Curves and surfaces with constant nonlocal mean curvature

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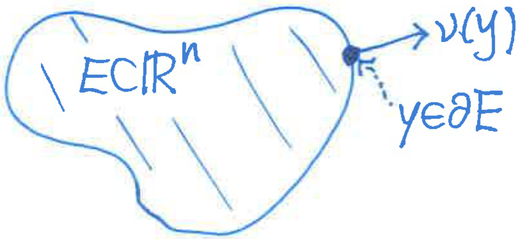
$E\subset\mathbb{R}^n$ bounded (and sufficiently smooth)



- (Standard) Perimeter :

$W^{1,1}$-seminorm

$$P(E) = \sup_{\|X\|_\infty \leq 1} \int_{\partial E} X(y)\cdot\nu(y)\,dy = \|\nabla \mathbb{1}_E\|_{L^1(\mathbb{R}^n)} = [\mathbb{1}_E]_{W^{1,1}(\mathbb{R}^n)}$$

$$\int_{\partial E} X(y)\cdot\nu(y)\,dy = \int_E (\operatorname{div} X)(y)\,dy = \int_{\mathbb{R}^n} (\operatorname{div} X)\cdot\mathbb{1}_E = \int_{\mathbb{R}^n} -X\cdot\nabla\mathbb{1}_E$$

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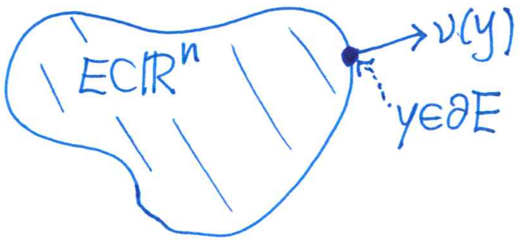

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- Fractional Perimeter: $\boxed{0<\alpha<1}$ $\longrightarrow$ Fractional Sobolev seminorm:

$$[\mathbb{1}_E]_{W^{\alpha,1}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\mathbb{1}_E(x) - \mathbb{1}_E(y)|}{|x-y|^{n+\alpha}}\, dx\, dy = 2 \int_E \int_{E^c} \frac{dx\, dy}{|x-y|^{n+\alpha}}$$

$$\hookrightarrow \quad P_\alpha(E) = C_{n,\alpha} \int_E \int_{E^c} \frac{dx\, dy}{|x-y|^{n+\alpha}}.$$

$0<\alpha<1 \leftarrow \alpha = 2s;\ s \in (0, \tfrac{1}{2})$

E bdd

- Fractional isoperimetric inequality : balls minimize fractional perimeter for a given volume [Frank-Seiringer, JAMS 2008]

Quantitative version : $\underbrace{\underbrace{\text{Fusco-Millot-Morini}}_{2011}\text{-Figalli-Maggi}}_{2014}$

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1st variation is NONLOCAL (or fractional) MEAN CURVATURE (NMC):

$E \subset \mathbb{R}^n$, $\partial E \in C^2$ (E perhaps unbounded and/or disconnected) $\rightsquigarrow$

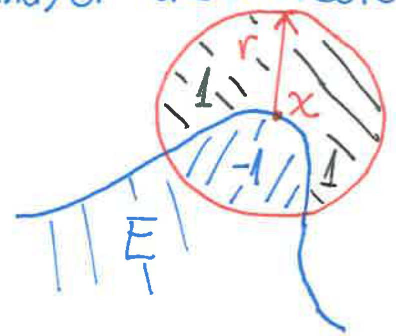
$$H_E(x) = H_{\alpha,E}(x) = P.V. \int_{\mathbb{R}^n} \frac{1_{E^c}(y) - 1_E(y)}{|x-y|^{n+\alpha}}\, dy$$

for $x \in \partial E$

(up to multiplicative ct)

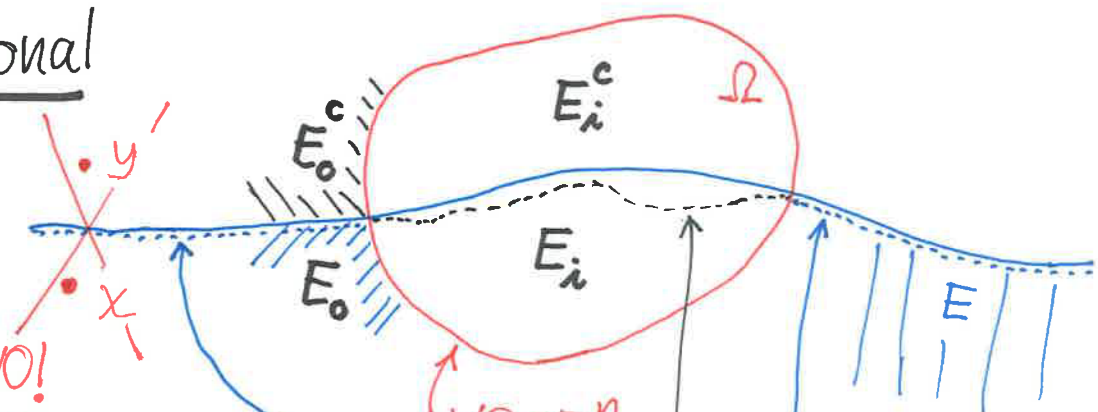
• integrable at ∞

• cancellation at $x=y$



- Fractional perimeter functional for E unbounded

NO!



$\forall \Omega \subset \mathbb{R}^n$ bounded

Dirichlet data in $\mathbb{R}^n \setminus \Omega$

Competitor for E in Ω.

$$P_\alpha(E) := \left\{ \int_{E_i}\int_{E_i^c} + \int_{E_i}\int_{E_o^c} + \int_{E_o}\int_{E_i^c} \right\} \frac{dx\,dy}{|x-y|^{n+\alpha}}$$

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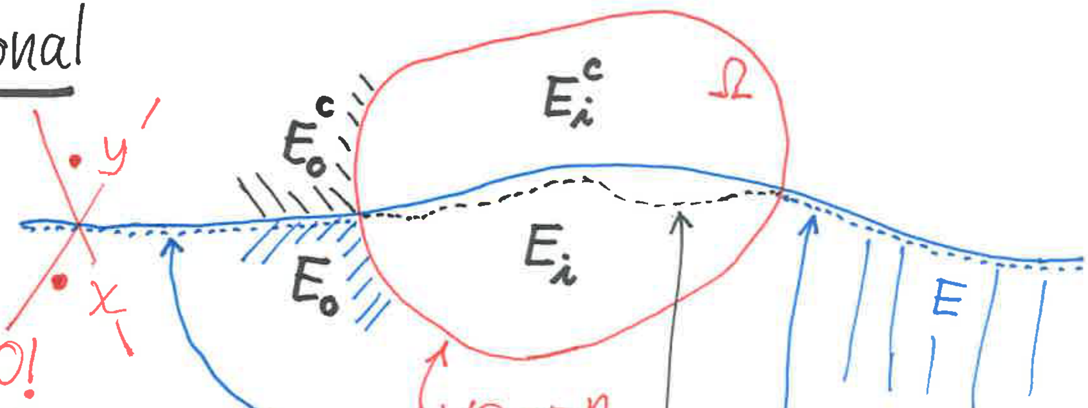

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[Caffarelli-Roquejoffre-Savin, CPAM '10]: Nonlocal minimal surfaces ($H_E \equiv 0$)

- Motivation for [Caff.-Roquej.-Savin '10] came from:

[Caffarelli-Souganidis 2008]:

↳ "cellular automata" or threshold dynamics $\xrightarrow[\delta t \downarrow 0]{}$ Motion by $\left\langle \begin{matrix} \text{classical} \\ \text{nonlocal} \end{matrix} \right\rangle$ mean curvature

$\mathbb{1}_{E^c} - \mathbb{1}_E$ as initial condition for the (classical or fractional) (linear) heat equation (≡ convolution with Gaussian or power decay distribution)

↳ small time step δt ⟶ New $E = E_{\delta t} = \{u(\cdot, \delta t) < 0\}$ & repeat process

• Nonlocal minimal surfaces : $E \subset \mathbb{R}^n$, $H_E(x) \equiv 0$ $\forall x \in \partial E$

[Caffarelli–Roquejoffre–Savin, CPAM '10] : For minimizing nonlocal minimal surfaces:

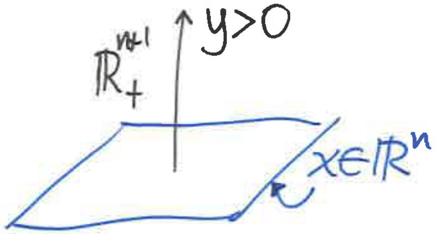
- Definition of variational pb. and existence of minimizer
- Density estimates
- The Euler-Lagrange eq'n in viscosity sense
- Improvement of flatness
- Extension problem and monotonicity formula
- Dimension reduction

• Thm [C-R-S, '10]
$E \subset \mathbb{R}^n$ minimizing nonlocal minimal set in B_1 $\Rightarrow$
$\Rightarrow \partial E \cap B_{1/2}$ is $C^{1,\alpha}$ except for a closed set of $\mathcal{H}^{n-2}$ dimension.

The extension problem [Caffarelli-Silvestre 2007]

$0<s<1$

$u:\mathbb{R}^n \to \mathbb{R}$ $\rightsquigarrow$
$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v(x,0) = u(x) & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$



$v = v(x,y)$.

Thm [Caff-Silv]

$$\boxed{-\lim_{y\downarrow 0} y^{1-2s} v_y = \frac{\partial v}{\partial y^s}(x,0) = \tilde{c}_{n,s}(-\Delta)^s u(x).}$$

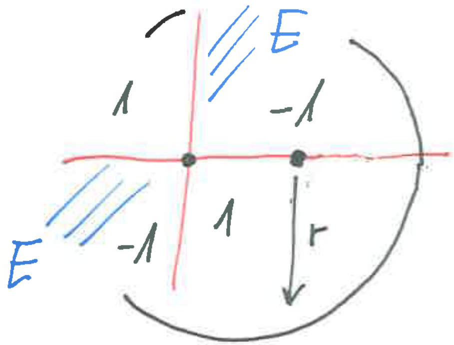
- Nonlocal minimal surfaces: $E \subset \mathbb{R}^n$, $H_E(x) \equiv 0 \quad \forall x \in \partial E$

- Thm [Barrios-Figalli-Valdinoci '12 & Figalli-Valdinoci '13]

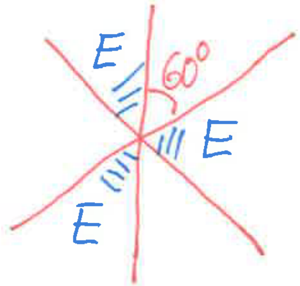
$\mathbb{R}^n \supset E$ minimizing nonlocal minimal set & $\partial E \in Lip \Rightarrow \partial E \in C^\infty$

- Thm [Savin-Valdinoci '12]

$\mathbb{R}^2 \supset E$ minimizing nonl. min. set $\Rightarrow E =$ half-plane ($\partial E =$ line)



$H_E \equiv 0$ BUT are not minimizers



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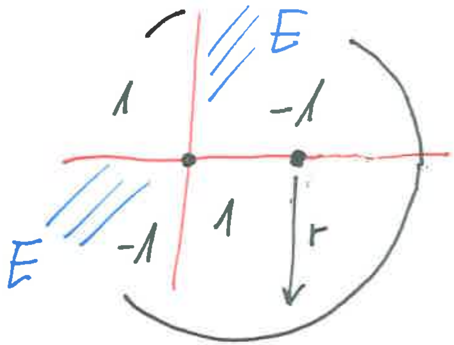

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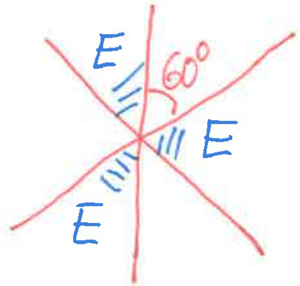


- **Thm** [Figalli-Valdinoci '13]

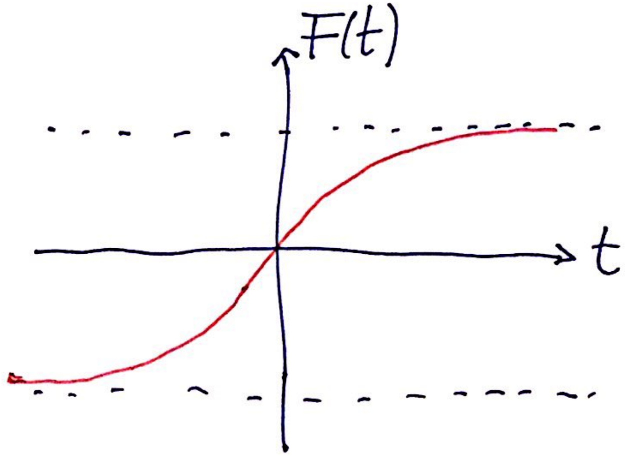
$\mathbb{R}^3 \supset \partial E = \{x_3 = \varphi(x_1, x_2),\ \varphi : \mathbb{R}^2 \to \mathbb{R}\}$ is a nonlocal minimal graph $\Rightarrow$ ∂E is a plane

$$\mathbb{R}^{n+1} \supset E = \{ x_{n+1} < \varphi(x) \,,\ x = (x_1, \dots, x_n) \in \mathbb{R}^n \}$$ a graph

$$\hookrightarrow \quad H_E(x, \varphi(x)) = 2 \int_{\mathbb{R}^n} F\left(\frac{u(x) - u(y)}{|x-y|} \right) \frac{dy}{|x-y|^{n+\alpha}}$$

where

$$F(t) := \int_0^t \frac{d\tau}{(1+\tau^2)^{\frac{n+1+\alpha}{2}}}$$



• **Thm** [Caffarelli-Valdinoci '11 '13]

$n \leq 7$ and α sufficiently close to 1 $\Rightarrow$

$\Rightarrow$ In $\mathbb{R}^n$, minimizing nonlocal minimal $<$ cones are flat / surfaces are smooth

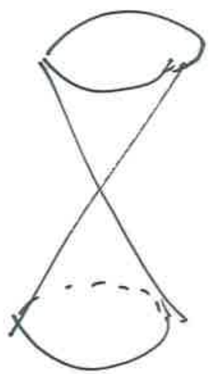
• [Davila-del Pino-Wei '14]

* cones in higher dimensions $\leftarrow$ are they stable ?

* Nonlocal catenoid ($\exists$)

$n \leq 6$ or 7 ? Role of $\alpha \in (0,1)$?

circular ones (numerics)



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  Assume: α sufficiently close to 1

- Corol [Cabré-Cinti-Serra '17]

  $\mathbb{R}^3 \supset \partial E$ is a stable nonlocal minimal surface $\Rightarrow$ ∂E is a plane

  Assume: α sufficiently close to 1

• Proofs in [Cabré-Cinti-Serra '17] based in three ingredients:

1) Second variation formula for nonlocal perimeter
(by [Figalli-Fusco-Maggi-Millot-Morini] and [Dávila-del Pino-Wei]:
∂E stable nonlocal minimal surface in $\mathbb{R}^n$ $\Rightarrow$ $\forall \xi \in C_c^2(\mathbb{R}^n)$

$$\int_{\partial E} c_{\partial E}^2(x)\, \xi^2(x)\, d\sigma(x) \leq \int_{\partial E}\int_{\partial E} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{n+\alpha}}\, d\sigma(x)\, d\sigma(y), \quad \text{where}$$

$$c_{\partial E}^2(x) := \int_{\partial E} \frac{|\nu_E(x)-\nu_E(y)|^2}{|x-y|^{n+\alpha}}\, d\sigma(y)$$

2) Behavior as $\alpha \to 1$ of the optimal cst in the fractional Hardy inequality.

3) • Thm [Cinti-Serra-Valdinoci, '16]
$\mathbb{R}^n \supset \partial E$ is a stable nonlocal minimal surface in B_r, $\partial E \in C^2$.
Then,

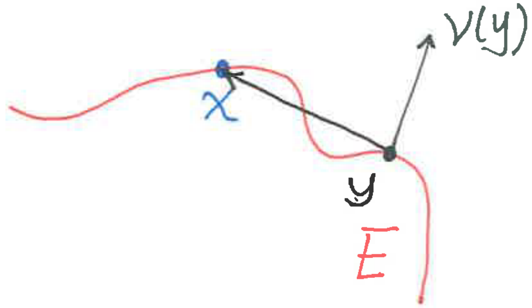
$$\mathrm{Per}_{B_{r/2}}(E) \leq \frac{C}{1-\alpha}\, r^{n-1} \qquad (C \text{ bdd as } \alpha \uparrow 1)$$

classical perimeter!!

- Curves and surfaces with constant nonlocal mean curvature

NMC (nonlocal mean curvature): $E\subset\mathbb{R}^n$, $\partial E\in C^2$ (E perhaps unbdd)

$$\boxed{H_E(x) \text{ for } x\in\partial E} = \int_{\mathbb{R}^n} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|x-y|^{n+\alpha}}\,dy = -\frac{2}{\alpha}\int_{\partial E} \frac{(x-y)\cdot\nu(y)}{|x-y|^{n+\alpha}}\,d\sigma(y)$$



Use:

$$\operatorname{div}_y \frac{x-y}{|x-y|^{n+\alpha}} = \alpha\,\frac{1}{|x-y|^{n+\alpha}}$$

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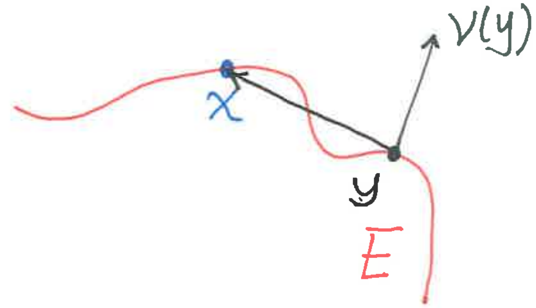
for $x \in \partial E$



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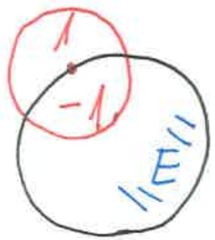

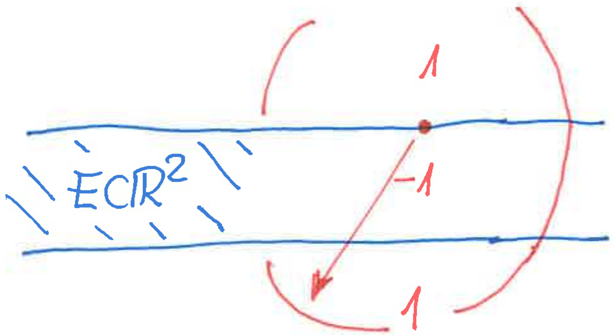

- E = hyperplane $\to H_E \equiv 0$
- E = ball in $\mathbb{R}^n \to H_E \equiv ctt > 0$
- E = band in $\mathbb{R}^2$ or cylinder in $\mathbb{R}^n \longrightarrow H_E \equiv ctt > 0$

- Classical mean curvature :

CMC surfaces : $H_E \equiv ctt$; are extremals of perimeter for given volume

- Thm [Aleksandrov 1958]

$E \subset \mathbb{R}^n$ bdd connected, $\partial E \in C^2$, ∂E CMC hypersurface

$\Rightarrow$ E is a ball

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$\Rightarrow E$ is a ball

- Thm [Delaunay 1841, JMPA]

In $\mathbb{R}^3$ (also in $\mathbb{R}^n, n \geq 3$), $\exists$ periodic CMC cylinders

Rolling an ellipse & tracing the FOCUS

called UNDULOID

(see them in many http)

(Do NOT exist in $\mathbb{R}^2$)

$\rightarrow$ extremals of fractional perimeter under volume constraint

$\boxed{\text{CNMC sets}}$ (sets with ctt nonlocal mean curvature H_E)

Joint work (arXiv 2015) with M.M. Fall, J. Solà-Morales & T. Weth

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- Thm 1 $\emptyset \neq E \subset \mathbb{R}^n$ bdd $C^{2,\beta}$ $(\beta > \alpha)$, $H_E \equiv$ ctt on ∂E
  $\Longrightarrow$ E is a ball.

$\leftarrow$ also proved by [Ciraolo-Figalli-Maggi-Novaga, arXiv 2015]
with a quantitative version : $B_s \subset E \subset B_t$ with $t-s$ small
if $\|H_E\|_{Lip(\partial E)}$ is small

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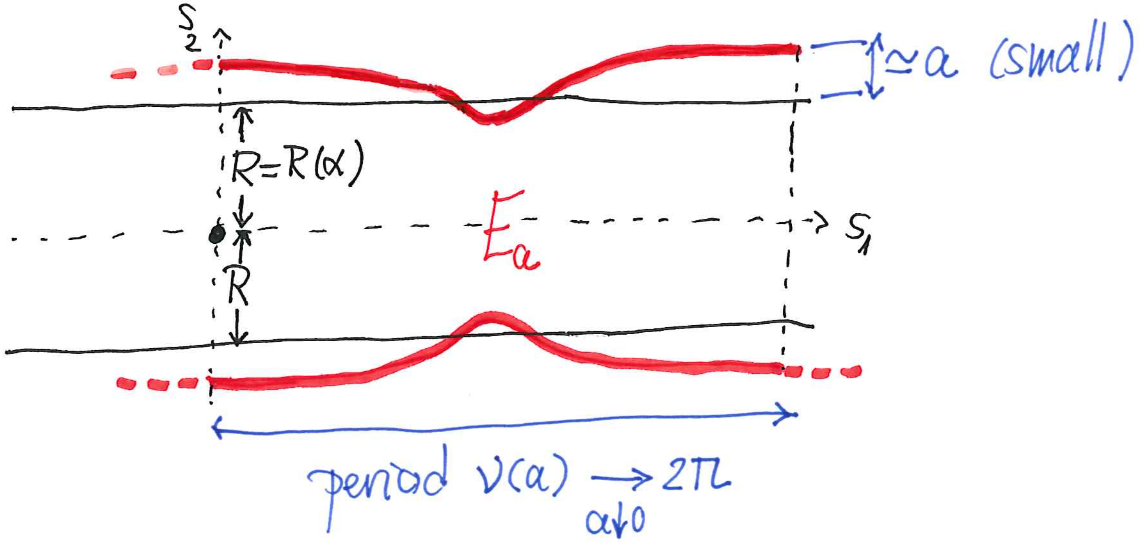
- Thm 2 $n=2 \to \exists R = R(\alpha)$ & $u_a : \mathbb{R} \to \mathbb{R}$ $\nu(a)$-periodic (a = small parameter) &
  $E_a = \{(s_1,s_2) \in \mathbb{R}^2 : -u_a(s_1) < s_2 < u_a(s_1)\}$
  have all same NMC $\equiv h_R > 0$ (ctt), $E_a \xrightarrow[a\downarrow 0]{} \{-R < s_2 < R\}$ = a band &
  $a \neq a' \Rightarrow E_a \neq E_{a'}$ & $\nu(a) \xrightarrow[a\downarrow 0]{} 2\pi$

- Hence:

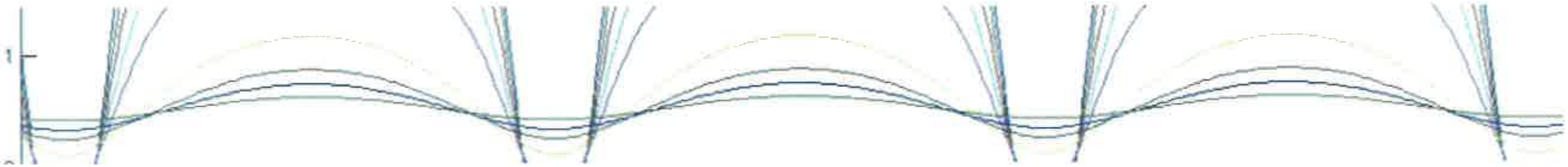
From a straight band in $\mathbb{R}^2$, a family of periodic bands $\{-u_a(s_1) < s_2 < u_a(s_1)\}$ bifurcate. They all have the same NMC (but their periods are different)

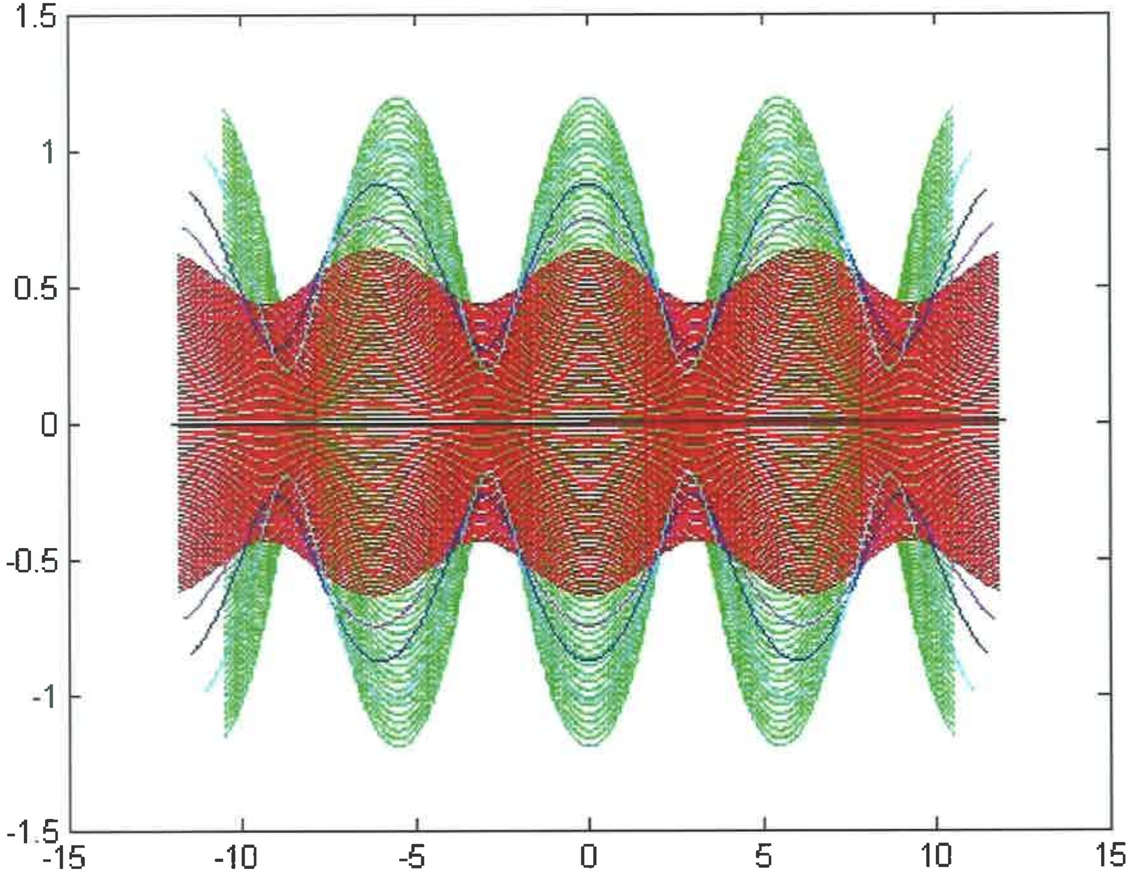


- Thm 3 [Cabré–Fall–Weth '16]:

The same holds in $\mathbb{R}^n$, $n \geq 3$ ← periodic CNMC cylinders

& the branch is C^∞.





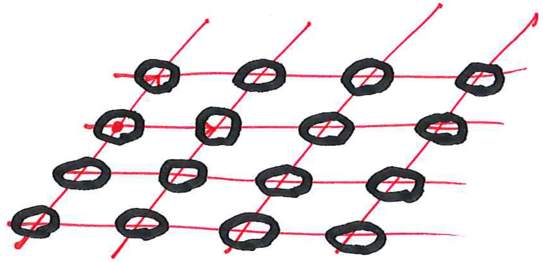
α=.5; bands from a=.1 (red) to a=.5 (green) with profiles of the intermediate values

• **Thm 4** [Cabré-Fall-Weth '16]

In $\mathbb{R}^N$, $\exists$ periodic lattices having ctt NMC and made of near-spheres :

$E = \bigcup \{\text{interior of near-balls}\}$

$\hookrightarrow \partial E$ has ctt NMC



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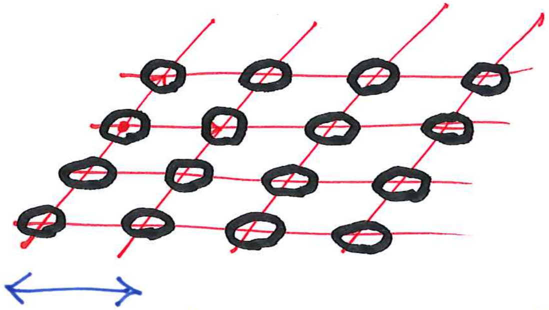

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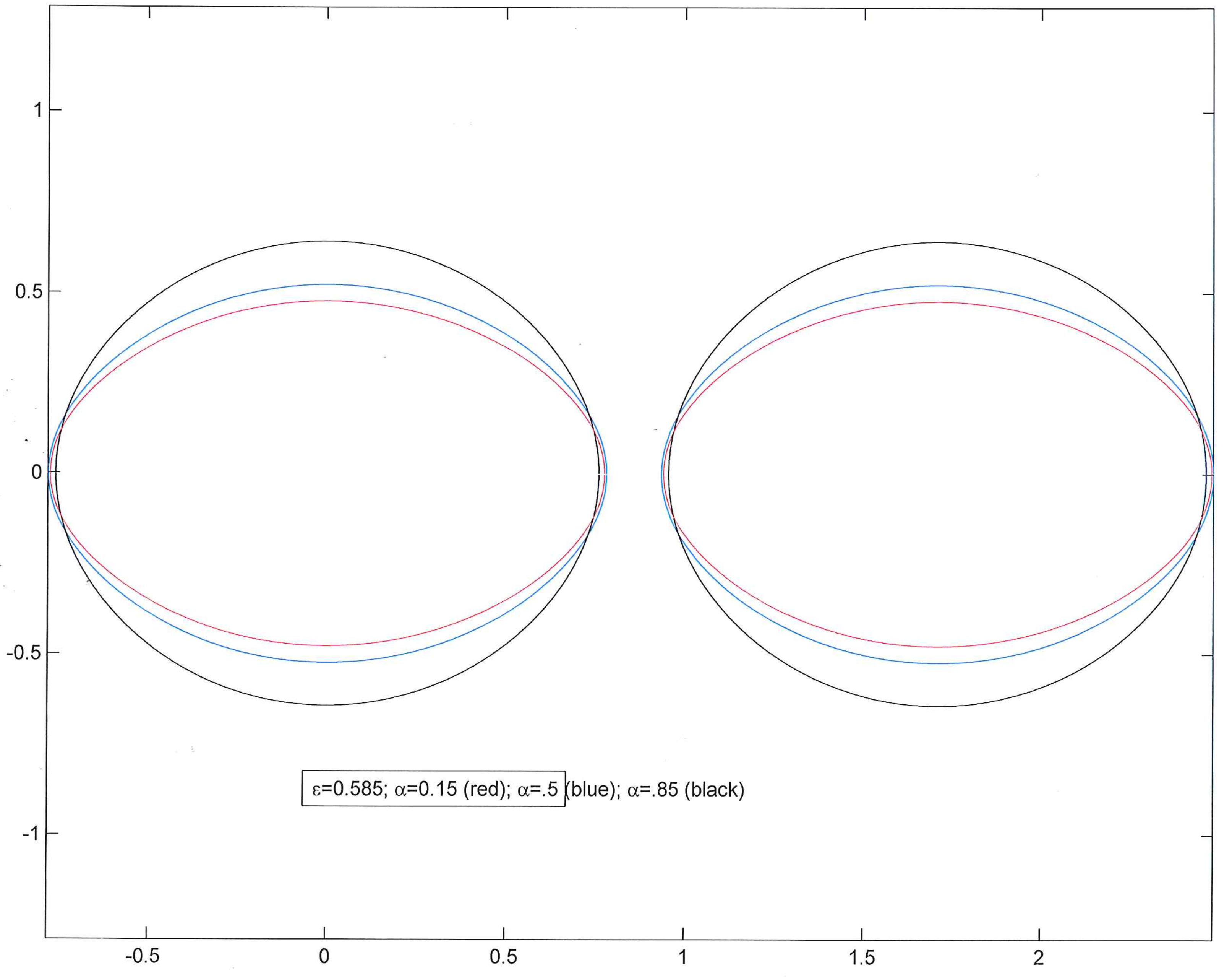
$\mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^{N-M}$
$\cup$
$\mathbb{R}^M \ni (x', 0)$
$\{a_1, \dots, a_M\}$ basis of $\mathbb{R}^M$

$\rightarrow$ $\mathcal{L} = \left\{ \sum_{k=1}^{M} k_i a_i : k \in \mathbb{Z}^M \right\}$

$S_\varphi = \left\{ (1+\varphi(\sigma))\sigma : \sigma \in S^{N-1} \right\}$

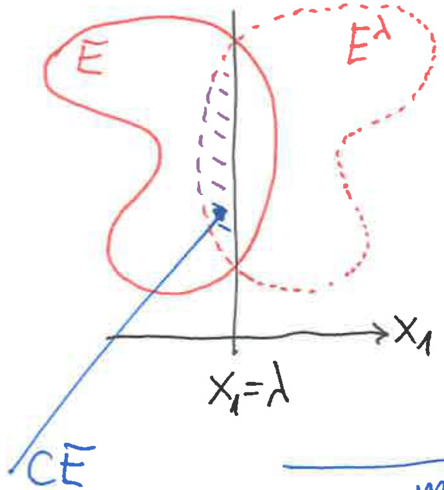
$\partial E = S_\varphi + r\mathcal{L}$ ($r > 0$ large)

Linearized operator at S^{N-1} ($\varphi \equiv 0, r = \infty$) $= L_\alpha - \lambda_1$, with $L_\alpha \varphi(\theta) = \int_{S^{N-1}} \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} d\sigma$, $\theta \in S^{N-1}$.


1
0.5
0
-0.5
-1
-0.5
0
0.5
1
1.5
2
ε=0.585; α=0.15 (red); α=.5 (blue); α=.85 (black)

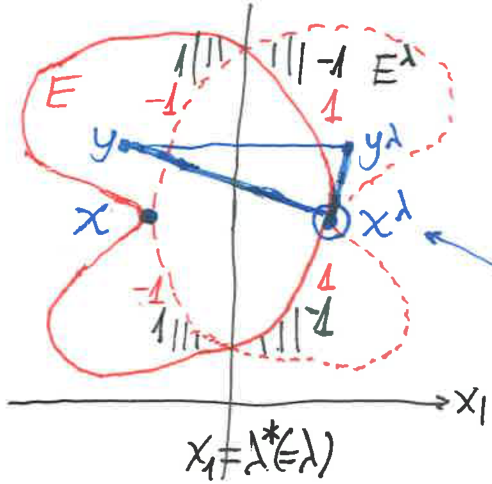
• Proof of Thm 1 (Aleksandrov) : use moving planes method for

$$H_E(x) = \int_{\mathbb{R}^n} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|x-y|^{n+\alpha}}\, dy \equiv ctt \quad \forall x \in \partial E$$



Two possible obstructions → $\boxed{1^{st}}$

continue: make $\lambda \downarrow$



$$H_E(x) = H_{E^\lambda}(x^\lambda)$$
$$\|$$
$$H_E(x^\lambda)$$

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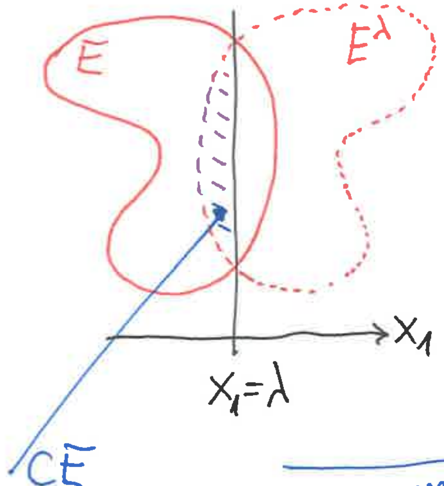

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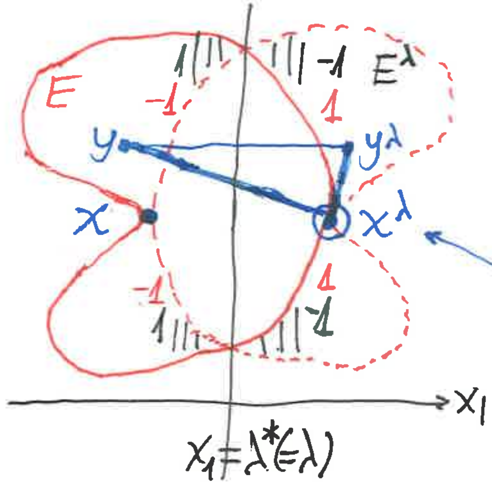

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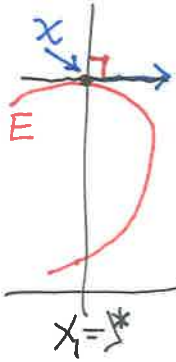
$$\parallel$$
$$H_E(x^\lambda)$$

2nd



$v\in T_x\partial E$ Lemma

$$\partial_v H_E(x) = (n+\alpha)\int_{\mathbb{R}^n}(\mathbb{1}_E-\mathbb{1}_{E^c})(y)\,|x-y|^{-n-\alpha-2}(x-y)\cdot v\,dy$$

- **Proof of Thm 2** : $\exists$ of CNMC periodic bands

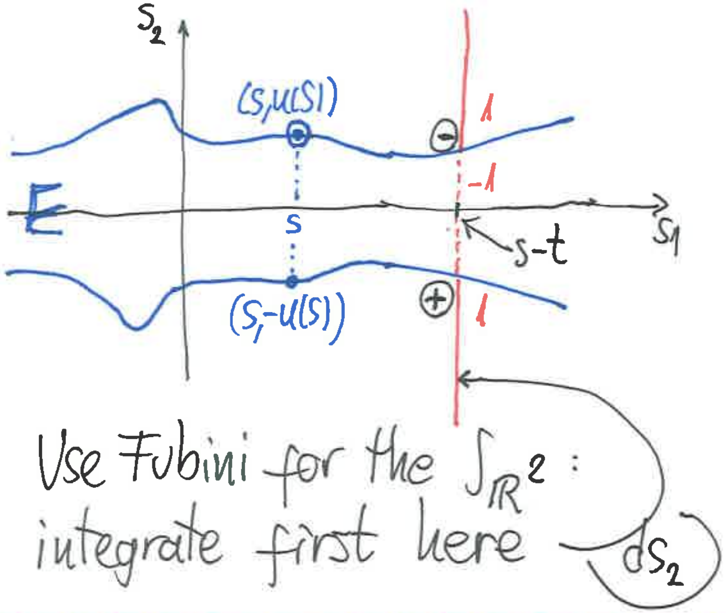
We use a Lyapunov-Schmidt reduction $\oplus$ Implicit Function Theorem applied to a quasilinear-type fractional elliptic equation.

- **STEP 1** : The setting, equation, and functional spaces:

$u:\mathbb{R}\to\mathbb{R}_+$, $0<m_1\le u\le m_2$

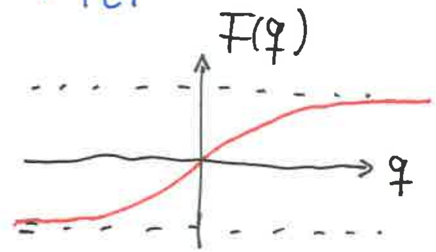
$E=\{-u(s_1)<s_2<u(s_1)\}\subset\mathbb{R}^2 \Rightarrow$

$$\boxed{\begin{aligned}\frac{1}{2}H_E(s,u(s)) &=: \frac{1}{2}H(u)(s) = \\ &= \int_{\mathbb{R}} F\left(\frac{u(s)-u(s-t)}{|t|}\right)\frac{dt}{|t|^{1+\alpha}} \\ &\quad -\int_{\mathbb{R}}\left\{F\left(\frac{u(s)+u(s-t)}{|t|}\right)-F(+\infty)\right\}\frac{dt}{|t|^{1+\alpha}}\end{aligned}}$$



Use Fubini for the $\int_{\mathbb{R}^2}$: integrate first here

where $F(q)=\int_0^q \frac{d\tau}{(1+\tau^2)^{\frac{2+\alpha}{2}}}$

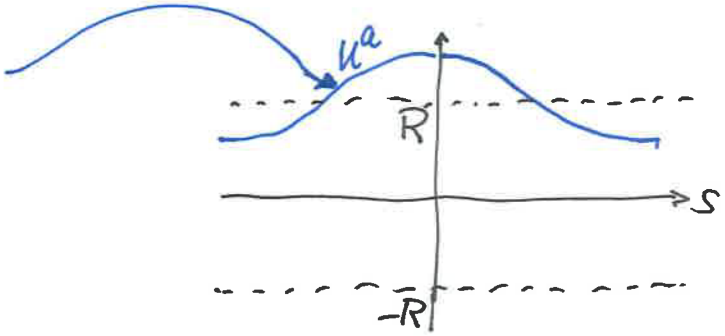


Want $\underline{H(u_a)(s) \equiv ctt \text{ indep. of } a}$

where $\quad u_a(s) := R + \frac{a}{\lambda}\left\{\cos(\lambda s) + v_a(\lambda s)\right\}$

want $\quad \lambda = \lambda(a) \quad \& \quad v_a = v(a)$

$\underline{\text{Period} = \frac{2\pi}{\lambda(a)}}$ $\qquad\qquad$ v_a $\underline{\text{even}}$ f'cn



Rescale $\overset{s_1}{\underset{\shortparallel}{s}}$-variable & the $\overset{s_2}{\underset{\shortparallel}{u}}$-variable to make all fcn's $\underline{2\pi\text{-periodic}}$ $\Big\}$ NMC rescales like λ^α

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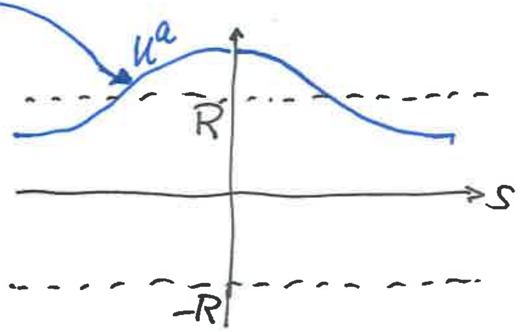

Period $= \frac{2\pi}{\lambda(a)}$

v_a even f'cn

Rescale $\overset{s_1}{\underset{\shortparallel}{s}}$-variable & the $\overset{s_2}{\underset{\shortparallel}{u}}$-variable to make all fcn's 2π-periodic

NMC rescales like λ^α

NEW EQUATION after dividing by a ; as in [CRANDALL - RABINOWITZ]

$$u(s) = \lambda R + a\{\cos(s) + v_a(s)\}$$
$$= \lambda R + a\,\varphi(s)$$

$$\Phi(a,\lambda,\varphi)(s) := \int_{\mathbb{R}} \frac{1}{a} F\left(a\,\frac{\varphi(s)-\varphi(s-t)}{|t|}\right)\frac{dt}{|t|^{1+\alpha}}$$
$$- \int_{\mathbb{R}} \frac{1}{a}\left\{F\left(\frac{2\lambda R + a(\varphi(s)+\varphi(s-t))}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right)\right\}\frac{dt}{|t|^{1+\alpha}} = 0$$

WANT

Solve $\quad 0 = \Phi(a,\lambda,\varphi)(s) := \int_{\mathbb{R}} \frac{1}{a} F\left(a\, \frac{\varphi(s)-\varphi(s-t)}{|t|}\right) \frac{dt}{|t|^{1+\alpha}}$

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Get $\begin{cases} \lambda = \lambda(a) \\ v = v(a) \end{cases}$ for $|a|$ small.

Spaces:

$X = C^{1,\beta}_{p,e} = \{ \varphi : \mathbb{R} \to \mathbb{R},\ C^{1,\beta}(\mathbb{R}),\ 2\pi\text{-periodic, even} \}$

$Y = C^{0,\beta-\alpha}_{p,e} = \{ \tilde{\varphi} : \mathbb{R} \to \mathbb{R},\ C^{0,\beta-\alpha}(\mathbb{R}),\ 2\pi\text{-periodic, even} \}$

$$\Phi(0,\lambda,\varphi)(s) = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\, dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}} F'\left(\frac{2\lambda R}{|t|}\right) dt .$$

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- **Lemma.** $\exists! R$ such that $\Phi(a=0, \lambda=1, \varphi=\cos(\cdot)) = 0$

before scaling, bands have period $2\pi/\lambda$

■ STEP 2 : Linearization at $(a=0, \lambda=1, \varphi=\cos(\cdot))$

$2+\alpha = 1 + 2\,\frac{1+\alpha}{2}$

This term is $\boxed{c_\alpha (-\Delta)^{\frac{1+\alpha}{2}} \varphi(s)}$ loose $1+\alpha$ derivatives

$$\Phi(0,\lambda,\varphi)(s) = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\,dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}} F'\left(\frac{2\lambda R}{|t|}\right) dt.$$

$$\underline{D_\lambda \Phi(0,1,\cos(\cdot))}(s) = -\int_{\mathbb{R}} \{\cos(s)+\cos(s-t)\} \frac{2R\, F''(2R/|t|)}{|t|^{3+\alpha}}\,dt$$

$$= \underline{c_\alpha \cdot \cos(s)} \quad !!!$$

$$Lw(s) := D_\varphi \Phi(0,1,\cos(\cdot))\cdot w(s) = \Phi(0,1,w)(s)$$

$$= \left\{ \underline{c_\alpha(-\Delta)^{\frac{1+\alpha}{2}} w} - \left(\int_{\mathbb{R}} P_R\right) \underline{w} - \underline{P_R * w} \right\}(s)$$

$P_R \in (L^1 \cap L^\infty)(\mathbb{R})$ even fcn

$$\Phi(0,\lambda,\varphi)(s) = \int_{\mathbb{R}} \frac{\varphi(s)-\varphi(s-t)}{|t|^{2+\alpha}}\,dt - \int_{\mathbb{R}} \frac{\varphi(s)+\varphi(s-t)}{|t|^{2+\alpha}}\, F'\!\left(\frac{2\lambda R}{|t|}\right) dt.$$

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$P_R \in (L^1 \cap L^\infty)(\mathbb{R})$ even fcn

- <u>Lemma</u> $w_k(s) = \cos(ks)$, $k = 0,1,2,3,\ldots$ are eigenfunctions of L in X with eigenvalues

$$\lambda_0 < \underline{0 = \lambda_1} < \lambda_2 < \lambda_3 < \ldots \nearrow +\infty \quad \& \quad \frac{\lambda_k}{k^{1+\alpha}} \xrightarrow[k\uparrow\infty]{} ct > 0$$

$$\Updownarrow$$

choice of R to have $\Phi(0,1,\cos(\cdot)) = 0$.

$$\Rightarrow \quad X \underset{\longleftarrow}{\overset{L}{\longrightarrow}} Y$$

$w_1 = \cos(\cdot)$ is in the kernel of $D_\varphi \Phi(0,1,\cos(\cdot))$ & does not belong to its image,

BUT is in the image of $D_\lambda \Phi(0,1,\cos(\cdot))$!!

$$\Big\downarrow \text{Lyapunov-Schmidt, I.F.Thm}$$

$$\begin{cases} \lambda = \lambda(a) \\ \varphi = \varphi(a) \end{cases} \text{ for } |a| \text{ small.} \qquad \blacksquare$$

$w_1 = \cos(\cdot)$ is in the kernel of $D_\varphi \Phi(0,1,\cos(\cdot))$ & does not belong to its image, BUT is in the image of $D_\lambda \Phi(0,1,\cos(\cdot))$!!

$\downarrow$ Lyapunov-Schmidt, I.F. Thm

$\begin{cases} \lambda = \lambda(a) \\ \varphi = \varphi(a) \end{cases}$ for $|a|$ small. $\blacksquare$

■ Step 3 : $\Phi = \Phi(a, \lambda, \varphi)$ is C^1 from $A \subset \mathbb{R} \times \mathbb{R} \times X$ into Y

• Proposition

Hölder spaces $C^{1,\beta}$ & $C^{0,\beta-\alpha}$ of even & 2π-periodic fcns

Main part of Φ is :

$$\Phi_1(a,\varphi)(s) := \int_{\mathbb{R}} F_1\left(a, \frac{\varphi(s)-\varphi(s-t)}{|t|}\right) \frac{dt}{|t|^{1+\alpha}}$$

where

$$F_1(a,q) = \int_0^q \frac{d\tau}{(1+a^2\tau^2)^{\frac{2+\alpha}{2}}}$$

$$= \frac{1}{2} \int_{\mathbb{R}} \frac{2\varphi(s) - \varphi(s-t) - \varphi(s+t)}{|t|^{2+\alpha}} F_3\left(a, \frac{\varphi(s)-\varphi(s-t)}{|t|}, \frac{\varphi(s)+\varphi(s+t)}{|t|}\right)$$

$(1+\alpha)$-deriv.

1st-deriv

with F_3 similar to F_1.

elliptic **quasi-linear** operator.