Scattering Amplitudes

LECTURE 3

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Review of Lectures 1-2
What does the blob represent?
Standard picture:
Feynman diagrams
Feynman diagrams

* Yang-Mills Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sim (\partial A)^2 + A^2 \partial A + A^4 \]

\[ \sim f^{abc} g_{\mu\nu} p_\alpha \]

\[ \sim f^{abe} f^{cde} g_{\mu\nu} g_{\alpha\beta} \]

* Draw diagrams

Feynman rules

Sum everything
Parke-Taylor formula

- Process $gg \rightarrow gggg$

- 220 Feynman diagrams, $\sim 100$ pages of calculations
Parke-Taylor formula

Our result has sucessfully passed both these numerical checks. Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

- Surprisingly simple expression for the final answer:

\[ M_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} \]
Amplitude: unique object

What is the scattering amplitude?

Feynman diagrams

Unique object fixed by physical properties

Was not successful (1960s)

Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory
Locality and tree-level unitarity

- Only poles: Feynman propagators

\[
\frac{1}{P^2} \quad \text{where} \quad P = \sum_{k \in \mathcal{P}} p_k
\]

- On the pole

Unitarity

\[
\mathcal{M} \xrightarrow{P^2=0} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R
\]

Feynman diagrams recombine on both sides into amplitudes
Loop unitarity

- Analogue of tree-level unitarity at one-loop

\[ \mathcal{M}^{1-\text{loop}} \xrightarrow{\ell^2 = (\ell + Q)^2 = 0} \mathcal{M}^{\text{tree}}_L \xrightarrow{\frac{1}{\ell^2 (\ell + Q)^2}} \mathcal{M}^{\text{tree}}_R \]

Unitarity cut

- In general \( \text{Cut} \leftrightarrow \ell^2 = 0 \)
New viewpoint

- Rigidity of the final answer after we provide an input
- Feynman diagrams: input = Lagrangian
- New methods: locality, unitarity and gauge invariance
- Amplitude is a unique gauge invariant function which factorizes properly on all factorization channels
Unitarity methods

(Bern, Dixon, Kosower)

- Expansion of the amplitude

\[ \mathcal{M}^{\ell\text{-loop}} = \sum_j a_j \int dI_j \]

Cuts give product of trees
Linear combinations of coefficients \( a_j \)

- Very successful method for loop amplitudes in different theories

- Practical problems:
  - Find basis of integrals
  - Solve (long) system of equations
One-loop unitarity

Higher cuts

**Triple cut**
\[ \ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = 0 \]

**Quadruple cut**
\[ \ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = (\ell + Q_3)^2 = 0 \]
BCFW recursion relations
(Britto, Cachazo, Feng, Witten, 2005)

\[ A_n = - \sum_j A_L(z_j) \frac{1}{P_j^2} A_R(z_j) \]

\[ z_j = \frac{P_j^2}{2 \langle 1 | P_j | 2 \rangle} \]

Chosen such that internal line is on-shell

Sum over all distributions of legs keeping 1,2 on different sides
New starting point

- Both methods are very efficient
- Based on conservative ideas of applying general principles to uniquely fix the answer
- Main goal of this effort (at least for me): completely new picture for Quantum Field Theory
- No locality, unitarity — we need new starting point
What is next?
Three point kinematics

Two options

\[ \lambda_1 \sim \lambda_2 \sim \lambda_3 \]

\[ \tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3 \]

Spinor helicity variables

\[ p^\mu = \sigma^\mu_{\dot{a}a} \lambda_a \tilde{\lambda}_{\dot{a}} \]

\[ \langle 12 \rangle = \epsilon_{ab} \lambda_{1a} \lambda_{2b} \]

\[ [12] = \epsilon_{\dot{a}\dot{b}} \lambda_{1\dot{a}} \lambda_{2\dot{b}} \]

Two solutions for 3pt kinematics

\[ p_1^2 = p_2^2 = p_3^2 = (p_1 + p_2 + p_3) = 0 \]
Three point amplitudes

Two solutions for amplitudes

\[ A_3 = [12] + h_1 + h_2 - h_3 [23] - h_1 + h_2 + h_3 [31] + h_1 - h_2 + h_3 \]

\[ h_1 + h_2 + h_3 \geq 0 \]

\[ A_3 = \langle 12 \rangle - h_1 - h_2 + h_3 \langle 23 \rangle + h_1 - h_2 - h_3 \langle 31 \rangle - h_1 + h_2 - h_3 \]

\[ h_1 + h_2 + h_3 \leq 0 \]

Supersymmetry: amplitudes of super-fields
(all component fields included)
Three point amplitudes

- In N=4 SYM: no need to specify helicities

\[ A_3^{(1)} = \frac{\delta^4(p_1 + p_2 + p_3)\delta^4([23][\tilde{\eta}_1] + [31][\tilde{\eta}_2] + [12][\tilde{\eta}_3])}{[12][23][31]} \]

\[ A_3^{(2)} = \frac{\delta^4(p_1 + p_2 + p_3)\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \]

Easy book-keeping

Fully fixed in any QFT up to coupling
On-shell diagrams

- Draw arbitrary graph with three point vertices

- Products of 3pt amplitudes: gauge invariant functions

- Well defined in any Quantum Field Theory
On-shell diagrams

- Draw arbitrary graph with three point vertices

Question: Can we build amplitude from on-shell diagrams?
Recursion relations

- Six point example

\[
\sum_{L,R} L_R n = 2 \begin{array}{c}
3 \quad 4 \\
2 \\
1 \\
\end{array} + 2 \begin{array}{c}
3 \quad 4 \\
2 \\
1 \\
\end{array} + 2 \begin{array}{c}
3 \quad 4 \\
2 \\
1 \\
\end{array}
\]

- Implementation of known method in this language
On-shell diagrams

- On-shell diagrams: natural gauge invariant objects
- Based on the complete rigidity of 3pt amplitudes
- Recursion relations in this language, hopefully in the future also at loops in more generality
On-shell diagrams

- Input: 3pt amplitude = fixed by Lorentz group and helicities of particles
- Still the same physics origin as Feynman diagrams, but implemented in much better language
- However, very surprisingly they are also starting point to a completely new story which brings us to the world of geometry
Hydrogen atom of gauge theories
Toy models

- Hard to make progress on difficult questions in full generality: time-proven method - choose toy model

- Long history of “integrable models”: exactly solvable

- **Kepler problem:**
  - orbits do not precess
  - Runge-Lenz vector

\[ \vec{A} = \frac{1}{2} \left( \vec{p} \times \vec{L} - \vec{L} \times \vec{p} \right) - \mu \frac{\lambda}{4\pi} \frac{x}{|x|} \]
Toy models

- **Hydrogen atom:** \[ H = \frac{1}{2m}p^2 - \frac{k}{r} \]
  - Hidden symmetry: Runge-Lenz-Pauli vector
  - Allows to find spectrum

- **Toy model for QFT: planar N=4 SYM theory**
  - (Brink, Schwarz, Scherk) (1984)
    - Theory of quarks and gluons, similar to QCD but no confinement
    - Hidden symmetry: Yangian - connection to 2d integrable models
      - (Drummond, Henn, Plefka, Korchemsky, Sokatchev) (2007)
    - Great theory to test new ideas in QFT
Hydrogen atom of gauge theories

- Useful playground for many theoretical ideas

- Integrability
  - Yangian
  - AdS/CFT
  - Strong coupling

- Symbols
  - Hexagon bootstrap
  - (Goncharov, Spradlin, Volovich,..)

- OPE expansion
  - (Drummond, Henn, Plefka)
  - (Korchemsky, Sokatchev)

- BDS ansatz
  - (Basso, Sever, Vieira)

- (Alday, Maldacena)

- (Beisert, Eden, Staudacher)

- (Bern, Dixon, Smirnov)

- (Dixon, von Hippel,..)
Amplitudes in N=4 SYM

- **N=4 superfield**

\[ \Phi = G_+ + \tilde{\eta}_A \Gamma_A + \frac{1}{2} \tilde{\eta}^A \tilde{\eta}^B S_{AB} + \frac{1}{6} \epsilon_{ABCD} \tilde{\eta}^A \tilde{\eta}^B \tilde{\eta}^C \Gamma^D + \frac{1}{24} \epsilon_{ABCD} \tilde{\eta}^A \tilde{\eta}^B \tilde{\eta}^C \tilde{\eta}^D G_- \]

- **Superamplitudes:**

\[ A_n = \sum_{k=2}^{n-2} A_{n,k} \]

Component amplitudes with power \( \tilde{\eta}^{4k} \)

- **Planarity:** limit \( N \to \infty \) - simplification
Dual variables

- Generally, each diagram has its own variables
  - No global loop momenta
  - Each diagram: its own labels

Planar limit: dual variables

\[
\begin{align*}
  k_1 &= (x_1 - x_2) \quad k_2 = (x_2 - x_3) \\
  \ell_1 &= (x_3 - y_1) \quad \ell_2 = (y_2 - x_3)
\end{align*}
\]

etc

Global variables
Using these variables: define a single function

\[ M = \int d^4 y_1 \ldots d^4 y_L \mathcal{I}(x_i, y_j) \]

Ideal object to study: rational function, no divergencies

Hidden dual conformal symmetry in these variables

(There is a hidden symmetry in QCD at tree-level)
Momentum twistors

(Hodges 2009)

* New variables: points in \( \mathbb{P}^3 \)

\[
Z = \begin{pmatrix}
\lambda_a \\
x_{a\dot{a}} \tilde{\lambda}_{\dot{a}}
\end{pmatrix}
\]

Dual Space–Time

Momentum Twistor Space

Cyclic ordering crucial

\[ p_j = x_{j+1} - x_j \]
Momentum twistors

- Dual conformal: SL(4) on momentum twistors

- Dual conformal invariants:
  \[ \langle 1234 \rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d \]

- Loop momenta: \( \ell \leftrightarrow Z_A Z_B \)

\[
\frac{d^4 \ell \, st}{\ell^2 (\ell + k_1)^2 (\ell + k_1 + k_2)^2 (\ell - k_4)^2} \]

\[
\frac{\langle AB d^2 A \rangle \langle AB d^2 B \rangle \langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB41 \rangle}
\]
Back to on-shell diagrams
Historic coincidence

- Same diagrams appeared in mathematics around 2005
- Very different motivation:

\[
\begin{pmatrix}
* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
* & * & \cdots & * \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * \\
\end{pmatrix} \geq 0
\]

- Goal: find algorithm for writing real matrices with positive minors (mod GL(k)): positive Grassmannian
Plabic graphs

- Draw a graph with two types of three point vertices
- Associate variables with the faces of diagram

![Graph Diagram]

with the property

$$\prod_{j} f_j = -1$$
Perfect orientation

- Arrows on all edges
- Not unique, always exists at least one
- Two (k) incoming, two (n-k) outgoing

Thus, the final relations involving the $e$'s is encoded by the matrix $C$:

$C = \begin{pmatrix}
  c_{13} & c_{14} \\
  c_{23} & c_{24}
\end{pmatrix}$.

Notice that only certain combinations of edge-weights appear in the equations. This happens for a very simple—and by now familiar—reason. Think of the GL(1)-redundancy of each vertex as a gauge-group, with the variable of a directed edge charged as a "bi-fundamental" of the GL(1)$\times$GL(1) of the vertices it connects.

Since the configuration $C$ must be invariant under these "gauge groups", only gauge-invariant combinations of the edge variables can appear. And just as we saw in the previous subsection, these combinations are those familiar from lattice gauge theory and can be viewed as encoding the flux through each closed loop in the graph—that is, each of its faces. Fixing the orientation of each face to be clockwise, the flux through it is given by the product of $\epsilon$ for a (anti-)aligned edge along its boundary. For future convenience, we define the face variables $f_i$ to be minus this product.

Applying this to the example above, we find:

$\begin{align*}
  f_1 &= \epsilon_1 \epsilon_5 \\
  f_2 &= \epsilon_1 \epsilon_6 \\
  f_3 &= \epsilon_3 \epsilon_4 \\
  f_4 &= \epsilon_3 \epsilon_8 \epsilon_1 \\
  f_0 &= \epsilon_5 \epsilon_6 \epsilon_7 \epsilon_8
\end{align*}$

The boundary-measurements $c_{\alpha a}$ can then be expressed in terms of the faces by

$c_{\alpha a} = \sum f_{b \in \text{clockwise closure of } \alpha a} (f)$,

where $b$ is the 'clockwise' closure of $\alpha$. (If there are any closed, directed loops, the geometric series of faces enclosed should be summed.) The faces of course over-count the degrees of freedom by one, and this is reflected by the fact that $Q_i(f_i) = 1$.
Entries of matrix

- Define elements of \((k \times n)\) matrix

\[
c_{ab} = - \sum_{\Gamma} \prod_{j} (-f_j)
\]

Incoming

If \(b\) incoming
- \(c_{aa} = 1\)
- \(c_{ab} = 0\)

Example: \(c_{11} = c_{22} = 1\), \(c_{12} = c_{21} = 0\)
- \(c_{13} = *, c_{14} = *, c_{23} = *, c_{24} = *\)

Perfect orientation
- White vertex: one in, two out
- Black vertex: two in, one out
- Sum over all allowed paths

Product of all face variables to the right of the path
Entries of matrix

Apply on our example

\[ c_{ab} = - \sum_{\Gamma} \prod_{j} (-f_j) \]

\[ -c_{13} = -f_0 f_3 f_4 \]

\[ -c_{14} = f_0 f_4 \]

\[ -c_{23} = f_0 f_1 f_3 f_4 \]

\[ -c_{24} = f_0 f_1 f_4 \]
Entries of matrix

* The matrix is

\[
C = \begin{pmatrix}
1 & 0 & f_0 f_3 f_4 & f_4(1 - f_0) \\
0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4
\end{pmatrix}
\]

* There always exists choice of signs for \( f_i \) such that \( C \in G_+(k, n) \)

* For our case:

\[
\begin{align*}
m_{12} &= 1 \\
m_{13} &= -f_0 f_1 f_3 f_4 \\
m_{14} &= -f_0 f_1 f_4 \\
m_{23} &= -f_0 f_3 f_4 \\
m_{24} &= -f_4(1 - f_0) \\
m_{34} &= f_0 f_1 f_3 f_4^2
\end{align*}
\]

\[
\begin{align*}
f_2 &\text{ eliminated} \\
f_0 &< 0 \\
f_1 &< 0 \\
f_3 &> 0 \\
f_4 &< 0 \\
\text{All minors positive}
\end{align*}
\]
Positive Grassmannian from on-shell diagram

- On-shell diagram: method how to generate $C \in G_+(k, n)$

All such matrices generated using on-shell diagrams
It is very interesting that the same objects appear in physics and mathematics.

But is it useful for something?
Physics from Grassmannian
Connection

\[ R = M_1^{\text{tree}} M_2^{\text{tree}} M_3^{\text{tree}} M_4^{\text{tree}} \]

\[ C = \begin{pmatrix}
  1 & 0 & f_0 f_3 f_4 & f_4 (1 - f_0) \\
  0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 
\end{pmatrix} \]
Connection

\[ R = M_1^{\text{tree}} M_2^{\text{tree}} M_3^{\text{tree}} M_4^{\text{tree}} \]

\[ C = \begin{pmatrix}
1 & 0 & f_0 f_3 f_4 & f_4 (1 - f_0) \\
0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4
\end{pmatrix} \]

\[ R = \frac{df_0}{f_0} \frac{df_1}{f_1} \frac{df_2}{f_2} \frac{df_3}{f_3} \delta(C \cdot Z) \]
Momentum conservation

\[ \delta(C \cdot Z) = \delta(C \cdot \tilde{\lambda}) \delta(C^\perp \cdot \lambda). \]

- Simple motivation: linearize momentum conservation

\[ \delta(P) = \delta \left( \sum_a \lambda_a \tilde{\lambda}_a \right) \]

- We want to write it as two linear factors

\[ \delta \left( C_{ab} \tilde{\lambda}_b \right) \delta \left( D_{ab} \lambda_b \right) \]

and get the condition: \( D_{ab} = C_{ab}^\perp \)
Dual picture for on-shell diagrams

For arbitrary on-shell diagram

- Label face variables
- Find perfect orientation
- Construct the Grassmannian matrix
- Write a logarithmic form

\[
R = \frac{df_0}{f_0} \frac{df_1}{f_1} \frac{df_2}{f_2} \ldots \frac{df_d}{f_d} \delta(C \cdot Z) \leftrightarrow M^\text{tree}_1 M^\text{tree}_2 \ldots M^\text{tree}_m
\]
Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

Planar N=4 SYM:

\[ \Omega = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \ldots \frac{d\alpha_n}{\alpha_n} \delta(C \cdot Z) \]
Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

General QFT: $\Omega = F(\alpha) \, \delta(C \cdot Z)$

- In a sense $F(\alpha)$ defines a theory (as Lagrangian does)
At least for planar N=4 SYM we established
Hopefully for other theories in following years....
Even for planar N=4 SYM not completely satisfactory: sum of objects given by unitarity

Search for a single object
Prelude
Volume of polyhedron

(Hodges 2009)

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Fig. 2: Configurations contributing to the six-gluon amplitude $A_6(1^-2^-3^-4^+5^+6^+)$. Note that (a) and (c) are related by a flip of indices composed with a conjugation. (b) vanishes for either helicity configuration of the internal line.

This is shown in fig. 2. Note that for this helicity configuration, the middle graph vanishes. Therefore, we are left with only two graphs to evaluate. Moreover, the two graphs are related by a flip of indices composed with a conjugation. Therefore, only one computation is needed.

Let us compute in detail the contribution coming from the first graph in fig. 2(a). The contribution of this term is given by the product of two MHV amplitudes times a propagator,

This formula can be simplified by noting that

This formula can be simplified by noting that

Using (2.7) it is straightforward to find (2.6)
Volume of polyhedron

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

![Diagram](attachment:image.png)

\[
\begin{align*}
\langle 1345 \rangle^3 & \quad \langle 1234 \rangle \langle 1245 \rangle \langle 2345 \rangle \langle 1235 \rangle \\
\langle 1356 \rangle^3 & \quad \langle 1235 \rangle \langle 1256 \rangle \langle 2356 \rangle \langle 1236 \rangle
\end{align*}
\]
Volume of polyhedron

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Volume of tetrahedron in momentum twistor space!

$\langle 1345 \rangle^3 \quad \langle 1234 \rangle \langle 1245 \rangle \langle 2345 \rangle \langle 1235 \rangle \quad \langle 1356 \rangle^3 \quad \langle 1235 \rangle \langle 1256 \rangle \langle 2356 \rangle \langle 1236 \rangle$

Each face labeled by $\langle abcd \rangle$

(Hodges 2009)
Volume of polyhedron

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Volume of tetrahedron in momentum twistor space!

Each face labeled by $\langle abcd \rangle$

$$\langle 1345 \rangle^3$$

$$\langle 1234 \rangle \langle 1245 \rangle \langle 2345 \rangle \langle 1235 \rangle$$

$$\langle 1356 \rangle^3$$

$$\langle 1235 \rangle \langle 1256 \rangle \langle 2356 \rangle \langle 1236 \rangle$$
Volume of polyhedron

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Amplitude is a volume of polyhedron

$\langle 1345 \rangle^3$

$\langle 1234 \rangle \langle 1245 \rangle \langle 2345 \rangle \langle 1235 \rangle$

$\langle 1356 \rangle^3$

$\langle 1235 \rangle \langle 1256 \rangle \langle 2356 \rangle \langle 1236 \rangle$
“Conjecture”

Amplitudes are volumes of some regions in some space
“Conjecture”

Amplitudes are volumes of some regions in some space

Must be related to positive Grassmannian
Strategy

- Simple intuitive geometric ideas
- Use suitable mathematical language to describe them
- Generalize to more complicated (non-intuitive) cases
Inside of the triangle
Let us consider three points in a projective plane.

\[ Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \sim tZ_j \]

We can also fix:

\[ Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix} \]
Inside of the triangle

Point inside the triangle

\[ Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \]

\[ c_1, c_2, c_3 > 0 \]

Projective: one of $c_j$ can be fixed to 1
Inside of the triangle

- Point inside the triangle

\[ Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \]

On the boundary
\[ c_3 = 0 \]
Inside of the triangle

- Point inside the triangle

\[ Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \]

On the boundary
\[ c_1 = 0 \]
Inside of the triangle

- Point inside the triangle

\[ Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \]

On the boundary
\[ c_2 = 0 \]
Inside of the triangle

- Point inside the triangle

\[ Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \]

On the boundary

\[ c_2 = c_3 = 0 \]
Logarithmic form

- Point inside the triangle

- Form with logarithmic singularities on boundaries

\[ Y = Z_1 + c_2 Z_2 + c_3 Z_3 \]

\[ \Omega = \frac{dc_2}{c_2} \frac{dc_3}{c_3} \]
Logarithmic form

- Point inside the triangle

- Form with logarithmic singularities on boundaries

\[
\Omega = \frac{dc_2}{c_2} \cdot \frac{dc_3}{c_3} \rightarrow \frac{dc_2}{c_2}
\]

\[
c_3 = 0
\]

\[
Y = Z_1 + c_2 Z_2 + c_3 Z_3
\]
Logarithmic form

- Point inside the triangle

\[ Y = Z_1 + c_2 Z_2 + c_3 Z_3 \]

- Form with logarithmic singularities on boundaries

\[ \Omega = \frac{dc_2}{c_2} \frac{dc_3}{c_3} \rightarrow \frac{dc_3}{c_3} \]
Logarithmic form

- Point inside the triangle

- Form with logarithmic singularities on boundaries

\[ \Omega = \frac{dc_2}{c_2} \frac{dc_3}{c_3} \rightarrow \frac{dc_3}{c_3} \rightarrow 1 \]

- Other boundaries can correspond to \( c_2, c_3 \rightarrow \infty \)

\[ Y = Z_1 + c_2 Z_2 + c_3 Z_3 \]

\[ c_2 = c_3 = 0 \]
Logarithmic form

• Form with logarithmic singularities on boundaries

\[ \Omega = \frac{dc_2 \ dc_3}{c_2 \ c_3} \]

\[ \langle X_1 X_2 X_3 \rangle = \epsilon_{abc} X_1^a X_2^b X_3^c \]

\[ d^2Y = dc_2 \ dc_3 Z_2 Z_3 \]

• Solve for \( c_2, c_3 \) from \( Y = Z_1 + c_2 Z_2 + c_3 Z_3 \)

\[ \Omega = \frac{\langle Y \ d^2Y \rangle \langle 123 \rangle^2}{\langle Y 12 \rangle \langle Y 23 \rangle \langle Y 31 \rangle} \]

Projective in all variables
Polygon
Point inside the polygon

- Consider a point inside a polygon in projective plane

\[ Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n \]
\[ c_j > 0 \]
interior of the polygon

- Convex polygon: condition on points \( Z_i \)

\[
Z = \begin{pmatrix}
Z_1 & Z_2 & Z_3 & \ldots & Z_n \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{pmatrix}
\]

All main minors positive

\[
\begin{vmatrix}
* & * & * \\
* & * & * \\
* & * & * \\
\end{vmatrix} > 0
\]
Logarithmic form

Easiest way how to write the form is to triangulate

\[ Y = Z_1 + c_2 Z_2 + c_3 Z_3 \]
\[ \Omega_1 = \frac{dc_2 \ dc_3}{c_2 \ c_3} \quad c_2, c_3 \geq 0 \]

\[ Y = Z_1 + c_3 Z_3 + c_4 Z_4 \]
\[ \Omega_2 = \frac{dc_3 \ dc_4}{c_3 \ c_4} \quad c_3, c_4 \geq 0 \]

\[ Y = Z_1 + c_4 Z_4 + c_5 Z_5 \]
\[ \Omega_3 = \frac{dc_4 \ dc_5}{c_4 \ c_5} \quad c_4, c_5 \geq 0 \]

\[ Y = Z_1 + c_5 Z_5 + c_6 Z_6 \]
\[ \Omega_4 = \frac{dc_5 \ dc_6}{c_5 \ c_6} \quad c_5, c_6 \geq 0 \]

How to sum them?
Logarithmic form

- Easiest way how to write the form is to triangulate

\[
Y = Z_1 + c_2 Z_2 + c_3 Z_3
\]
\[
\Omega_1 = \frac{\langle Y d^2 Y \rangle \langle 123 \rangle^2}{\langle Y 12 \rangle \langle Y 23 \rangle \langle Y 31 \rangle}
\]

\[
Y = Z_1 + c_3 Z_3 + c_4 Z_4
\]
\[
\Omega_2 = \frac{\langle Y d^2 Y \rangle \langle 134 \rangle^2}{\langle Y 13 \rangle \langle Y 34 \rangle \langle Y 41 \rangle}
\]

\[
Y = Z_1 + c_4 Z_4 + c_5 Z_5
\]
\[
\Omega_3 = \frac{\langle Y d^2 Y \rangle \langle 145 \rangle^2}{\langle Y 14 \rangle \langle Y 45 \rangle \langle Y 51 \rangle}
\]

\[
Y = Z_1 + c_5 Z_5 + c_6 Z_6
\]
\[
\Omega_4 = \frac{\langle Y d^2 Y \rangle \langle 156 \rangle^2}{\langle Y 15 \rangle \langle Y 56 \rangle \langle Y 61 \rangle}
\]

Write in projective form
Logarithmic form

- Now it makes sense to sum them

\[
\Omega = \frac{\langle Y d^2 Y \rangle \langle 123 \rangle^2}{\langle Y_{12} \rangle \langle Y_{23} \rangle \langle Y_{31} \rangle} + \frac{\langle Y d^2 Y \rangle \langle 134 \rangle^2}{\langle Y_{13} \rangle \langle Y_{34} \rangle \langle Y_{41} \rangle} + \frac{\langle Y d^2 Y \rangle \langle 145 \rangle^2}{\langle Y_{14} \rangle \langle Y_{45} \rangle \langle Y_{51} \rangle} + \frac{\langle Y d^2 Y \rangle \langle 156 \rangle^2}{\langle Y_{15} \rangle \langle Y_{56} \rangle \langle Y_{61} \rangle}
\]

- Boundaries of the polygon are \( \langle Y_{ii + 1} \rangle = 0 \)

Spurious poles
Cancel in the sum
Logarithmic form

Now it makes sense to sum them

\[ \Omega = \frac{\langle Y d^2 Y \rangle \langle 123 \rangle^2}{\langle Y_{12} \rangle \langle Y_{23} \rangle \langle Y_{31} \rangle} + \frac{\langle Y d^2 Y \rangle \langle 134 \rangle^2}{\langle Y_{13} \rangle \langle Y_{34} \rangle \langle Y_{41} \rangle} + \frac{\langle Y d^2 Y \rangle \langle 145 \rangle^2}{\langle Y_{14} \rangle \langle Y_{45} \rangle \langle Y_{51} \rangle} + \frac{\langle Y d^2 Y \rangle \langle 156 \rangle^2}{\langle Y_{15} \rangle \langle Y_{56} \rangle \langle Y_{61} \rangle} \]

Boundaries of the polygon are \( \langle Y i i + 1 \rangle = 0 \)
From Y to supersymmetry

- Let us take the form for the triangle

\[ \Omega = \frac{\langle Y \, d^2 Y \rangle \langle 123 \rangle^2}{\langle Y 12 \rangle \langle Y 23 \rangle \langle Y 31 \rangle} \]

- Rewrite external Z:

\[ Z_j = \begin{pmatrix} z_j^{(1)} \\ z_j^{(2)} \\ (\phi \cdot \eta_j) \end{pmatrix} \quad \text{for} \quad z_j \in \mathbb{P}^2 \quad \text{bosonic} \]

\[ \eta_j^A \quad \text{fermionic} \quad A = 1, 2 \]

\[ \phi^A \quad \text{auxiliary} \]

- Also define

\[ Y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]
From Y to supersymmetry

We plug them into the form for triangle:

\[
\frac{\langle Y \ d^2 Y \rangle \langle 123 \rangle^2}{\langle Y_{12} \rangle \langle Y_{23} \rangle \langle Y_{31} \rangle} \rightarrow \frac{\langle 12 \rangle (\phi \cdot \eta_3) + \langle 23 \rangle (\phi \cdot \eta_1) + \langle 31 \rangle (\phi \cdot \eta_2))^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}
\]

Final step: integrate over \( \phi \):

\[
\int d^2 \phi \int \Omega \ \delta(Y - Y_0) = \frac{\langle 12 \rangle \eta_3 + \langle 23 \rangle \eta_1 + \langle 31 \rangle \eta_2)^2}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}
\]
Definition

space specified by a set of inequalities \( \Omega \) logarithmic singularities \( \mathcal{M}^{\ell}\text{-loop}_{n,k} \)

Full definition of Amplituhedron

- Definitions of objects:
  \[ \mathcal{Y} = \mathcal{C} \cdot Z \]
  \[ \mathcal{Y} = \begin{pmatrix} Y^{(1)} \\ \vdots \\ Y^{(\ell)} \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C^{(1)} \\ \vdots \\ C^{(\ell)} \end{pmatrix}, \quad Z = \begin{pmatrix} Z^{(1)} \\ \vdots \\ Z^{(\ell)} \end{pmatrix} \]

- Positivity conditions:
  \[ Z \in \mathcal{M}_{+}(k + 4, n), \quad C \in \mathcal{G}_{+}(k, n) \]
  \[ C^{(1)} \in \mathcal{G}_{+}(k + 2m, n), \quad Z^{(\ell)} = \mathcal{G}(2, n) \]

- \( \Omega_{n,k,\ell} \): form with logarithmic singularities on boundaries of \( \mathcal{Y} \)

- The amplitude is:
  \[ \mathcal{M}_{n,k,\ell} = \int d^4 \phi_1 \, d^4 \phi_2 \ldots d^4 \phi_k \, \Omega_{n,k,\ell} \bigg|_{\mathcal{Y} = (1, 0, \ldots, 0)} \]
Triangulation

Triangulate in terms of “simplices”

\[ \Omega_0 \sim \frac{dx}{x} \quad \text{for each} \]

Space specified by a set of inequalities

Set of regions:
- cover the whole space
- each region specified by \( f_j \in (0, \infty) \)

Sum them

\[ \Omega \rightarrow \mathcal{M}^{\ell-\text{loop}}_{n,k} \]

Logarithmic singularities
Triangulation

space specified by a set of inequalities

\[ \Omega \]

logarithmic singularities

\[ \mathcal{M}^{\ell-\text{loop}}_{n,k} \]

sum them

\[ \Omega_0 \sim \frac{dx}{x} \text{ for each} \]
High school problem  \[ gg \rightarrow gg \]

* Positive quadrant
High school problem $gg ightarrow gg$

- Positive quadrant
- Vectors

$\vec{a}_1 = \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right)$  \hspace{1cm}  $\vec{b}_1 = \left( \begin{array}{c} z_1 \\ w_1 \end{array} \right)$

$Vol(1) = \frac{dx_1}{x_1} \frac{dy_1}{y_1} \frac{dz_1}{z_1} \frac{dw_1}{w_1}$
High school problem \( gg \rightarrow gg \)

- Positive quadrant
- Vectors

\[
\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \\
\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}
\]

\[
[\text{Vol (1)}]^2 = \frac{dx_1}{x_1} \frac{dy_1}{y_1} \frac{dz_1}{z_1} \frac{dw_1}{w_1} \frac{dx_2}{x_2} \frac{dy_2}{y_2} \frac{dz_2}{z_2} \frac{dw_2}{w_2}
\]
High school problem $gg \rightarrow gg$

- Positive quadrant

- Vectors

\[
\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \\
\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}
\]

- Impose:

\[
(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1) \leq 0 \quad \phi > 90^\circ
\]

Subset of configurations allowed: triangulate
High school problem \( gg \rightarrow gg \)

- Positive quadrant
- Vectors

\[
\begin{align*}
\vec{a}_1 &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} & \vec{b}_1 &= \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \\
\vec{a}_2 &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} & \vec{b}_2 &= \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}
\end{align*}
\]

\[
\text{Vol (2)} = \frac{dx_1}{x_1} \frac{dy_1}{y_1} \frac{dz_1}{z_1} \frac{dw_1}{w_1} \frac{dx_2}{x_2} \frac{dy_2}{y_2} \frac{dz_2}{z_2} \frac{dw_2}{w_2} \left[ \frac{\vec{a}_1 \cdot \vec{b}_2 + \vec{a}_2 \cdot \vec{b}_1}{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1)} \right]
\]
High school problem  \( gg \rightarrow gg \)

- **Positive quadrant**

- **Vectors**
  \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell \quad \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_\ell \)

- **Conditions**
  \[
  (\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) \leq 0
  \]
  for all pairs \( i, j \)

Let me know if you solve it!
Why true?

- No QFT proof because it is not QFT but geometry
- It is correct: the result satisfies locality and unitarity
- Totally different approach: same answer
- Many open questions: triangulations, mathematical structure.....
Physics vs geometry

- Dynamical particle interactions in 4-dimensions

- Static geometry in high dimensional space
What is scattering amplitude?
What is scattering amplitude?
Step 1.1.1.

- It is very early to say if/how this can generalize
- Some encouraging news but more work needed
- New formulation of QFT?
  - Integrals
  - Masses
  - RG flow
  - Correlation functions
  - Beyond perturbation theory

Establish as an efficient computational tool
Beyond understanding QFT better there is one more motivation
Beyond understanding QFT better there is one more motivation
We have a theory of quantum gravity: string theory
New geometric picture for string theory?
Amplitudes as a new field

- This is one of the directions in fast developing field
- More: scattering equations, BCJ duality, string amplitudes, supergravity finiteness, hexagon bootstrap, cluster polylogarithms, worldsheet models, integration techniques, LHC calculations,.....
- Zeroth order problems open, many chances for young people to make big discoveries!
Resources

Books and reviews

https://arxiv.org/abs/1308.1697
https://arxiv.org/abs/1610.05318

Conferences

Summer school in July in Edinburgh, you can still apply!

https://higgs.ph.ed.ac.uk/workshops/amplitudes-2017-summer-school
Thank you for your attention