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Decay of entropy and the Kac master equation

Lecture V

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$$(\mathcal{N}_{A_k(\underline{\alpha},\underline{\theta})}h)(\mathbf{v}) = \int_{\mathbb{R}^M} h(A_k(\underline{\alpha},\underline{\theta})\mathbf{v} + \left(I_M - A_k(\underline{\alpha},\underline{\theta})A_k(\underline{\alpha},\underline{\theta})^T\right)^{1/2}\mathbf{w})e^{-\pi|\mathbf{w}|^2}d\mathbf{w}$$

Orthogonal Singular Value decomposition

$$A_k(\underline{\alpha},\underline{\theta}) = U_k(\underline{\alpha},\underline{\theta})\Gamma_k(\underline{\alpha},\underline{\theta})V_k(\underline{\alpha},\underline{\theta})^T$$

$$\Gamma_k(\underline{\alpha},\underline{\theta}) = [\gamma_1, \cdots, \gamma_M] \ , \ 0 \le \gamma_j \le 1$$

Theorem

Let $h \in L^1(\mathbb{R}^M, e^{-\pi|\mathbf{v}|^2} d\mathbf{v})$ and assume that $S(h) < \infty$. Then

$$S(\mathcal{N}_A h) \leq \sum_{\sigma \subset \{1,\ldots,M\}} \Pi_{i\in\sigma^{\mathrm{c}}}\gamma_i^2 \Pi_{j\in\sigma}(1-\gamma_j^2) \int_{\mathbb{R}^{\sigma^{\mathrm{c}}}} h_U^\sigma(\mathbf{u}) \log h_U^\sigma(\mathbf{u}) e^{-\pi|\mathbf{u}|^2} d\mathbf{u}$$

where

$$h_U(\mathbf{v}) = h(U_k(\underline{\alpha}, \underline{\theta})\mathbf{v})$$

and the σ marginal h_U^σ is given by

$$h_U^\sigma(\mathbf{u}) = \int_{\mathbb{R}^\sigma} h(U(\mathbf{u}', \mathbf{u})) e^{-\pi|\mathbf{u}'|^2} d\mathbf{u}'$$

Note

$$\int_{\mathbb{R}^{\sigma^{\mathrm{c}}}} h_U^\sigma(\mathbf{u}) \log h_U^\sigma(\mathbf{u}) e^{-\pi|\mathbf{u}|^2} d\mathbf{u} = \int_{\mathbb{R}^M} h(\mathbf{v}) \log h_U^\sigma(P_{\sigma^c} U^T \mathbf{v}) e^{-\pi|\mathbf{v}|^2} d\mathbf{v}$$

Have to show that

$$\sum_{\alpha_1,\ldots,\alpha_k} \lambda_{\alpha_1}\cdots\lambda_{\alpha_k} \int \rho(\theta_1)d\theta_1\cdots\rho(\theta_k)d\theta_k \times$$

$$\times \sum_{\sigma\subset\{1,\ldots,M\}} \Pi_{i\in\sigma^{\mathrm{c}}}\gamma_{k,i}(\underline{\alpha},\underline{\theta})^2 \Pi_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha},\underline{\theta})^2)\times$$

$$\int_{\mathbb{R}^M} h(\mathbf{v}) \log h^{\sigma}_{U_k(\underline{\alpha},\underline{\theta})}(P_{\sigma^{\mathrm{c}}} U_k(\underline{\alpha},\underline{\theta})^T\mathbf{v}) e^{-\pi|\mathbf{v}|^2} d\mathbf{v} \;.$$

$$\leq \int_{\mathbb{R}^M} h(\mathbf{v}) \log(h(\mathbf{v})) e^{-\pi|\mathbf{v}|^2} d\mathbf{v}$$

Brascamp-Lieb inequalities

For $i = 1, \ldots K$, let $H_i \subset \mathbb{R}^M$ be subspaces of dimension d_i and $B_i : \mathbb{R}^M \to H_i$ be linear maps with the property that $B_i B_i^T = I_{H_i}$, the identity map on H_i. Assume further there are non-negative constants $c_i, i = 1, \ldots, K$ such that

$$\sum_{i=1}^{K} c_i B_i^T B_i = I_M \; . \tag{1}$$

Then for any non-negative functions $f_i : H_i \to \mathbb{R}$

$$\int_{\mathbb{R}^M} \Pi_{i=1}^K f_i^{c_i}(B_i \mathbf{v}) e^{-\pi|\mathbf{v}|^2} d\mathbf{v} \leq \Pi_{i=1}^K \left(\int_{H_i} f_i(u) e^{-\pi|u|^2} du \right)^{c_i} \; . \tag{2}$$

For any non-negative function $h \in L^1(\mathbb{R}^M, e^{-\pi|\mathbf{v}|^2} d\mathbf{v})$ with $\|h\|_1 = 1$

$$\int_{\mathbb{R}^M} h(\mathbf{v}) \log h(\mathbf{v}) e^{-\pi|\mathbf{v}|^2} d\mathbf{v}$$

$$\geq \sum_{i=1}^{K} c_i \left[h(\mathbf{v}) \log f_i(B_i \mathbf{v}) e^{-\pi|\mathbf{v}|^2} d\mathbf{v} - \log \int_{H_i} f_i(u) e^{-\pi u^2} du \right]$$

Apply the BL theorem to our problem

$$\sum_{\alpha_1,\ldots,\alpha_k} \lambda_{\alpha_1} \cdots \lambda_{\alpha_k} \int d\rho(\theta_1) \cdots d\rho(\theta_k) \times$$

$$\times \sum_{\sigma \subset \{1,\ldots,M\}} \Pi_{i \in \sigma^c} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \Pi_{j \in \sigma} (1 - \gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2) \times$$

$$\int_{\mathbb{R}^M} h(\mathbf{v}) \log h^{\sigma}_{U_k(\underline{\alpha},\underline{\theta})}(P_{\sigma^c} U_k(\underline{\alpha}, \underline{\theta})^T \mathbf{v}) e^{-\pi|\mathbf{v}|^2} d\mathbf{v} \ .$$

$$f_i(u) \Leftrightarrow h_U^{\sigma}(\mathbf{u}) \ , \ \int_{\mathbb{R}^{\sigma^c}} h_U^{\sigma}(\mathbf{u}) e^{-\pi|\mathbf{u}|^2} d\mathbf{u} = 1$$

$$H_i \Leftrightarrow \mathbb{R}^{\sigma^{\mathrm{c}}}$$

$$B_i \Leftrightarrow P_{\sigma^{\mathrm{c}}} U_k(\underline{\alpha}, \underline{\theta})^T$$

$$\int \rho(\theta_1) d\theta_1 \cdots \rho(\theta_k) d\theta_k \sum_{\sigma \subset \{1,\ldots,M\}} \Pi_{i \in \sigma^{\mathrm{c}}} \gamma_{k,i}(\underline{\alpha}, \underline{\theta})^2 \Pi_{j \in \sigma} (1 - \gamma_{k,j}(\underline{\alpha}, \underline{\theta})^2) \times$$

$$U_k(\underline{\alpha}, \underline{\theta}) P_{\sigma^{\mathrm{c}}}^T P_{\sigma^{\mathrm{c}}} U_k(\underline{\alpha}, \underline{\theta})^T = I_M C_{k,M} \ .$$

where

$$C_{k,M} = \left[\frac{M}{N+M} + \frac{N}{N+M} (1 - \mu_{\rho} \frac{N+M}{N\Lambda})^k \right]$$

$$\int \rho(\theta_1)d\theta_1 \cdots \rho(\theta_k)d\theta_k \sum_{\sigma\subset\{1,\ldots,M\}} \Pi_{i\in\sigma^{\mathrm{c}}}\gamma_{k,i}(\underline{\alpha},\underline{\theta})^2 \Pi_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha},\underline{\theta})^2) \times$$

$$U_k(\underline{\alpha},\underline{\theta})P^T_{\sigma^{\mathrm{c}}}P_{\sigma^{\mathrm{c}}}U_k(\underline{\alpha},\underline{\theta})^T$$

$$= \int \rho(\theta_1)d\theta_1 \cdots \rho(\theta_k)d\theta_k$$

$$U_k(\underline{\alpha},\underline{\theta})\left\{\sum_{\sigma\subset\{1,\ldots,M\}} \Pi_{i\in\sigma^{\mathrm{c}}}\gamma_{k,i}(\underline{\alpha},\underline{\theta})^2 \Pi_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha},\underline{\theta})^2)P^T_{\sigma^{\mathrm{c}}}P_{\sigma^{\mathrm{c}}}\right\}U_k(\underline{\alpha},\underline{\theta})^T$$

$$= \int \rho(\theta_1)d\theta_1 \cdots \rho(\theta_k)d\theta_k U_k(\underline{\alpha},\underline{\theta})\Gamma^2_k(\underline{\alpha},\underline{\theta})U_k(\underline{\alpha},\underline{\theta})^T$$

$$= \int \rho(\theta_1)d\theta_1 \cdots \rho(\theta_k)d\theta_k A_k(\underline{\alpha},\underline{\theta})A_k(\underline{\alpha},\underline{\theta})^T = C_{k,M}$$

$$
\sum_{\alpha_1,\ldots,\alpha_k} \lambda_{\alpha_1}\cdots\lambda_{\alpha_k}\int d\rho(\theta_1)\cdots d\rho(\theta_k)\times
$$

$$
\times \sum_{\sigma\subset\{1,\ldots,M\}} \Pi_{i\in\sigma^{\mathrm{c}}}\gamma_{k,i}(\underline{\alpha},\underline{\theta})^2\Pi_{j\in\sigma}(1-\gamma_{k,j}(\underline{\alpha},\underline{\theta})^2)\times
$$

$$
\int_{\mathbb{R}^M} h(\mathbf{v})\log h^{\sigma}_{U_k(\underline{\alpha},\underline{\theta})}(P_{\sigma^{\mathrm{c}}}U_k(\underline{\alpha},\underline{\theta})^T\mathbf{v})e^{-\pi|\mathbf{v}|^2}d\mathbf{v}\ .
$$

$$
\leq \left[\frac{M}{N+M}+\frac{N}{N+M}(1-\mu_\rho\frac{N+M}{N\Lambda})^k\right]S(h)
$$

Kac Model

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Brascamp-Lieb inequalities

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