Villani's program on constructive rate of convergence to the equilibrium : Part II - Hypocoercivity estimates

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Outline of the talk

1. Introduction and main result
   - Villani's program
   - Boltzmann and Landau equation
   - Quantitative trend to the equilibrium
   - First step: quantitative coercivity estimates
   - Second step: (quantitative) hypocoercivity estimates
2. H^1 hypocoercivity estimates
   - The torus
   - The Fokker-Planck operator with confinement force
3. L^2 hypocoercivity estimates
   - The relaxation operator with confinement force
   - The linearized Boltzmann/Landau operator in a domain
   - The linearized Boltzmann operator with harmonic confinement force

Outline of the talk

Here is the program (Villani's Notes on 2001 IHP course, Section 8. Toward exponential convergence)

1. Find a constructive method for bounding below the spectral gap in $L^2(M^{-1})$, the space of self-adjointness, say for the Boltzmann operator with hard spheres.

$\triangleright$ CIRM, April 2017 : coercivity estimates

3. Find a constructive argument to overcome the degeneracy in the space variable, to get an exponential decay for the linear semigroup associated with the linearized spatially inhomogeneous Boltzmann equation; something similar to hypo-ellipticity techniques.

$\triangleright$ Trieste, June 2017 : hypocoercivity estimates

2. Find a constructive argument to go from a spectral gap in $L^2(M^{-1})$ to a spectral gap in L^1, with all the subtleties associated with spectral theory of non-self-adjoint operators in infinite dimension ...

4. Combine the whole things with a perturbative and linearization analysis to get the exponential decay for the nonlinear equation close to equilibrium.

$\triangleright$ Granada, June 2017 : extension of spectral analysis and nonlinear problem

Existence near the equilibrium and trend to the equilibrium (a general picture) :

- Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995): non-constructive method for HS Boltzmann equation in the torus
- Desvillettes, Villani (2001 & 2005) if-theorem by entropy method
- Villani, 2001 IHP lectures on "Entropy production and convergence to equilibrium" (2008)
- Guo and Guo' school (issues 1,2,3,4)

  2002–2008: high energy (still non-constructive) method for various models

  2010–...: Villani's program for various models and geometries
- Mouhot and collaborators (issues 1,2,3,4)

  2005–2007: coercivity estimates with Baranger and Strain

  2006–2015: hypocoercivity estimates with Neumann, Dolbeault and Schmeiser

  2006–2013: $L^p(m)$ estimates with Gualdani and M.
- Carrapatoso, M., Landau equation for Coulomb potentials, 2017

Consider the Boltzmann/Landau equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$
$$F(0, .) = F_0$$

on the density of the particle $F = F(t, x, v) \geq 0$, time $t \geq 0$, velocity $v \in \mathbb{R}^3$, position $x \in \Omega$

- $\Omega = \mathbb{T}^3$ (torus);
- $\Omega \subset \mathbb{R}^3$ + boundary conditions;
- $\Omega = \mathbb{R}^3$ + force field confinement (open problem in general?).

Q = nonlinear (quadratic) Boltzmann or Landau collisions operator
: conservation of mass, momentum and energy

Around the H-theorem

We recall that $\varphi = 1, v, |v|^2$ are collision invariants, meaning

$$\int_{\mathbb{R}^3} Q(F,F)\varphi \, dv = 0, \quad \forall\, F.$$

$\Rightarrow$ laws of conservation

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \int_{\mathbb{T}^3\times\mathbb{R}^3} F_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

We also have the H-theorem, namely

$$\int_{\mathbb{R}^3} Q(F,F) \log F \begin{cases} \le 0 \\ = 0 \ \Rightarrow \ F = \text{Maxwellian} \end{cases}$$

From both pieces of information, we expect

$$F(t,x,v) \underset{t\to\infty}{\longrightarrow} M(v) := \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

Existence, uniqueness and stability in small perturbation regime in large space and with constructive rate

Theorem 1. (Gualdani-M.-Mouhot; Carrapatoso-M.; Briant-Guo)

Take an "admissible" weight function m such that

$$\langle v\rangle^{2+3/2} \prec m \prec e^{|v|^2}.$$

There exist some Lebesgue or Sobolev space $\mathcal{E}$ associated with the weight m and some $\varepsilon_0 > 0$ such that if

$$\|F_0 - M\|_{\mathcal{E}(m)} < \varepsilon_0,$$

there exists a unique global solution F to the Boltzmann/Landau equation and

$$\|F(t) - M\|_{\mathcal{E}(\tilde{m})} \le \Theta_m(t),$$

with optimal rate

$$\Theta_m(t) \simeq e^{-\lambda t^\sigma} \text{ or } t^{-K}$$

with $\lambda > 0$, $\sigma \in (0,1]$, $K > 0$ depending on m and whether the interactions are "hard" or "soft".

Conditionally (up to time uniform strong estimate) exponential H-Theorem

- $(F_t)_{t\geq 0}$ solution to the inhomogeneous Boltzmann equation for hard spheres interactions in the torus with strong estimate

$$\sup_{t\geq 0}\left(\|F_t\|_{H^k} + \|F_t\|_{L^1(1+|v|^s)}\right) \leq C_{s,k} < \infty.$$

- [Desvillettes, Villani, 2005] proved: for any $s \geq s_0$, $k \geq k_0$

$$\forall\, t \geq 0 \qquad \int_{\Omega\times\mathbb{R}^3} F_t \log \frac{F_t}{M(v)}\, dvdx \leq C_{s,k}\,(1+t)^{-\tau_{s,k}}$$

with $C_{s,k} < \infty$, $\tau_{s,k} \to \infty$ when $s, k \to \infty$

Corollary. (Gualdani-M.-Mouhot)

$\exists\, s_1, k_1$ s.t. for any $a > \lambda_2$ exists C_a

$$\forall\, t \geq 0 \qquad \int_{\Omega\times\mathbb{R}^3} F_t \log \frac{F_t}{M(v)}\, dvdx \leq C_a\, e^{\frac{a}{2}\,t},$$

with $\lambda_2 < 0$ (2^{nd} eigenvalue of the linearized Boltzmann eq. in $L^2(M^{-1})$).

We define the linearized Boltzmann / Landau operator in the space homogeneous framework

$$\mathcal{S}f := \frac{1}{2}\Big\{Q(f,M) + Q(M,f)\Big\}$$

and the orthogonal projection π in $L^2(M^{-1})$ on the eigenspace

$$\text{Span}\{(1, v, |v|^2)M\}.$$

Theorem 2. (..., Guo, Mouhot, Strain)

There exist two Hilbert spaces $\mathfrak{h} = L^2(M^{-1})$ and $\mathfrak{h}_*$ and constructive constants $\lambda, K > 0$ such that

$$(-\mathcal{S}h, g)_{\mathfrak{h}} = (-\mathcal{S}g, h)_{\mathfrak{h}} \le K\|g\|_{\mathfrak{h}_*}\|h\|_{\mathfrak{h}_*}$$

and

$$(-\mathcal{S}h, h)_{\mathfrak{h}} \ge \lambda\,\|\pi^\perp h\|^2_{\mathfrak{h}_*}, \quad \pi^\perp = I - \pi$$

The space $\mathfrak{h}_*$ depends on the operator (linearized Boltzmann or Landau) and the interaction parameter $\gamma \in [-3, 1]$, $\gamma = 1$ corresponds to (Boltzmann) hard spheres interactions and $\gamma = -3$ corresponds to (Landau) Coulomb interactions.

Second step in Villani's program: (quantitative) hypocoercivity estimates

In a Hilbert space $\mathcal{H}$, we consider now an operator

$$\mathcal{L} = \mathcal{S} + \mathcal{T}$$

with

$$\mathcal{S}^* = \mathcal{S} \leq 0, \quad \mathcal{T}^* = -\mathcal{T}.$$

More precisely, $\mathcal{H} \supset \mathcal{H}_x \otimes \mathcal{H}_v$, $\mathcal{S}$ acts on the v variable space $\mathcal{H}_v$ with null space $N(\mathcal{S})$ of finite dimension, we denote π the projection on $N(\mathcal{S})$.
As a consequence, in the two variables space $\mathcal{H}$ the operator $\mathcal{S}$ is degenerately / partially coercive

$$(-\mathcal{S}f, f) \gtrsim \|f^\perp\|_*^2, \quad f^\perp = f - \pi f$$

For the initial Hilbert norm, we get the same degenerate / partial positivity of the Dirichlet form

$$D[f] := (-\mathcal{L}, f) \gtrsim \|f^\perp\|_*^2, \quad \forall f.$$

That information is not strong enough in order to control the longtime behavior of the dynamic of the associated semigroup !!

▷ Find a new Hilbert norm by twisting

$$|||f|||^2 := \|f\|^2 + 2(Af, Bf)$$

such that the new Dirichlet form is coercive:

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) + (A\mathcal{L}f, Bf) + (Af, B\mathcal{L}f) \\ &\gtrsim \|f^\perp\|^2 + \|\pi f\|^2. \end{aligned}$$

▷ We destroy the nice symmetric / skew symmetric structure and we have also to be very careful with the "remainder terms".
▷ That functional inequality approach is equivalent (and more precise if constructive) to the other more dynamical approach (called "Lyapunov" or "energy" approach).

Theorem. (for strong coercive operators in both variables, in particular $\mathfrak{h}_* \subset \mathfrak{h}$)

There exist some new but equivalent Hilbert norm $|||\cdot|||$ and a (constructive) constant $\lambda > 0$ such that the associated Dirichlet form satisfies

$$D[f] \gtrsim |||f|||^2, \quad \forall\, f,\ \langle \pi f\rangle = 0.$$

▷ It implies $|||e^{\mathcal{L}t} f||| \le e^{-\lambda t} |||f|||$ and then $\|e^{\mathcal{L}t} f\| \le C e^{-\lambda t}\|f\|$, $\forall\, f,\ \langle \pi f\rangle = 0$.

- Fourier approach and hypocoercivity : Kawashima
- Non constructive spectral analysis approach : Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995)
- Non constructive estimate and hypoellipticity : Eckmann, Pillet, Rey-Bellet (1999)
- Constructive entropy approach: Desvillettes-Villani (2001-2005)
- Energy (in high order Sobolev space) approach : Guo and Guo' school [2002-..]
- Micro-Macro approach : Shizuta, Kawashima (1984), Liu, Yu (2004), Yang, Guo, Duan, ...
- Constructive estimate and hypoelliptícty : Hérau, Nier, Helffer, Eckmann, Hairer (2003-2005), Villani (2009)

  2006–2015: hypocoercivity estimates with Neumann, Dolbeault and Schmeiser
- Carrapatoso, M., Landau equation for Coulomb potentials, 2017

Other problems (not tackled here):

- The case $\mathfrak{h}_* \not\subset \mathfrak{h}$
- The whole space with weak confinement
- The whole space without any confinement
- uniform estimate in the macroscopic limit
- uniform estimate in the grazing collisions limit

Several issues

Geometry of the domain:

- the torus
- the whole space with confinement force
- bounded domain

Collisions operator

- elliptic operator (Fokker-Planck operator)
- relaxation operator (no additional derivative)
- linearized Boltzmann/Landau : more than one invariant (velocity)

Steps

- H^1 estimate : torus and Fokker-Planck in the whole space
- macroscopic projection : domain and relaxation operator in the whole space
- H^1+ micro-macro decomposition : Boltzmann in the whole space

Outline of the talk

We consider

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

for "any" linear collision term $\mathcal{S}$ "of hard potential type" and

$$\mathcal{T} g := -v \cdot \nabla_x g, \quad \Omega := \mathbb{T}^d.$$

We work in the flat space L^2. We define the twisted H^1 norm

$$|||g|||^2 := \|g\|^2_{L^2} + \eta_x \|\nabla_x g\|^2_{L^2} + 2\eta(\nabla_v g, \nabla_x g)_{L^2} + \eta_v \|\nabla_v g\|^2_{L^2},$$

by choosing $\eta^2 < \eta_x \eta_v$ and then the Dirichlet form

$$\begin{aligned} D(g) &= ((-\mathcal{L} g, g)) \\ &= (-\mathcal{L} g, g) - \eta_x (\nabla_x \mathcal{L} g, \nabla_x g) \\ &\quad -\eta(\nabla_v \mathcal{L} g, \nabla_x g) - \eta(\nabla_v g, \nabla_x \mathcal{L} g) - \eta_v (\nabla_v \mathcal{L} g, \nabla_v g). \end{aligned}$$

Theorem 3. ([Villani 2009] after [Mouhot, Neuman 2006])

For convenient choices of $1 \geq \eta_x > \eta > \eta_v > 0$ there holds (with explicit constants)

$$D(g) \gtrsim \|g\|^2_{H^1_{xv}} \gtrsim |||g|||^2, \quad \forall g, \ \langle \pi g \rangle = 0.$$

A possible choice is $\eta_x = 1$, $\eta = \varepsilon^2$, $\eta_v = \varepsilon^3$, $\varepsilon > 0$ small enough.

• The crucial information comes from the third term (in blue). More precisely, throwing away the contribution of the collision operator $\mathcal{S}$, we compute:

$$\begin{aligned} D_{3,1} &:= -\eta(\nabla_v \mathcal{T} g, \nabla_x g) - \eta(\nabla_v g, \nabla_x \mathcal{T} g) \\ &= -\eta(\nabla_v \mathcal{T} g, \nabla_x g) - \eta(\nabla_v g, \mathcal{T} \nabla_x g) \quad \text{because } [\mathcal{T}, \nabla_x] = 0 \\ &= \eta([\nabla_v, -\mathcal{T}] g, \nabla_x g) \\ &= \eta(\nabla_x g, \nabla_x g) \\ &= \eta \|\nabla_x g\|^2. \end{aligned}$$

• Another key remark is that for any g such that $\langle \pi g \rangle = 0$, we have

$$D_{3,1} = \|\nabla_x g\|^2 \gtrsim \|\nabla_x \pi g\|^2 \gtrsim \|\pi g\|^2,$$

where we have used the Poincaré(-Wirtinger) inequality in the torus in the last inequality. Together with the first term

$$D_1 = (-\mathcal{L} g, g) = (-\mathcal{S} g, g) \geq \|g^\perp\|_*^2 \geq \|g^\perp\|^2,$$

we get

$$D(g) \gtrsim ... + \|g^\perp\|^2 + \|\pi g\|^2 = ... + \|g\|_{L^2}.$$

For clarity (?) we introduce some abstract framework. More precisely, we introduce the usual notation

$$A := \nabla_v, \quad B := \nabla_x$$

and we observe that

$$A^* = -A, \quad B^* = -B, \quad [A, B] = 0, \quad [\mathcal{S}, B] = 0, \quad [\mathcal{T}, B] = 0, \quad [\mathcal{T}, A] = B.$$

We also introduce the additional assumptions on the collisional operator

$$\mathfrak{h}_* \subset \mathfrak{h}$$

and

$$(A(-\mathcal{S}g), Ag) \gtrsim (-\mathcal{S}Ag, Ag) + |Ah|^2 - |h|^2$$

which is fulfilled by the Fokker-Planck operator, the standard relaxation operator and the linearized Boltzmann and Landau operator (for hard interaction potentials).

- Because of $\mathcal{T}^* = -\mathcal{T}$, the first term is (partially) dissipative

$$D_1(g) := (-\mathcal{L}g, g) = (-\mathcal{S}g, g) \gtrsim \|g^\perp\|^2.$$

- Using the hypothesis on the collision operator, the second term gives

$$\begin{aligned} D_2(g) &:= \eta_v(A(-\mathcal{S})g, Ag) + \eta_v(A(-\mathcal{T})g, Ag) \\ &\gtrsim \eta_v(-\mathcal{S}Ag, Ag) + \eta_v|Ah|^2 - \eta_v|h|^2 - \eta_v|Bg|\,|Ag|. \end{aligned}$$

- With the help of the above "key computation", the third term is

$$\begin{aligned} D_3(g) &:= -\eta(A\mathcal{T}g, Bg) - \eta(Ag, B\mathcal{T}g) - \eta(A\mathcal{S}g, Bg) - \eta(Ag, B\mathcal{S}g) \\ &= \eta|Bg|^2 + \eta([\mathcal{T}, B]g, Ag) - \eta(\mathcal{S}B^*g, A^*g) - \eta(\mathcal{S}Bg, Ag). \end{aligned}$$

- For the last term, using again $\mathcal{T}^* = -\mathcal{T}$ and also $[B, \mathcal{S}] = 0$, we get

$$\begin{aligned} D_4(g) &:= -\eta_x(B\mathcal{T}g, Bg) - \eta_x(B\mathcal{S}g, Bg) \\ &= -\eta_x([B, \mathcal{T}]g, Bg) - \eta_x(\mathcal{S}Bg, Bg). \end{aligned}$$

We put all the terms together. We kill the blue term by taking $\eta_v << \eta$ together with the magenta terms and we use the specific (commutation) properties of the torus framework, so that in particular the red terms vanish. We get

$$
\begin{aligned}
D(g) \quad \gtrsim \quad & \|g^\perp\|^2 \\
& + \eta_v(-\mathcal{S}Ag, Ag) + \eta_v |Ag|^2 - \eta_v |g|^2 \\
& + \eta \|Bg\|^2 - 2\eta(\mathcal{S}Bg, Ag) \\
& + \eta_x(-\mathcal{S}Bg, Bg).
\end{aligned}
$$

Taking $\eta^2 << \eta_x \eta_v$ and using the Cauchy-Schwarz inequality

$$|(\mathcal{S}Bg, Ag)| \le (-\mathcal{S}Ag, Ag)^{1/2}(-\mathcal{S}Bg, Bg)^{1/2},$$

we get rid of the non necessary positive red term and we end up with

$$
\begin{aligned}
D(g) \quad & \gtrsim \quad \eta\|g^\perp\|^2 + \eta_v|Ag|^2 - \eta_v|g|^2 \ + \ \eta\|Bg\|^2 \\
& \gtrsim \quad \eta\|g^\perp\|^2 + \eta_v|Ag|^2 + \eta_v|g|^2 \ + \ \eta\|Bg\|^2.
\end{aligned}
$$

In the last line in order to change the $-$ into a $+$, we have used the Poincaré inequality in the torus and $\eta >> \eta_v$. It is here that we need $\langle \pi g \rangle = 0$.

We consider the "kinetic Fokker-Planck" operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where

$$\mathcal{T} h := -v \cdot \nabla_x h + \nabla_x V \cdot \nabla_v h, \quad \Omega := \mathbb{R}^d,$$

with a smooth confinement potential $V \sim |x|^\gamma$, $\gamma \geq 1$, and $\mathcal{S}$ is the Fokker-Planck operator which is (for this unknown)

$$\mathcal{S} h := \Delta h - v \cdot \nabla_v h$$

We introduce the probability measure

$$G := e^{-V} M(v), \quad M(v) := (2\pi)^{-d/2} e^{-|v|^2/2}.$$

We work in the Hilbert spaces $\mathfrak{h} := L^2_v(M)$ and $H := L^2_{xv}(G)$. We observe that $h = 1$ is the unique normalized positive steady state and the associated projector is

$$\pi h := (h, 1)_{\mathfrak{h}} 1 = \langle hM \rangle.$$

We introduce the Hilbert norm

$$|||h|||^2 := \|h\|_H^2 + \eta_x \|\nabla_x h\|_H^2 + 2\eta(\nabla_v h, \nabla_x h)_H + \eta_v \|\nabla_v h\|_H^2,$$

with $\eta^2 < \eta_x \eta_v$ and then the Dirichlet form

$$\begin{aligned} D(h) &= ((-\mathcal{L}h, h)) \\ &= (-\mathcal{L}h, h) - \eta_x(\nabla_x \mathcal{L}h, \nabla_x h) \\ &\quad -\eta(\nabla_v \mathcal{L}h, \nabla_x h) - \eta(\nabla_v h, \nabla_x \mathcal{L}h) - \eta_v(\nabla_v \mathcal{L}h, \nabla_v h). \end{aligned}$$

Theorem 4. ([Villani 2009] after [Nier, Hérau, Helffer 2004, 2005])

For convenient choices of $1 > \eta_v > \eta > \eta_x > 0$ there holds (with explicit constants)

$$D(h) \gtrsim \|h\|^2_{H^1_{xv}} \gtrsim |||h|||^2, \quad \forall\, h,\ \langle \pi h\, e^{-V} \rangle = 0.$$

A possible choice is $\eta_v = \varepsilon^5$, $\eta = \varepsilon^7$, $\eta_v = \varepsilon^8$, $\varepsilon > 0$ small enough, instead of $1 = \eta_x > \eta > \eta_v > 0$ in Theorem 3.

Proof of Theorem 4.

Still in the abstract framework

$$A := \nabla_v, \quad B := \nabla_x,$$

we start with the same expression as for Theorem 3

$$\begin{aligned} D(h) \quad := \quad & (-\mathcal{S}h, h) \\ & + \eta_v (A(-\mathcal{S})h, Ah) + \eta_v (Bh, Ah) \\ & + \eta \|Bh\|^2 + \eta([\mathcal{T}, B]h, Ah) - \eta(A\mathcal{S}h, Bh) - \eta(Ah, B\mathcal{S}h) \\ & - \eta_x (\mathcal{S}Bh, Bh) - \eta_x([B, \mathcal{T}]h, Bh), \end{aligned}$$

where now

$$[B, \mathcal{T}] = D^2 V \nabla_v \neq 0!$$

We observe that (in H) we have

$$A^* = v - \nabla_v, \quad \mathcal{S} = -A^* A,$$

and because of the Poincaré inequality in the whole space

$$\int |\nabla_x u|^2 e^{-V} dx \gtrsim \int \langle \nabla V \rangle^2 u^2 e^{-V} dx, \quad \forall\, u, \ \langle u e^{-V} \rangle = 0,$$

(e.g.nice proof by [Bakry, Barthe, Cattiaux, Guillin, 2008]) we have

$$[B, \mathcal{T}] \lesssim \langle \nabla V \rangle \nabla_v \lesssim BA.$$

Using the two above pieces of information in the previous identity and killing the blue term by taking $\eta_v^2 << \eta$ together with the magenta terms , we get

$$
\begin{aligned}
D(h) \quad \gtrsim \quad & |Ah|^2 \\
& + \eta_v |A^* A h|^2 \\
& + \eta |Bh|^2 - \eta(BAh, Ah) + \eta(A^* Ah, A^* Bh) + \eta(B^* Ah, A^* Ah) \\
& + \eta_x |ABh|^2 - \eta_x (BAh, Bh).
\end{aligned}
$$

Because $[A, A^*]$ is "negligible", we simplify the argument by replacing A^* by A (in other words, we assume $[A, A^*] = 0$) and similarly we replace B^* by B. We also kill the last term by assuming $\eta_x << \eta$ and using the positive terms in the third and fourth lines. As a consequence, we get

$$
\begin{aligned}
D(h) \quad \gtrsim \quad & |Ah|^2 \\
& + \eta_v |A^2 h|^2 \\
& + \eta |Bh|^2 - \eta |BAh|\, |Ah| - 2\eta |A^2 h| |BAh| \\
& + \eta_x |BAh|^2.
\end{aligned}
$$

We conclude by choosing $\eta^2 << \eta_x$ in order to kill the first red term and by choosing $\eta^2 << \eta_x \eta_v$ in order to kill the second red term.

Outline of the talk

We consider the kinetic "standard relaxation" operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where

$$\mathcal{T} f := -v \cdot \nabla_x f + \nabla_x V \cdot \nabla_v f, \quad \Omega := \mathbb{R}^d,$$

with a smooth confinement potential $V \sim |x|^\gamma$, $\gamma \geq 1$, and $\mathcal{S}$ is the "standard" relaxation operator which is (for this unknown)

$$\mathcal{S} f := \langle f \rangle M - f.$$

We introduce the probability measure

$$G := e^{-V} M(v), \quad M(v) := (2\pi)^{-d/2} e^{-|v|^2/2}.$$

We work in the Hilbert spaces $\mathfrak{h} := L^2_v(M^{-1})$ and $\mathcal{H} := L^2_{xv}(G^{-1})$. We observe that $f = G$ is the unique normalized positive steady state and the associated projector is

$$\pi f := (f, G)_{\mathfrak{h}} \, M = \langle f \rangle M.$$

In the previous H^1 estimate, we fundamentally used the positive term $|D_v^2 f|$ in order to get rid of the bad term $|D^2 V \nabla_v f|$ produced by the non symmetric part of the norm and the transport term. Such a trick cannot be used in the present situation.

We rather introduce the Hilbert norm

$$|||f|||^2 := \|f\|_{\mathcal{H}}^2 + 2\eta(\rho, \nabla_x \Delta_x^{-1} j)_{\mathcal{H}}$$

with $1 >> \eta > 0$ and then the Dirichlet form

$$\begin{aligned} D(f) &= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) - \eta(\rho_f, \Delta_x^{-1} \nabla_x j[\mathcal{L}f]) - (\rho[\mathcal{L}f], \Delta_x^{-1} \nabla_x j_f). \end{aligned}$$

Here

$$\begin{aligned} \rho &:= \rho_f = \rho[f] = \langle f \rangle, \\ j &:= j_f = j[f] = \langle f\, v \rangle. \end{aligned}$$

Theorem 5. ([Dolbeault, Mouhot, Schmeiser 2015] after [Hérau 2006])

For a convenient choice of $1 >> \eta > 0$ there holds (with explicit constants)

$$D(f) \gtrsim \|f\|_{\mathcal{H}}^2 \gtrsim |||f|||^2, \quad \forall f, \ \langle \pi f \rangle = 0.$$

Why do we choose that norm?
From the partial dissipativity of the collision operator we control $f^\perp = \rho M - f$. We next have control ρ in order to get an estimate on the full density f.

In the case of the torus (so that $\mathcal{T} := -v \cdot \nabla_x$), we compute

$$\partial_t \rho = \rho[\mathcal{L} f] = \langle \mathcal{T} f \rangle = -\nabla_x j,$$

which is useless and next

$$\begin{aligned} \partial_t j &= j[\mathcal{L} f] = \langle v \mathcal{T} \pi f \rangle + \langle v \mathcal{L} f^\perp \rangle \\ &= -\nabla \rho + \langle v \mathcal{L} f^\perp \rangle \end{aligned}$$

As a consequence,

$$\begin{aligned} \frac{d}{dt} (\Delta^{-1} \nabla j, \rho) &= (\Delta^{-1} \nabla j[\mathcal{L} f], \rho) + ... \\ &= -(\Delta^{-1} \Delta \rho, \rho) + ... \\ &= -(\rho, \Delta \Delta^{-1} \rho) + ... \\ &= -\|\rho\|^2 + ... \end{aligned}$$

and the other terms are $\mathcal{O}(\|\rho\| \|f^\perp\| + \|f^\perp\|^2)$.

We rather define the macroscopic operator

$$
\begin{aligned}
\Delta_V u &:= \operatorname{div}(\nabla u + \nabla V u) = \nabla(e^{-V}\nabla(u e^V)) \\
\Delta_V^* u &:= \Delta u - \nabla V \cdot \nabla u = e^V \nabla(e^{-V}\nabla u)
\end{aligned}
$$

and the twisted L^2 scalar product

$$
((f,g)) \quad = \quad (f,g)_{\mathcal{H}} + \eta(\Delta_V^{-1}\nabla j_f, \rho_g e^V)_{L^2} + \eta(\rho_f e^V, \Delta_V^{-1}\nabla j_g)_{L^2}.
$$

The associated Dirichlet form splits into three parts. The first term is

$$
\begin{aligned}
D_1 &:= (-\mathcal{L}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}} \\
&= \int (f - \rho M) f M^{-1} \, e^V \\
&= \int (f - \rho M)^2 M^{-1} \, e^V = \|f^\perp\|^2_{\mathcal{H}}
\end{aligned}
$$

The second term is

$$D_2 \quad := \quad \eta(\Delta_V^{-1}\nabla j[-\mathcal{L}f], \rho_f e^V).$$

We split

$$j[-\mathcal{L}f] = j[-\mathcal{T}\pi f] + j[-\mathcal{L}f^\perp]$$

and we observe that

$$\begin{aligned} j[-\mathcal{T}\pi f] &= j[v\cdot\nabla_x\rho_f M - \nabla_x V\cdot\nabla_v\rho_f M] \\ &= j[Mv\cdot(\nabla_x\rho_f + \nabla_x V\rho_f)] \\ &= \nabla_x\rho_f + \nabla_x V\rho_f = e^{-V}\nabla(\rho_f e^V). \end{aligned}$$

As a consequence, the leader term is D_2 is

$$\begin{aligned} D_{2,1} &:= \eta(\Delta_V^{-1}\nabla e^{-V}\nabla_x(\rho_f e^V), \rho_f e^V) \\ &= \eta(\Delta_V^{-1}\Delta_V\rho_f, \rho_f e^V) \\ &= \eta(\rho_f, \Delta_V^*\Delta_V^{*-1}\rho_f e^V) = \eta\,\|\rho_f\|^2_{L^2(e^{-V})}. \end{aligned}$$

We consider the linearized Boltzmann/Landau operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where

$$\mathcal{T} f := -v \cdot \nabla_x f, \quad x \in \Omega \subset \mathbb{R}^3 \text{ bounded},$$

with boundary condition
- diffusion reflection;
- specular reflection;
- Maxwell reflection (a mix of both).

For simplicity, we rather consider the case of the torus but the proof may be adapted to a pure diffusion or a Maxwell reflection (not clear for a pure specular reflection).

The difficulty comes from the dimension ($= 5$) of the null space $N(\mathcal{S})$. We define

$$\begin{aligned}
a &:= a_f = a[f] = \langle f \rangle =: \bar\pi_0 f = \bar\pi_0, \\
b &:= b_f = b[f] = \langle fv \rangle =: (\bar\pi_\beta f)_{1\le\beta\le3} = (\bar\pi_\beta)_{1\le\beta\le3}, \\
c &:= c_f = c[f] = \langle f(|v|^2 - 3)/6 \rangle =: \bar\pi_4 f = \bar\pi_4,
\end{aligned}$$

and the orthogonal projection operator on $N(\mathcal{S})$ by

$$\pi f := aM + b \cdot vM + c\,(|v|^2 - 3)M = \sum_{\beta=0}^{4} \hat\varphi_\beta \bar\pi_\beta, \quad \hat\varphi_\beta = \varphi_\beta M.$$

We define the twisted L^2 norm

$$|||f|||^2 := \|f\|^2_{\mathcal{H}} + 2\eta(\bar{\pi}[f], \Delta^{-1}\nabla\widetilde{\pi}[f])_{L^2}$$

where the last term is a shorthand for

$$\sum_{\alpha,k} 2\eta_\alpha(\bar{\pi}_\alpha, \Delta^{-1}\partial_{x_k}\widetilde{\pi}_{\alpha k})$$

and the macroscopic quantities

$$\bar{\pi}_\alpha := \langle f\varphi_\alpha M\rangle, \qquad \widetilde{\pi}_{\alpha k} := \langle f\widetilde{\varphi}_{\alpha k}\rangle.$$

We define the Dirichlet form

$$D(f) \;\; = \;\; (-\mathcal{L}f, f) - \eta(\widetilde{\pi}[\mathcal{L}f], \nabla\Delta^{-1}\bar{\pi}[f]) - \eta(\widetilde{\pi}[f], \nabla\Delta^{-1}\bar{\pi}[\mathcal{L}f]).$$

Theorem 6. ([M. book in preparation] after [Guo, Briant 2010, 2016] presented as a more involved dynamical argument)

For a convenient choice of $(\widetilde{\varphi}_{\alpha k})$ and (η_α) there holds (with explicit constants)

$$D(f) \gtrsim \|f\|^2_{\mathcal{H}} \gtrsim |||f|||^2, \quad \forall f, \ \langle\pi f\rangle = 0.$$

The two leader terms are

$$\begin{aligned} D_{2,1} &:= \eta(\widetilde{\pi}[\mathcal{T}\pi f], \nabla\Delta^{-1}\bar{\pi}[f]) \\ D_{3,1} &:= \eta(\widetilde{\pi}[\pi f], \nabla\Delta^{-1}\bar{\pi}[\mathcal{T}\pi f]), \end{aligned}$$

all the other terms are $\mathcal{O}(\|\bar{\pi}\| \, \|f^\perp\| + \|f^\perp\|^2)$.

We take for $1 \le k \le 3$

$$\begin{aligned} &\text{for } \alpha = 0: &&\widetilde{\varphi}_{0k} := \frac{1}{5}(10 - |v|^2)\, v_k; \\ &\text{for } \alpha \in \{1,2,3\}: &&\widetilde{\varphi}_{\alpha\alpha} := \frac{1}{2}[1 + 2v_\alpha^2 - |v|^2], \quad \widetilde{\varphi}_{\alpha k} := \frac{1}{7}|v|^2 v_i v_k \ \text{ if } \ k \neq \alpha: \\ &\text{for } \alpha = 4: &&\widetilde{\varphi}_{4k} := \frac{\sqrt{6}}{13}(|v|^2 - 5)\, v_k. \end{aligned}$$

For any $(\xi_{\alpha\beta k\ell})_{0\le\alpha,\beta\le4,1\le k,\ell\le3}$ and $0\le\alpha\le4$, we compute

$$\sum_{k=1}^{3}\sum_{\beta=0}^{4}\sum_{\ell=1}^{3}\langle\widetilde{\varphi}_{\alpha k}\varphi_\beta v_\ell M\rangle\,\xi_{\alpha\beta k\ell}=\sum_{k=1}^{3}\xi_{\alpha\alpha kk}+\mathbf{1}_{1\le\alpha\le3}\sum_{k\ne\alpha}(\xi_{\alpha kk\alpha}-\xi_{\alpha k\alpha k})$$

As a consequence, we have

$$\begin{aligned}
D_{2,1} &= \sum_\alpha \eta_\alpha \sum_{\beta,k,\ell}\langle\widetilde{\varphi}_{\alpha k}\varphi_\beta v_\ell M\rangle(\partial_{x_\ell}\bar{\pi}_\beta,\partial_{x_k}\Delta^{-1}\bar{\pi}_\alpha)\\
&= \sum_\alpha \eta_\alpha \sum_k(\partial_{x_k}\bar{\pi}_\alpha,\partial_{x_k}\Delta^{-1}\bar{\pi}_\alpha)\\
&\quad + \sum_{1\le\alpha\le3}\eta_\alpha\sum_{k\ne\alpha}\Big\{(\partial_{x_\alpha}\bar{\pi}_k,\partial_{x_k}\Delta^{-1}\bar{\pi}_\alpha)-(\partial_{x_k}\bar{\pi}_k,\partial_{x_\alpha}\Delta^{-1}\bar{\pi}_\alpha)\Big\}\\
&= \sum_\alpha \eta_\alpha(\bar{\pi}_\alpha,\sum_k \partial^2_{x_kx_k}\Delta^{-1}\bar{\pi}_\alpha)\\
&= \sum_\alpha \eta_\alpha\|\bar{\pi}_\alpha\|^2_{L^2}
\end{aligned}$$

what is exactly what we need. Here we have used in a crucial way the "commutation property" $\partial^*_{x_\alpha}\partial_{x_k}-\partial^*_{x_k}\partial_{x_\alpha}=0$.

By orthogonality and with obvious notations, it happens that

$$\begin{aligned} D_{3,1} &:= \eta(\widetilde{\pi}[\pi f], \nabla\Delta^{-1}\bar{\pi}[\mathcal{T}\pi f]) \\ &= \eta_0(\widetilde{\pi}[\pi_{123} f], \nabla\Delta^{-1}\bar{\pi}_0[\mathcal{T}\pi_{123} f]) \\ &\quad + \sum_{\alpha=1}^{3} \eta_\alpha(\widetilde{\pi}[\pi_4 f], \nabla\Delta^{-1}\bar{\pi}_\alpha[\mathcal{T}\pi_{04} f]) \\ &\lesssim \eta_0\|\bar{\pi}_{123}\|^2 + \sum_{\alpha=1}^{3} \eta_\alpha\|\bar{\pi}_4\|\|\bar{\pi}_{04}\| \quad \text{no } \eta_4!. \end{aligned}$$

We conclude by taking $\eta_1 = \eta_2 = \eta_3$, $\eta_0 << \eta_4$ and $\eta_1^2 << \eta_0\eta_4 << \eta_4^2$.

We consider the linearized kinetic Boltzmann/Landau operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where

$$\mathcal{T} g := -v \cdot \nabla_x g + \nabla_x V \cdot \nabla_v g, \quad \Omega := \mathbb{R}^d,$$

with harmonic potential $V = |x|^2$, and $\mathcal{S}$ is the linearized homogeneous Boltzmann operator (for hard spheres interactions).

The previous approach seems to fail because of $\partial^*_{x_\alpha} \partial_{x_k} - \partial^*_{x_k} \partial_{x_\alpha} \neq 0$ in the whole space (in the weighted $L^2(e^V)$ space).

The macroscopic conservations are

$$\int g \, (1, x, v, x \cdot v, x \times v, |x|^2, |v|^2) \, G^{1/2} \, dv dx = 0.$$

In particular,

$$\int a \, e^{-V/2} \, dx = \int b \, e^{-V/2} \, dx = \int c \, e^{-V/2} \, dx = \int b \times x \, e^{-V/2} \, dx = 0.$$

We work in the flat space L^2. We define the twisted H^1 + macroscopic correction norm

$$\begin{aligned} |||g|||^2 &:= \|g\|^2 + \|Xg\|^2 + \|Yg\|^2 \\ &\quad + (X_i c, E_i^\perp) + \eta_1 (X_i b_j + X_j b_i, \Gamma_{ij}^\perp + 2c\delta_{ij}) + \eta_2 (X_i a, b_i) \end{aligned}$$

where

$$X_i := \frac{1}{2}\partial_{x_i} V + \partial_{x_i}, \quad Y_i := \frac{1}{2} v_i + \partial_{v_i}$$

and

$$E_i^\perp := \langle (|v|^2 - 5) v_i M^{1/2} g^\perp \rangle, \quad \Gamma_{ij}^\perp := \langle (v_i v_j - 1) M^{1/2} g^\perp \rangle,$$

after having observed that

$$\begin{aligned} E_i^\perp (\mathcal{L} g) &= -\partial_i c + \mathcal{O}(\|g^\perp\|) \\ (\Gamma_{ij}^\perp + 2c\delta_{ij})(\mathcal{L} g) &= -(X_i b_j + X_j b_i) + \mathcal{O}(\|g^\perp\|). \end{aligned}$$

Theorem 7. ([Duan 2011])

For a convenient choice of (η_i) the associated Dirichlet form satisfies (with explicit constants)

$$D(g) \gtrsim |||g|||^2,$$

for any g satisfying the macroscopic conservations.

Because of of the choice of the harmonic potential, we almost have

$$(-\mathcal{L}g, g) - (X\mathcal{L}g, Xg) - (Y\mathcal{L}g, Yg) \gtrsim \|g^\perp\|^2 + \|Xg^\perp\|^2 + \|Yg^\perp\|^2$$

We have to control the macroscopic quantities and the main issue is the control the b term. That comes form

$$\eta_1(X_i b_j + X_j b_i, (\Gamma_{ij}^\perp + 2c\delta_{ij})(-\mathcal{L}g)) \geq \eta_1\|X_i b_j + X_j b_i\|^2 - \eta_1\mathcal{O}(\|g^\perp\|).$$

We finish the proof if we are able to prove the following Korn's lemma.

Lemma. ([Duan 2011, non constructive])

There exists a constant λ such that

$$\int |\nabla u|^2 e^{-V} \lesssim \int |\nabla^s u|^2 e^{-V}, \quad \nabla^s u := \frac{1}{2}(\nabla u + (\nabla u)^T),$$

for any u such that

$$\langle u e^{-V} \rangle = \langle \partial_i u_j - \partial_j u_i \rangle = 0.$$