# Nonlinear Fractional Parabolic Equations Problems in Bounded Domains

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"Nonlocal PDES and Applications to Physics, Geometry and Probability"

ICTP Trieste 29 May 2017

# Outline

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Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space. Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the stock market,...

These equations have occupied a large part of my research since 1980.

- The mathematical study of diffusion starts with the Heat Equation,

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- The heat example is generalized into the theory of linear parabolic equations, which is nowadays a basic topic in any advanced study of PDEs.

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# Nonlinear equations

- However, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very not-linear. A general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x,u,Du) + \mathcal{B}(x,t,u,Du)}$$

  with monotonicity conditions on $H$ and $\nabla_p \vec{\mathcal{A}}(x,t,u,p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)

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# Nonlinear heat flows

- Many specific examples, now considered the “classical nonlinear diffusion models”, have been investigated to understand in detail the qualitative features and to introduce the quantitative techniques, that happen to be many and from very different origins
- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), Hele-Shaw Problem (potential flow in a thin layer between solid plates), Porous Medium Equation: $u_t = \Delta(u^m)$, Evolution P-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.
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# Recent Direction. Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties.

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$$\boxed{u_t + (-\Delta)^s u = 0}$$

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# The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator**.
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$$

- Singular integral operator:

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy$$

  With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s}u$. This one has kernel $C_1|x-y|^{n-2s}$, which is not integrable.

- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

  where $P_{ik}$ denotes the transition function which has a . tail (i.e, power decay with the distance $|i-k|$). In the limit you get an operator $A$ as the infinitesimal generator of a Levy process: if $X_t$ is the isotropic $\alpha$-stable Lévy process we have

$$Au(x) = \lim_{h\to 0} \mathbb{E}(u(x) - u(x+X_h))$$

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# The fractional Laplacian operator II

- The $\alpha$-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla \cdot (y^{1-\alpha} \nabla v) = 0 \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+; \quad v(x,0) = u(x), \quad x \in \mathbb{R}^n.$$

Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y \to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

When $s = 1/2$ i.e. $\alpha = 1$, the extended function $v$ is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre, 2007.

This construction is generalized to other differential operators, like the harmonic oscillator, by Stinga and Torrea, Comm. PDEs, 2010.

- The semigroup formula in terms of the heat flow generated by $\Delta$:

$$(-\Delta)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta} f(x) - f(x) \right) \frac{dt}{t^{1+s}}.$$

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# Fractional Laplacians on bounded domains

In $\mathbb{R}^n$ all the previous versions are equivalent. In a bounded domain $\Omega \subset \mathbb{R}^n$ we have to re-examine all of them. Two main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at the boundary or what is the domain of the functionals. There are more alternatives.

- The restricted Laplacian. It is the simplest option. Functions $f(x)$ defined in $\Omega$ are extended by zero to the complement and then the whole space hypersingular integral is used

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- The spectral Laplacian

$$(-\Delta_{sp})^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta_D} f(x) - f(x) \right) \frac{dt}{t^{1+s}} = \sum_{j=1}^{\infty} \lambda_j^s \hat{f}_j\, \varphi_j(x)\,,$$

where $(\lambda_j, \varphi_j), j = 1, 2, \ldots$ are the normalized spectral sequence of the standard Dirichlet Laplacian $\Delta_D$ on $\Omega$, $\hat{f}_j$ are the Fourier coeff. of $f$.

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We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This model represents the linear flow generated by the so-called Lévy processes in Stochastic PDEs, where the transition from one site $x_j$ of the mesh to another site $x_k$ has a probability that depends on the distance $|x_k - x_j|$ in the form of an inverse power for $j \neq k$. The power we take is $c\,|x_k - x_j|^{-n-2s}$. The range is $0 < s < 1$. The limit from random walk to the continuous equation is done by E. Valdinoci, in *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. 49 (2009), 33-44.

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$$u(x,t) = \int u_0(y) P_t(x-y)\, dy,$$

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$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(n+2s)/2}} \qquad \Rightarrow \text{look at the fat tail.}$$

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  solves the problem of existence and uniqueness of solutions when the initial data is a locally finite Radon measure with the condition

$$\int_{\mathbb{R}^n} (1+|x|)^{-(n+2s)}\, d\mu(x) < \infty\,. \tag{1}$$

  Moreover we prove that any constructed solution by convolution, or any very weak solution $u \ge 0$, has an initial trace $\mu$ which is a measure in the above class $\mathcal{M}_s$. So the result closes the problem of the Widder theory for the fractional heat equation posed in $\mathbb{R}^n$.

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# Nonlocal nonlinear diffusion model I

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case $p$ is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.

- We still need a closure relation to relate $u$ and $p$. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where $f$ is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of $u$ if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (around 20!).

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# Nonlocal diffusion model. The problem

- The diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, $p$ es related to $u$ through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel
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- The diffusion model with nonlocal effects is thus given by the system
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where $u$ is a function of the variables $(x,t)$ to be thought of as a density or concentration, and therefore nonnegative, while $p$ is the pressure, which is related to $u$ via a linear operator $\mathcal{K}$. $u_t = \nabla\cdot(u\,\nabla(-\Delta)^{-s}u)$
- The problem is posed for $x \in \mathbb{R}^n$, $n \ge 1$, and $t > 0$, and we give initial conditions
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where $u_0$ is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

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# Nonlocal diffusion model

- The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory as a large number works on elliptic equations, mainly of the linear or semilinear type (Caffarelli school; Bass, Kassmann, and others)
- There are many works on the subject. Here is a good reference to fractional elliptic work by a young Spanish author

  Xavier Ros-Otón. *Nonlocal elliptic equations in bounded domains: a survey*, Preprint in arXiv:1504.04099 [math.AP].

# Nonlocal diffusion Model I. Applications

- Modeling dislocation dynamics as a continuum. This has been studied by P. Biler, G. Karch, and R. Monneau (2008), and then other collaborators, following old modeling by A. K. Head on *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* (1972).
  This is a one-dimensional model. By integration in $x$ they introduce viscosity solutions a la Crandall-Evans-Lions. Uniqueness holds.
- Equations of the more general form $u_t = \nabla \cdot (\sigma(u)\nabla \mathcal{L}u)$ have appeared recently in a number of applications in particle physics. Thus, Giacomin and Lebowitz (J. Stat. Phys. (1997)) consider a lattice gas with general short-range interactions and a Kac potential, and passing to the limit, the macroscopic density profile $\rho(r,t)$ satisfies the equation
  $$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho)\nabla \frac{\delta F(\rho)}{\delta \rho}\right]$$
  See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

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# Extreme cases

- If we take $s = 0$, $\mathcal{K} =$ the identity operator, we get the standard porous medium equation, whose behaviour is well-known, see references later.
- In the other end of the $s$ interval, when $s = 1$ and we take $\mathcal{K} = -\Delta$ we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u. \tag{4}$$

In one dimension this leads to $u_t = u_x p_x - u^2$, $p_{xx} = -u$. In terms of $v = -p_x = \int u\,dx$ we have

$$v_t = up_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + vv_x = 0$ which generates shocks in finite time but only if we allow for $u$ to have two signs.

- HYDRODYNAMIC LIMIT. The case $s = 1$ in several dimensions is more interesting because it does not reduce to a simple Burgers equation.

$$u_t = \nabla \cdot (u\,\nabla p) = \nabla u \cdot \nabla p - u^2; \qquad , \qquad p = (-\Delta)^{-1} u\,,$$

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# Main estimates for this model

We recall that the equation of M1 is $\partial_t u = \nabla \cdot (u \nabla K(u))$, posed in the whole space $\mathbb{R}^n$.
We consider $K = (-\Delta)^{-s}$ for some $0 < s < 1$ acting on Schwartz class functions defined in the whole space. It is a positive essentially self-adjoint operator. We let $H = K^{1/2} = (-\Delta)^{-s/2}$.
*We do next formal calculations, assuming that $u \geq 0$ satisfies the required smoothness and integrability assumptions. This is to be justified later by approximation.*

- Conservation of mass

$$\frac{d}{dt}\int u(x,t)\,dx = 0. \tag{5}$$

- First energy estimate:

$$\frac{d}{dt}\int u(x,t)\log u(x,t)\,dx = -\int |\nabla H u|^2\,dx. \tag{6}$$

- Second energy estimate

$$\frac{d}{dt}\int |Hu(x,t)|^2\,dx = -2\int u|\nabla K u|^2\,dx. \tag{7}$$

# Main estimates

- Conservation of positivity: $u_0 \geq 0$ implies that $u(t) \geq 0$ for all times.
- $L^\infty$ estimate. We prove that the $L^\infty$ norm does not increase in time.

  *Proof.* At a point of maximum of $u$ at time $t = t_0$, say $x = 0$, we have

  $$u_t = \nabla u \cdot \nabla P + u\,\Delta K(u).$$

  The first term is zero, and for the second we have $-\Delta K = L$ where $L = (-\Delta)_q$ with $q = 1 - s$ so that

  $$\Delta K u(0) = -Lu(0) = -\int \frac{u(0) - u(y)}{|y|^{n+2(1-s)}}\,dy \leq 0.$$

  This concludes the proof.
- We did not find a clean comparison theorem, a form of the usual maximum principle is not proved for Model 1. Good comparion works for Model 2 to be presented below, actually, it helps produce a very nice theory.
- Finite propagation is true for model M1. Infinite propagation is true for model M2.

  $$\partial_t u + (-\Delta)^s u^m = 0,$$

  the most recent member of the family, that we love so much.

# Boundedness

- Solutions are bounded in terms of data in $L^p$, $1 \le p \le \infty$.
  For Model 1 Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
  Or use energy estimates based on the properties of the quadratic and bilinear forms associated to the fractional operator, and then the iteration technique

- **Theorem (for M1)** *Let $u$ be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant $C$ such that for every $t > 0$*

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \tag{8}$$

  *with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant $C$ depends only on $n$ and $s$.*
  This theorem allows to extend the theory to data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$, with global existence of bounded weak solutions.

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# Energy and bilinear forms

- **Energy solutions:** The basis of the boundedness analysis is a property that goes beyond the definition of weak solution. The general energy property is as follows: for any $F$ smooth and such that $f = F'$ is bounded and nonnegative, we have for every $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} \int F(u(t_2))\,dx - \int F(u(t_1))\,dx \quad &= -\int_{t_1}^{t_2}\int \nabla[f(u)]u\nabla p\,dx\,dt = \\ &- \int_{t_1}^{t_2}\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx\,dt \end{aligned}$$

where $h$ is a function satisfying $h'(u) = uf'(u)$. We can write the last integral as a bilinear form

$$\int \nabla h(u)\nabla(-\Delta)^{-s}u\,dx = \mathcal{B}_s(h(u), u)$$

- This bilinear form $\mathcal{B}_s$ is defined on the Sobolev space $W^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{B}_s(v,w) = C_{n,s}\iint \nabla v(x)\frac{1}{|x-y|^{n-2s}}\nabla w(y)\,dx\,dy\,. \tag{9}$$

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where $\mathcal{N}_{-s}(x,y) = C_{n,s}|x-y|^{-(n-2s)}$ is the kernel of operator $(-\Delta)^{-s}$.

- After some integrations by parts we also have

$$\mathcal{B}_s(v,w) = C_{n,1-s} \iint (v(x)-v(y)) \frac{1}{|x-y|^{n+2(1-s)}} (w(x)-w(y))\, dx\, dy \tag{10}$$

since $-\Delta \mathcal{N}_{-s} = \mathcal{N}_{1-s}$.

- It is known (Stein) that $\mathcal{B}_s(u,u)$ is an equivalent norm for the fractional Sobolev space $W^{1-s,2}(\mathbb{R}^n)$.
We will need in the proofs that $C_{n,1-s} \sim K_n(1-s)$ as $s \to 1$, for some constant $K_n$ depending only on $n$.

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# Additional and Recent work, open problems

- The asymptotic behaviour as $t \to \infty$ is a very interesting topic developed in a paper with Luis Caffarelli. This was our first work (2008, published 2011). Rates of convergence are found in dimension $n = 1$ (Carrillo, Huang, Santos, JLV) but they are not available for $n > 1$, they are tied to some functional inequalities that are not known.
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# Outline

# FPME: Second model for fractional Porous Medium Flows

- An alternative natural equation is the equation that we will call FPME:

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{11}$$

- This model arises from stochastic differential equations when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling.
  Understanding the physical situation looks difficult to me , but the modelling on linear an non linear fractional heat equations is done by Stefano Olla, Milton Jara and collaborators, see for instance
  *M. D. Jara, T. Komorowski, S. Olla,* Ann. Appl. Probab. **19** (2009), no. 6, 2270–2300. *M. Jara, C. Landim, S. Sethuraman,* Probab. Theory Relat. Fields **145** (2009), 565–590.

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# Mathematical theory of the FPME, Model 2

- The Problem is

$$\boxed{u_t + (-\Delta)^s(|u|^{m-1}u) = 0}$$

We take $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 1$, with initial data in $u_0 \in L^1(\mathbb{R}^n)$. Normally, $u_0, u \geq 0$.

This second model, M2 here, represents another type of nonlinear interpolation, this time between

$$\boxed{u_t - \Delta(|u|^{m-1}u) = 0} \quad \text{and} \quad \boxed{u_t + |u|^{m-1}u = 0}$$

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In the classical Bénilan-Brezis-Crandall style, a semigroup of weak energy solutions is constructed, the $L^1 - L^\infty$ smoothing effect works,

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# Outline of work done for model M2

*Comparison of models M1 and M2 is quite interesting*

- Existence of self-similar solutions, paper JLV, JEMS 2014. The fractional Barenblatt solution is constructed:

  $$U(x,t) = t^{-\alpha} F(xt^{-\beta})$$

  The difficulty is to find $F$ as the solution of an elliptic nonlinear equation of fractional type.

  $F$ has behaviour like a Blumental tail $F(r) \sim r^{-(n+2s)}$ for $m \geq 1$, but not for some fast diffusion $m < 1$. Asymptotic behaviour follows: the Barenblatt solution is an attractor.

- A priori upper and lower estimates of intrinsic, local type. Paper with Matteo Bonforte in Advances Math., 2014 for problems posed in $\mathbb{R}^n$.

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- A detailed account on such progress is obtained in the papers (cf. arxiv) and in the following reference that is meant as a survey for two-year progress on Model M2

  *Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators*, by Juan Luis Vázquez. In "Nonlinear elliptic and parabolic differential equations", Disc. Cont. Dyn. Syst. - S **7,** no. 4 (2014), 857–885.

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# Some future Directions

- Other nonlocal linear operators (hot topic)
- Elliptic theory (main topic, by many authors)
- Geostrophic flows (this is more related to Fluid Mechanics)
- Reaction-diffusion and blowup
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- Chemotaxis systems, ....

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# Outline

# Operators and Equations in Bounded Domains

- This work is recent and needs a different lecture. It comes from long time collaboration with Matteo Bonforte, and recently with Yannick Sire and Alessio Figalli.
- We develop a new programme for Existence, Uniqueness, Positivity, A priori bounds and Asymptotic behaviour for fractional porous medium equations on bounded domains, after examining very carefully the set of suitable concepts of FLO in a bounded domain.
- But the main issue is how many natural definitions we find of the FLO in a bounded domain.
- Then we use the "dual" formulation of the problem and the concept of weak dual solution. In brief, we use the linearity of the operator $L$ to lift the problem to a problem for the potential function

$$U(x,t) = \int_\Omega u(y,t)G(x,y)dy$$

Where $G$ is the elliptic Green function for $L$.

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# Fractional Laplacian operators on bounded domains

- The Restricted Fractional Laplacian operator (RFL) is defined via the hypersingular kernel in $\mathbb{R}^n$, "restricted" to functions that are zero outside $\Omega$.

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{n+2s}}\,\mathrm{d}z\,, \qquad \text{with } \operatorname{supp}(g)\subset\overline{\Omega}\,.$$

  where $s\in(0,1)$ and $c_{n,s}>0$ is a normalization constant.
- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0<\overline{\lambda}_1\le\overline{\lambda}_2\le\ldots\le\overline{\lambda}_j\le\overline{\lambda}_{j+1}\le\ldots$ and $\overline{\lambda}_j\asymp j^{2s/N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j\in C^s(\overline{\Omega})$.
- Lateral boundary conditions for the RFL:

$$u(t,x)=0\,, \qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N\setminus\Omega\right).$$

- The Green function $G$ of RFL satisfies a strong behaviour condition (K4)

$$G(x,y)\asymp\frac{1}{|x-y|^{N-2s}}\left(\frac{\delta_\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma}\wedge 1\right), \qquad \text{with } \gamma=s$$

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where $\Delta_\Omega$ is the classical Dirichlet Laplacian on the domain $\Omega$,

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- Lateral boundary conditions for the SFL. They are better defined by using the Caffarelli-Silvestre extension adapted to bounded domain as a cylinder. Then we put $U = 0$ on the lateral boundary $x \in \partial\Omega$, $y > 0$.
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- Censored Fractional Laplacians (CFL)
  This is another option that has been introduced in 2003 by Bogdan, Burdzy and Chen. Definition
  $$\mathcal{L}f(x) = \text{P.V.} \int_{\Omega} (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,, \qquad \text{with } \frac{1}{2} < s < 1 \,,$$
  where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

- The Green function $G(x,y)$ satisfies condition $K4$, proven by Chen, Kim and Song (2010), in the form
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- We have presented 3 models of Dirichlet fractional Laplacian. Put $\left[\, a(x,y) = const \right]$ in the last case. The estimates (K4) show that they are of course **not equivalent**.

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◊

The detailed analysis of existence and uniqueness of solutions for a large class of integro-differential operators, plus sharp decay and decay and boundary behaviour is done in the last paper.

It is reported in the following Talk:

Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains,
given by Matteo Bonforte, matteo.bonforte@uam.es

at

*2016-17 Warwick EPSRC Symposium: Non-local Equations and Fractional Diffusion*, Mathematical Institute, University of Warwick, UK
May 26, 2017

# Thank you for your attention