / (t-s)^{1+\alpha} ds dt = *$$

How do we treat the term ***u(t)u(t) – u(t)u(s)*** *?*

We rewrite it as

$$1/2\, u(t)^2 + [1/2(u(t)^2 - u(t)u(s) + 1/2(u(s)^2] - u(s)^2, \text{ then}$$

$$* = \int^T \int^t [½((u(t)-u(s))^2 + u(t)^2 - u(s)^2]/(t-s)^{1+\alpha}\, dtds =$$

$$= \int^T \int^t 1/2[(u(t)-u(s)]^2/(t-s)^{1+\alpha}\, dtds + \int u(t)^2/(T-t)^\alpha\, dt$$

We solve by implicit discretization in time that makes clear the natural cancelations in the integral

For monotone h(u),= H'(u)?,
H convex , we note that for the energy formula

$$\int h(u(t)) \int^t (u(t)-u(s)) / (t-s)^{1+\alpha} ds dt$$

H'(u(t))(u(t) – u(s)) *> H(u(t)) -H(u(s))*
(by convexity)

We solve by implicit discretization in time that makes clear the natural cancelations in the integral

“ Porous media with memory”

In the case of porous media flow, there are different reservoir models where the memory term is incorporated in different ways some times in the pressure others in the density, or both (see for instance the extensive work of Caputo)
Also master equation type models

“

Back to the beginning:
Master equation and Levy walks:

A density u evolves in time taking into account input from the past and shredding into the future according to the balance law:

The instantaneous rate of change u_t *satisfies then (gain – loss)*

$$u_t = \iint (u(y,t-s)-u(x,t))\, K(x,y,t,s)\, dy\, ds$$

If K is symmetric in x and y the problem has variational structure (has associated a bilinear form and can be treated by energy methods)).
If the equation has the form:

$$u_t = \iint (u(y+x,t-s)-u(x,t))K(x,y,t,s)dy\, ds$$

it has non divergence structure (master equation)

For the non divergence master equation (with rough kernels, no continuity assumptions) we can show existence and Holder regularity. (C. Silvestre)

This implies higher regularity for " quasilinear " master equations

Some non local geometric and optimal control equations

Non local mean curvature and phase transition problems
(Chen and Fife)

Non local fully non linear equations

Non-local minimal surfaces

- *Non local models of phase transition in solids (Chen and Fife)*

- *A discrete way of generating movement by mean curvature (Merriman, Bence, Osher)*

Fractional minimal surfaces

For s bigger than ½, the integral
Diverges even for a smooth surface .
For an angle, the integral diverges for
any positive s
In particular, for s bigger than ½ a
further rescale gives classical movement
by mean curvature, anf for s less than ½
this integral seems to detect some
intermediate regularity

Phase transitions

Allen Cahn and Chrn Fife

Thank you for your attention

Guided bythe ideas of the second order theory solutions are locally C^{1+h} surfaces, except for a negligeable set

Main steps follow the second order approach:

Positive density, monotonicity formula, global cones, improvement of flatness

For flat surfaces this area linearizes to the corresponding fractional Laplacean, as in the second order case

The obstacle problem for fractional Laplacian

Some basic ideas to attack phase transition problems :

Search and exploit scaling invariance, classify global solutions

Isolate degenerate configurations

Classify singular points

Final result: Optimal regularity of solutions, and regularity of the free boundary except at singular points

Classification of singularities

While the space term will read (once on the left side)

$$-\iint h(u)\,\mathrm{div}\, u(x,t)\ \nabla\left(\int u(y)\, V(x-y)\, dy\right) dx\, dt =$$

$$\iint \left(\nabla h(u(x,t)) u(x,t)\ \nabla\left(\int u(y,t)\, V(x-y)\, dy\right) dx\, dt \right. .$$

The choice $F'(u) = h'(u)u,$ *yields the energy integral*

$$\iint \nabla F(u(x,t)\ \nabla\left(\int u(y,t)\, V(x-y)\, dy\right) dx\, dt$$

A good candidate is $h=\log u$ $(F'=1)$. *It linearizes the elliptic part*

$$\iint \nabla u(x,t) \int V(x-y)\ \nabla u(y,t)\, dy\, dx\, dt$$