Scattering Amplitudes

LECTURE 1

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Particle experiments: our probe to fundamental laws of Nature
Theorist’s perspective: scattering amplitude
Tool how to learn about the dynamics: interactions, theories, symmetries
What does the blob really represent?
What does the blob really represent?

but there is more than that.....
It can be for example a sum of different pictures

Fig. 2: Configurations contributing to the six-gluon amplitude $A_{(1-2-3-4+5+6)}$. Note that (a) and (c) are related by a flip of indices composed with a conjugation. (b) vanishes if the helicity configuration of the internal line.

This is shown in fig. 2. Note that for this helicity configuration, the middle graph vanishes. Therefore, we are left with only two graphs to evaluate. Moreover, the two graphs are related by a flip of indices composed with a conjugation. Therefore, only one computation is needed.

Let us compute in detail the contribution coming from the first graph in Fig. 2(a). The contribution of this term is given by the product of two MHV amplitudes times a propagator,

$$
\left(\langle 2 \hat{3} \rangle \langle \hat{3} \hat{P} \rangle \langle \hat{P} 2 \rangle \right)^{1/2} \left(\langle 1 \hat{P} \rangle \langle \hat{P} \hat{4} \rangle \langle \hat{4} 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle \right)^{1/2}.
$$

This formula can be simplified by noting that

$$
\lambda \hat{3} = \lambda 3,
$$

$$
\lambda \hat{4} = \lambda 4 - t^{[2]} 2 \langle 3 2 \rangle ^{[2]} 4 \lambda 3,
$$

$$
\langle \cdot \hat{P} \rangle = -\langle \cdot | 2+3 | 4 \rangle \hat{P} 4.
$$

Using (2.7) it is straightforward to find (2.6)

$$
\langle 1 | 2+3 | 4 \rangle 3 ^{[2]} \langle 5 6 \rangle \langle 6 1 \rangle t^{[3]} 2 \langle 5 | 3+4 | 2 \rangle.
$$

(2.8)
And in a special case
even something more surprising
Overview of lectures

- **Lecture 1: Review of scattering amplitudes**
  - Motivation
  - On-shell amplitudes
  - Kinematics of massless particles

- **Lecture 2: New methods for amplitudes**
  - Recursion relations for tree-level amplitudes
  - Unitarity methods for loop amplitudes
  - On-shell diagrams

- **Lecture 3: Geometric formulation**
  - Toy model: N=4 SYM theory
  - Positive Grassmannian
  - Amplituhedron
Motivation
Our theoretical framework to describe Nature

Compatible with two principles

- Special relativity
- Quantum mechanics

\[ H(t)|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle \]
Perturbative QFT

(Dirac, Heisenberg, Pauli; Feynman, Dyson, Schwinger)

- Fields, Lagrangian, Path integral

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \not{D} \psi - m \bar{\psi} \psi \]

\[ \int D\!A \, D\!\psi \, D\!\bar{\psi} \, e^{i S(A, \psi, \bar{\psi}, J)} \]

- Feynman diagrams: pictures of particle interactions

  Perturbative expansion: trees, loops
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron

1928

\[ g_e = 2 \]

\[ g_e \sim 2 \]
Great success of QFT

- QFT has passed countless tests in last 70 years

- Example: Magnetic dipole moment of electron
  
  1947
  
  Theory: $g_e = 2.00232$
  
  Experiment: $g_e = 2.0023$
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron
  - 1957 Theory: $g_e = 2.0023193$
  - 1972 Experiment: $g_e = 2.00231931$
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron
  - Theory: $g_e = 2.0023193044$
  - Experiment: $g_e = 2.00231930438$

1990
Dualities

- At strong coupling: perturbative expansion breaks

- Surprises: dual to weakly coupled theory
  - Gauge-gauge dualities
    (Montonen-Olive 1977, Seiberg-Witten 1994)
  - Gauge-gravity duality
    (Maldacena 1997)
Incomplete picture

- Our picture of QFT is incomplete
- Also, tension with gravity and cosmology

If there is a new way of thinking about QFT, it must be seen even at weak coupling

- Explicit evidence: scattering amplitudes
Colliders at high energies

- Proton scattering at high energies

Needed: amplitudes of gluons for higher multiplicities

\[ gg \rightarrow gg \ldots g \]
Early 80s

- Status of the art: $gg \rightarrow ggg$

Brute force calculation
24 pages of result

$$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$$
New collider

- 1983: Superconducting Super Collider approved
- Energy 40 TeV: many gluons!
- Demand for calculations, next on the list: $gg \rightarrow gggg$
Parke-Taylor formula

(Parke, Taylor 1985)

- Process $gg \rightarrow gggg$

- 220 Feynman diagrams, $\sim 100$ pages of calculations

- 1985: Paper with 14 pages of result
Parke-Taylor formula

- Process $gg \rightarrow gggg$
- 220 Feynman diagrams, $\sim 100$ pages of calculations
Our result has successfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.
Parke-Taylor formula

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Within a year they realized

\[
M_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}
\]

Spinor-helicity variables

\[
p^\mu = \sigma^\mu_{\dot{a}\dot{a}} \lambda_a \bar{\lambda}_{\dot{a}}
\]

\[
\langle 12 \rangle = \epsilon_{ab} \lambda_a^{(1)} \lambda_b^{(2)}
\]

\[
[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}}^{(1)} \tilde{\lambda}_{\dot{b}}^{(2)}
\]

(Mangano, Parke, Xu 1987)
Parke-Taylor formula

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� Within a year they realized

$$M_n = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \ldots \langle n1 \rangle}$$
Problems with Feynman diagrams

- Particles on internal lines are not real
  - Individual diagrams not gauge invariant
- Obscure simplicity of the final answer
  - Most of the terms in each diagram cancels
- Lesson: work with gauge invariant quantities with fixed spin structure
Birth of amplitudes

- New field in theoretical particle physics

  - New methods and effective calculations
  - Uncovering new structures in QFT

“Road map”

- Explicit calculation
- New structure discovered
- New method which exploits it
What are scattering amplitudes
Scattering process

- Interaction of elementary particles
- Initial state $|i\rangle$ and final state $|f\rangle$
- Scattering amplitude $\mathcal{M}_{if} = \langle i | f \rangle$
- Example: $e^+ e^- \rightarrow e^+ e^-$ or $e^+ e^- \rightarrow \gamma \gamma$ etc.
- Cross section: $\sigma = \int |\mathcal{M}|^2 \, d\Omega$ probability
Scattering amplitude in QFT

- Scattering amplitude depends on types of particles and their momenta
  \[ M_{if} = F(p_i, s_i) \]

- Theoretical framework: calculated in some QFT

- Specified by Lagrangian: interactions and couplings
  \[ L = L(O_j, g_k) \]

- Example: QED
  \[ L_{int} = e \overline{\psi} \gamma_\mu \psi A^\mu \]
Perturbation theory

- Weakly coupled theory

\[ \mathcal{M} = \mathcal{M}_0 + g \mathcal{M}_1 + g^2 \mathcal{M}_2 + g^3 \mathcal{M}_3 + \ldots \]

- Representation in terms of Feynman diagrams

- Perturbative expansion = loop expansion

\[ \mathcal{M} = \mathcal{M}^{\text{tree}} + \mathcal{M}^{1\text{-loop}} + \mathcal{M}^{2\text{-loop}} + \ldots \]
Divergencies

- Loop diagrams are generally UV divergent

\[ \sim \int_{-\infty}^{\infty} \frac{d^4\ell}{(\ell^2 + m^2)[(\ell + p)^2 + m^2]} \sim \log \Lambda \]

- IR divergencies: physical effects, cancel in cross section

- Dimensional regularization: calculate integrals in 
  \[ 4 + \epsilon \] dimensions
  \[ \text{Divergencies} \sim \frac{1}{\epsilon^k} \]
Renormalizable theories

- Absorb UV divergencies: counter terms
  - Finite number of them: renormalizable theory
  - Infinite number: non-renormalizable theory
- Mostly only renormalizable theories are interesting
- Exceptions: effective field theories
  Example: Chiral perturbation theory - derivative expansion
  \[ \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \ldots \]
  Different loop orders are mixed
Analytic structure of amplitudes

- Tree-level: rational functions
  \[
  \sim \frac{g^2}{(p_1 + p_2)^2}
  \]

- Only poles

- Loops: polylogarithms and more complicated functions
  \[
  \sim \log^2(s/t)
  \]

- Branch cuts
Kinematics of massless particles
Massless particles

Parameters of elementary particles of spin $S$

- Spin  $s = (-S, S)$  On-shell (physical) particle  $p^2 = m^2$
- Mass  $m$
- Momentum  $p^\mu$

Massless particle:  $m = 0$  $p^2 = 0$

spin = helicity: only two extreme values  $h = \{-S, S\}$

Example: photon  $h = (+, -)$  $s = 0$  missing
Spin functions

- At high energies particles are massless. Fundamental laws reveal there...

- Spin degrees of freedom: spin function
  - $s=0$: Scalar - no degrees of freedom
  - $s=1/2$: Fermion - spinor $u$
  - $s=1$: Vector - polarization vector $\epsilon^\mu$
  - $s=2$: Tensor - polarization tensor $h^{\mu\nu}$
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Polarization vectors

- Spin 1 particle is described by vector $\epsilon^{\mu}$
- Null condition: $\epsilon \cdot \epsilon^* = 0$ 3 degrees of freedom left
- We further impose: $\epsilon \cdot p = 0$ Identification
- Feynman diagrams depend on gauge dependence

$\epsilon_\mu \sim \epsilon_\mu + \alpha p_\mu$
Spinor helicity variables

- Standard SO(3,1) notation for momentum
  \[ p^\mu = (p_0, p_1, p_2, p_3) \quad p_j \in \mathbb{R} \]

- We use SL(2,C) representation
  \[ p^2 = p_0^2 + p_1^2 + p_2^2 - p_3^2 \]

\[ p_{ab} = \sigma^\mu_{ab} p_\mu = \begin{pmatrix} p_0 + ip_1 & p_2 + p_3 \\ p_2 - p_3 & p_0 - ip_1 \end{pmatrix} \]

On-shell: \[ p^2 = \text{det}(p_{ab}) = 0 \]

\[ \text{Rank} (p_{ab}) = 1 \]
Spinor helicity variables

- We can then write \( p_{ab} = \lambda_a \kappa_b \)

- \( \text{SL}(2,\mathbb{C}): \) dotted notation \( p_{\dot{a} \dot{b}} = \lambda_a \tilde{\lambda}_b \)

- \( \tilde{\lambda} \) is complex conjugate of \( \lambda \)

- Little group transformation
  
  \[ \lambda \rightarrow t\lambda \]
  
  \[ \tilde{\lambda} \rightarrow \frac{1}{t} \tilde{\lambda} \]

  leaves momentum unchanged
  
  \( p \rightarrow p \)

  3 degrees of freedom
Spinor helicity variables

- Momentum invariant
  \[ (p_1 + p_2)^2 = (p_1 \cdot p_2) \]
  \[ p_1^\mu = \sigma_\mu^{a\dot{a}} \lambda_1^a \tilde{\lambda}_{1\dot{a}} \]
  \[ p_2^\mu = \sigma_\mu^{b\dot{b}} \lambda_2^b \tilde{\lambda}_{2\dot{b}} \]

- Plugging for momenta
  \[ (p_1 \cdot p_2) = (\sigma_\mu^{a\dot{a}} \sigma_\mu^{b\dot{b}}) (\lambda_1^a \lambda_2^b) (\tilde{\lambda}_{1\dot{a}} \tilde{\lambda}_{2\dot{b}}) \]
  \[ \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} = (\epsilon_{ab} \lambda_1^a \lambda_2^b) (\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_{1\dot{a}} \tilde{\lambda}_{2\dot{b}}) \]

Define:

\[ \langle 12 \rangle \equiv \epsilon_{ab} \lambda_1^a \lambda_2^b \]
\[ [12] \equiv \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_{1\dot{a}} \tilde{\lambda}_{2\dot{b}} \]
Invariant products

- **Momentum invariant**

\[ s_{ij} = (p_i + p_j)^2 = \langle ij \rangle [ij] \]

Square brackets

**Angle brackets**

\[ \langle ij \rangle = \epsilon_{ab} \lambda_{ia} \lambda_{jb} \]

**Antisymmetry**

\[ \langle 21 \rangle = -\langle 12 \rangle \]

\[ [21] = -[12] \]

**More momenta**

\[ (p_1 + p_2 + p_3)^2 = \langle 12 \rangle [12] + \langle 23 \rangle [23] + \langle 13 \rangle [13] \]
Invariant products

- **Shouten identity**

\[
\langle 13 \rangle \langle 24 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 14 \rangle \langle 23 \rangle
\]

- **Mixed brackets**

\[
\langle 1 | 2 + 3 | 4 \rangle \equiv \langle 12 \rangle [24] + \langle 13 \rangle [34]
\]

- **Momentum conservation**

\[
\sum_{i=1}^{n} \lambda_{ia} \tilde{\lambda}_{i\dot{a}} = 0
\]

Non-trivial conditions:

**Quadratic relation between components**
Polarization vectors

- Two polarization vectors
  \[ \epsilon_+^\mu = \sigma_{a \dot{a}}^\mu \frac{\eta_a \lambdabar_{\dot{a}}}{\langle \eta \lambda \rangle} \quad \epsilon_-^\mu = \sigma_{a \dot{a}}^\mu \frac{\lambda_a \eta_{\dot{a}}}{[\tilde{\eta} \tilde{\lambda}]} \]
  where \( \eta, \tilde{\eta} \) are auxiliary spinors

- Freedom in choice of \( \eta, \tilde{\eta} \) corresponds to
  \[ \epsilon^\mu \sim \epsilon^\mu + \alpha p^\mu \]

- Gauge redundancy of Feynman diagrams

Note that \( (\epsilon_+ \cdot \epsilon_-) = 1 \)
Scaling of amplitudes

- Consider some amplitude \( A(−+−−+⋯−) \)

\[
A = (\epsilon_1 \epsilon_2 \cdots \epsilon_n) \cdot Q
\]

- Little group scaling

\[
\lambda \to t\lambda \quad p \to p \quad \epsilon_+ \to \frac{1}{t^2} \cdot \epsilon_+ \quad \epsilon_- \to t^2 \cdot \epsilon_-
\]

\[
A(i^−) \to t^2 \cdot A(i^−) \quad A(i^+) \to \frac{1}{t^2} \cdot A(i^+)
\]

depends only on momenta
Back to Parke-Taylor formula

- Let us consider \( A(1^-2^-3^+4^+5^+6^+) \)

- Scaling
  \[
  A \left( t\lambda_i, \frac{1}{t} \bar{\lambda}_i \right) = t^2 \cdot A(\lambda_i, \bar{\lambda}_i) \quad \text{for particles 1,2}
  \]

- Scaling
  \[
  A \left( t\lambda_i, \frac{1}{t} \bar{\lambda}_i \right) = \frac{1}{t^2} \cdot A(\lambda_i, \bar{\lambda}_i) \quad \text{for particles 3,4,5,6}
  \]

- Check for explicit expression
  \[
  A_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}
  \]

  If only \( \langle ij \rangle \) allowed the form is unique
Helicity amplitudes

- In Yang-Mills theory we have + or - “gluons”
  \[ A_6(1^-2^-3^+4^+5^+6^+) \]
- We denote k: number of - helicity gluons
- Some amplitudes are zero

\[
\begin{align*}
A_n(++++\cdots+) &= 0 & \text{First non-trivial: } k=2 \\
A_n(+-+\cdots+) &= 0 & A_n(--+\cdots+) \\
A_n(--\cdots-) &= 0 & \text{Parke-Taylor formula for tree level} \\
A_n(+-\cdots-) &= 0
\end{align*}
\]
Three point amplitudes
Three point kinematics

- Gauge invariant building blocks: on-shell amplitudes
  \[ p_1^2 = p_2^2 = p_3^2 = 0 \quad p_1 + p_2 + p_3 = 0 \]
  
- Plugging second equation into the first
  \[ (p_1 + p_2)^2 = (p_1 \cdot p_2) = 0 \]

- Similarly we get for other pairs
  \[ (p_1 \cdot p_2) = (p_1 \cdot p_3) = (p_2 \cdot p_3) = 0 \]

  These momenta are very constrained!
Three point kinematics

- Use spinor helicity variables trivializes on-shell condition
  \[ p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad p_3 = \lambda_3 \tilde{\lambda}_3 \]

- The mutual conditions then translate to
  \[ (p_1 \cdot p_2) = \langle 12 \rangle [12] = 0 \]

- And similarly for other two pairs
  \[ (p_1 \cdot p_3) = \langle 13 \rangle [13] = 0 \quad (p_2 \cdot p_3) = \langle 23 \rangle [23] = 0 \]
Two solutions

- We want to solve conditions

\[ \langle 12 \rangle [12] = \langle 13 \rangle [13] = \langle 23 \rangle [23] = 0 \]

- Solution 1: \( \langle 12 \rangle = 0 \) which implies \( \lambda_2 = \alpha \lambda_1 \)

Then we also have \( \langle 23 \rangle = \alpha \langle 13 \rangle \)

And we set \( \langle 13 \rangle = 0 \) by demanding \( \lambda_3 = \beta \lambda_1 \)

\[ \lambda_1 \sim \lambda_2 \sim \lambda_3 \]
Two solutions

Solution 2: \([12] = [23] = [13] = 0\)

\[\tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3\]

Let us take this solution

\[p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \alpha \lambda_2 \tilde{\lambda}_1, \quad p_3 = (-\lambda_1 - \alpha \lambda_2) \tilde{\lambda}_1\]

complex momenta

No solution for real momenta
Three point amplitudes

- Gauge theory: scattering of three gluons (not real)
- Building blocks: $\langle 12 \rangle, \langle 23 \rangle, \langle 13 \rangle, [12], [23], [13]$
- Mass dimension: each term $\sim m$
- Three point amplitude $A_3 \sim \epsilon^3 p \sim p \sim m$
Three point amplitudes

Two options

\[ A_3^{(1)} = \langle 12 \rangle^{a_1} \langle 13 \rangle^{a_2} \langle 23 \rangle^{a_3} \quad A_3^{(2)} = [12]^{b_1} [13]^{b_2} [23]^{b_3} \]

Apply to \( A_3(1^-, 2^-, 3^+) \)

\[ A_3(t\lambda_1, t^{-\frac{1}{2}}\tilde{\lambda}_1) = t^{a_1+a_2} \cdot A_3 \quad a_1 + a_2 = 2 \quad a_1 = 3 \]
\[ A_3(t\lambda_2, t^{-\frac{1}{2}}\tilde{\lambda}_2) = t^{a_1+a_3} \cdot A_3 \quad \rightarrow \quad a_1 + a_3 = 2 \quad a_2 = -1 \]
\[ A_3(t\lambda_3, t^{-\frac{1}{2}}\tilde{\lambda}_3) = t^{a_2+a_3} \cdot A_3 \quad a_2 + a_3 = -2 \quad a_3 = -1 \]
Three point amplitudes

Similarly for \( A_3(1^+, 2^+, 3^-) \)

Two fundamental amplitudes

\[
A_3(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle}
\]

\[
A_3(1^+, 2^+, 3^-) = \frac{[12]^3}{[13][23]}
\]

This is true to all orders: just kinematics
Three point amplitudes

- Collect all \((- - +)\) amplitudes

\[
A_3(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle}
\]

\[
A_3(1^-, 2^+, 3^-) = \frac{\langle 13 \rangle^3}{\langle 12 \rangle \langle 23 \rangle}
\]

\[
A_3(1^+, 2^-, 3^-) = \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 13 \rangle}
\]

- Similarly for \((+ + -)\) amplitudes

\[
\frac{\langle a \ b \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}
\]

where a,b are - helicity gluons

\[
\frac{[ab]^4}{[12][23][31]}
\]

where a,b are + helicity gluons
Three point amplitudes

- Using similar analysis we find for gravity

\[
\frac{\langle ab \rangle^8}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2}
\]

\[
\frac{[ab]^8}{[12]^2 [23]^2 [31]^2}
\]

where \(a, b\) are - helicity gravitons

where \(a, b\) are + helicity gravitons

- Note that there is no three-point scattering

\textbf{They exist only for complex momenta}

- Important input into on-shell methods
Two solutions for 3pt kinematics

\[ \lambda_1 \sim \lambda_2 \sim \lambda_3 \]

Under the little group rescaling:

\[ A_3(t\lambda_j, t^{-1}\tilde{\lambda}_j) \sim t^{2h_j} \cdot A_3 \]

Solve the system of equations

\[ \tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3 \]
Two solutions for amplitudes

\[ A_3 = [12] - h_1 - h_2 + h_3 [23] - h_2 - h_3 + h_1 [31] - h_1 - h_3 + h_2 \]

\[ A_3 = \langle 12 \rangle h_1 + h_2 - h_3 \langle 23 \rangle h_2 + h_3 - h_1 \langle 31 \rangle h_1 + h_3 - h_2 \]

Which one is correct?
General 3pt amplitudes

Two solutions for amplitudes

A_3 = [12]+h_1+h_2−h_3 [23]−h_1+h_2+h_3 [31]+h_1−h_2+h_3

h_1 + h_2 + h_3 \leq 0

Mass dimension must be positive!
All spins allowed

- Note that these formulas are valid for any spins.
- For example for amplitude $A_3(1^0, 2^{1^+}, 3^{2^+})$
  $$A_3 = \frac{\langle 23 \rangle \langle 31 \rangle^3}{\langle 12 \rangle^3}$$
- But we can also do higher spins $A_3(1^{3^+}, 2^{5^+}, 3^{12^-})$
  $$A_3 = \frac{\langle 23 \rangle^{10} \langle 31 \rangle^{14}}{\langle 12 \rangle^{20}}$$
- Completely fixed just by kinematics!
Tree-level amplitudes
Feynman diagrams

- Yang-Mills Lagrangian
  \[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sim (\partial A)^2 + A^2 \partial A + A^4 \]

  \[ \sim f^{abc} g_{\mu\nu} p_\alpha \]

  \[ \sim f^{abe} f^{cde} g_{\mu\nu} g_{\alpha\beta} \]

- Draw diagrams
- Feynman rules
- Sum everything
Change of strategy

What is the scattering amplitude?

Feynman diagrams

Unique object fixed by physical properties

Was not successful (1960s)

Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory
Locality and unitarity

- Only poles: Feynman propagators
  
  ![Locality Diagram](image)

  \[
  \frac{1}{P^2} \quad \text{where} \quad P = \sum_{k \in \mathcal{P}} p_k
  \]

- On the pole
  
  ![Unitarity Diagram](image)

  \[\mathcal{M} \xrightarrow{P^2=0} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R\]

  Feynman diagrams recombine on both sides into amplitudes
Factorization on the pole

On $P^2 = 0$  
Res $\mathcal{M} = \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

- For $P^2 = 0$ the internal leg: on-shell physical particle
- Both sub-amplitudes are on-shell, gauge invariant
- On-shell data: statement about on-shell quantities
On-shell constructibility

- Factorization of tree-level amplitudes

\[ \mathcal{M} \xrightarrow{P^2 = 0} \frac{1}{P^2} \mathcal{M}_L \mathcal{M}_R \]

- On-shell constructibility: factorizations fix the answer

- Write a proposal tree-level amplitude \( \mathcal{\tilde{M}} \)
  - On-shell gauge invariant function, correct weights
  - It factorizes properly on all channels

The amplitude is uniquely specified by these properties
On-shell constructibility

- This is obviously a theory specific statement
- Theories with contact terms might not be constructible
- Naively, this is false for Yang-Mills theory

\[ gg \rightarrow gg \]

Four point amplitude

Contact term
On-shell constructibility

- This is obviously a theory specific statement
- Theories with contact terms might not be constructible
- Naively, this is false for Yang-Mills theory
  \[ gg \rightarrow gg \]
  Four point amplitude

Contact term

Imposing gauge invariance fixes it
On-shell constructibility

• In gravity we have infinity tower of terms

\[ \mathcal{L} \sim \sqrt{g} \, R \sim h^2 + h^3 + h^4 + \ldots \]

• Only \( h^3 \) terms important, others fixed by diffeomorphism symmetry

• On-shell constructibility of Yang-Mills, GR, SM

  Only function which factorizes properly on all poles is the amplitude.
Four point test
From 3pt to 4pt

- Three point amplitudes exist for all spins
- For 4pt amplitude: we have a powerful constraint
  \[ A_4 \xrightarrow{s=0} \frac{1}{s} A_3 \]
  This must be true on all channels
- This will immediately kill most of the possibilities
- We are left with spectrum of spins: 0, \( \frac{1}{2} \), 1, \( \frac{3}{2} \), 2
Three point of spin $S$

- I will discuss amplitudes of single spin $S$ particle

- For 3pt amplitudes we get

  \[
  A_3 = \left( \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \right)^S
  \]

  minimal powercounting

  \[
  A_3 = \left( \frac{[12]^3}{[23][31]} \right)^S
  \]

- There exist also non-minimal amplitudes

  \[
  A_3 = (\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle)^S
  \]

  \[
  A_3 = ([12][23][31])^S
  \]
Four point amplitude

Let us consider a 4pt amplitude of particular helicities

\[ A_4(- - ++) \]

Mandelstam variables:

\[ s = (p_1 + p_2)^2 = \langle 12 \rangle [12] = \langle 34 \rangle [34] \]
\[ t = (p_1 + p_4)^2 = \langle 14 \rangle [14] = \langle 23 \rangle [23] \]
\[ u = (p_1 + p_3)^2 = \langle 13 \rangle [13] = \langle 24 \rangle [24] \]

One can show that the little group dictates:

\[ A_4 = (\langle 12 \rangle [34])^{2S} \cdot F(s, t) \]

It must be consistent with factorizations
s-channel constraint

The s-channel factorization dictates $P = 1 + 2 = -3 - 4$

$A_4 \rightarrow \left( \frac{\langle 12 \rangle^3}{\langle 1P \rangle \langle 2P \rangle} \right)^S \frac{1}{s} \left( \frac{[34]^3}{[3P][4P]} \right)^S$ on $s=0$

Note: $s = \langle 12 \rangle[12] = \langle 34 \rangle[34]$
The $s$-channel factorization dictates $P = 1 + 2 = -3 - 4$

$$A_4 \rightarrow \left( \frac{\langle 12 \rangle^3}{\langle 1P \rangle \langle 2P \rangle} \right)^s \frac{1}{s} \left( \frac{[34]^3}{[3P][4P]} \right)^s$$

on $s = 0$

Note: $s = \langle 12 \rangle[12] = \langle 34 \rangle[34]$
s-channel constraint

\[
\left( \frac{\langle 12 \rangle^3}{\langle 1P\rangle\langle 2P\rangle} \right)^S \frac{1}{s} \left( \frac{[34]^3}{[3P][4P]} \right)^S
\]

- **Rewrite using momentum conservation:**

  \[
  \langle 1P \rangle[3P] = -\langle 1|P|3 \rangle = -\langle 1|1 + 2|3 \rangle = \langle 12 \rangle[23] \\
  \langle 2P \rangle[4P] = -\langle 2|P|4 \rangle = \langle 2|3 + 4|4 \rangle = \langle 23 \rangle[34]
  \]

- **We get**

  \[
  \frac{1}{s} \left( \frac{(\langle 12 \rangle[34])^3}{\langle 1P\rangle[3P]\langle 2P\rangle[4P]} \right)^S
  \]
s-channel constraint

\[
\left( \frac{\langle 12 \rangle^3}{\langle 1P \rangle \langle 2P \rangle} \right)^S \frac{1}{s} \left( \frac{[34]^3}{[3P][4P]} \right)^S
\]

* Rewrite using momentum conservation:

\[
\langle 1P \rangle [3P] = -\langle 1|P|3 \rangle = -\langle 1|1 + 2|3 \rangle = \langle 12 \rangle [23]
\]

\[
\langle 2P \rangle [4P] = -\langle 2|P|4 \rangle = \langle 2|3 + 4|4 \rangle = \langle 23 \rangle [34]
\]

* We get

\[
\frac{1}{s} \left( \frac{\langle 12 \rangle [34]^3}{\langle 12 \rangle [23] \langle 23 \rangle [34]} \right)^S
\]
s-channel constraint

\[
\left( \frac{\langle 12 \rangle^3}{\langle 1P \rangle \langle 2P \rangle} \right)^S \frac{1}{s} \left( \frac{[34]^3}{[3P][4P]} \right)^S
\]

\begin{itemize}
  \item Rewrite using momentum conservation:
    \[
    \langle 1P \rangle [3P] = -\langle 1|P|3 \rangle = -\langle 1|1 + 2|3 \rangle = \langle 12 \rangle [23] \\
    \langle 2P \rangle [4P] = -\langle 2|P|4 \rangle = \langle 2|3 + 4|4 \rangle = \langle 23 \rangle [34]
    \]

  \item We get
    \[
    \frac{1}{s} \left( \frac{(\langle 12 \rangle [34])^2}{\langle 23 \rangle [23]} \right)^S
    \]
\end{itemize}
s-channel constraint

\[
\left( \frac{\langle 12 \rangle^3}{\langle 1P \rangle \langle 2P \rangle} \right)^S \frac{1}{s} \left( \frac{[34]^3}{[3P][4P]} \right)^S
\]

• Rewrite using momentum conservation:

\[
\langle 1P \rangle [3P] = -\langle 1|P|3 \rangle = -\langle 1|1 + 2|3 \rangle = \langle 12 \rangle [23] \\
\langle 2P \rangle [4P] = -\langle 2|P|4 \rangle = \langle 2|3 + 4|4 \rangle = \langle 23 \rangle [34]
\]

• We get

\[
\frac{1}{s} \left( \frac{\langle 12 \rangle [34]}{t} \right)^S
\]
s-channel constraint

\[ \left( \frac{|12\rangle^3}{\langle 1P\rangle\langle 2P\rangle} \right)^{S} \frac{1}{s} \left( \frac{|34\rangle^3}{[3P][4P]} \right)^{S} \]

- **Rewrite using momentum conservation:**
  \[ \langle 1P\rangle[3P] = -\langle 1|P|3\rangle = -\langle 1|1 + 2|3\rangle = \langle 12\rangle[23] \]
  \[ \langle 2P\rangle[4P] = -\langle 2|P|4\rangle = \langle 2|3 + 4|4\rangle = \langle 23\rangle[34] \]

- **We get**
  \[ (\langle 12\rangle[34])^{2S} \cdot \frac{1}{s \cdot t^S} \]

  “Trivial” helicity factor  Important piece  

Note:
\[ t = -u \]
Comparing channels

- On s-channel we got: \( A_4 \rightarrow (\langle 12 \rangle [34])^{2S} \cdot \frac{1}{s} \frac{1}{t} \frac{1}{u} \) and search for \( F(s, t, u) \)

- On t-channel we would get: \( A_4 \rightarrow (\langle 12 \rangle [34])^{2S} \cdot \frac{1}{t} \frac{1}{s} \frac{1}{u} \)
Comparing channels

- On s-channel we got: $A_4 \rightarrow (\langle 12 \rangle [34])^{2S} \cdot \begin{pmatrix} 1 \\ s \\ t \\ u \end{pmatrix}$
- On t-channel we would get: $A_4 \rightarrow (\langle 12 \rangle [34])^{2S} \cdot \begin{pmatrix} 1 \\ s \\ t \\ s' \end{pmatrix}$
- Require simple poles: $\frac{1}{s}, \frac{1}{t}, \frac{1}{u}$ and search for $F(s, t, u)$
- There are only two solutions:
  $$F(s, t, u) = \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \quad F(s, t, u) = \frac{1}{stu}$$
  spin 0 ($\phi^3$) spin 2 (GR)
Where are gluons (spin-1)?

- Need to consider multiplet of particles

\[
A_3 = \left(\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}\right)^S f_{a_1 a_2 a_3}, \quad A_3 = \left(\frac{[12]^3}{[23][31]}\right)^S f_{a_1 a_2 a_3}
\]

- The same check gives us S=1 and requires

\[
f_{a_1 a_2 a_P} f_{a_3 a_4 a_P} + f_{a_1 a_4 a_P} f_{a_2 a_3 a_P} = f_{a_1 a_3 a_P} f_{a_2 a_4 a_P}
\]

and the result corresponds to SU(N) Yang-Mills theory.
Power of 4pt check

- We can apply this check for cases with mixed particle content:
  - Spin >2 still not allowed
  - Spin 2 is special: only one particle and it couples universally to all other particles
  - We get various other constraints on interactions (of course all consistent with known theories)

- General principles very powerful
Thank you for attention!