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# Introduction to Ultracold Atoms

## Superfluid – Mott insulator transition

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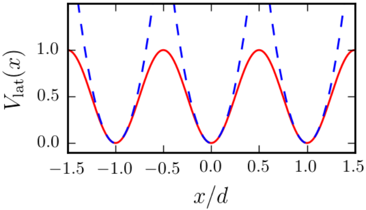
September 4, 2017

In a deep lattice potential, atoms are tightly trapped around the potential minima.

Harmonic approximation for each well :

$$V_{\rm lat}(x \approx x_i) \approx \frac{1}{2} m_a \omega_{\rm lat}^2 (x - x_i)^2,$$
$$\hbar\omega_{\rm lat} = 2\sqrt{V_0 E_R}.$$




The bands are centered around the energy $\overline{E}_n \approx (n + 1/2)\hbar\omega_{\rm lat}$.

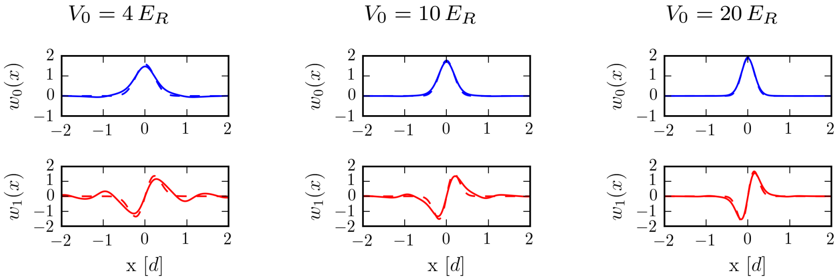
First correction : quantum tunneling across the potential barriers, as in tight-binding methods used in solid-state physics (Linear Combination of Atomic Orbitals)

This is best handled using a new basis, formed by so-called *Wannier functions*.

Wannier functions : discrete Fourier transforms with respect to the site locations of the Bloch wave functions,

$$w_n(x - x_i) = \frac{1}{\sqrt{N_s}} \sum_{q \in BZ1} e^{-iqx_i} \phi_{n,q}(x).$$

- All Wannier functions can be deduced from $w_n(x)$ by translation of $x_i = id$.
- There are exactly $N_s$ such functions per band (as many as Bloch functions).
- Wannier functions form a basis of Hilbert space (*not* an eigenbasis of $\hat{H}$).




Cautionary note: Bloch functions are defined up to a $q-$dependent phase which needs to be fixed to obtain localized Wannier functions [W. Kohn, Phys. Rev. (1959)].

Second quantized formalism (useful for the interacting case).

Bloch basis :

$$H = \sum_{n,k \in BZ1} \varepsilon_n(k) \hat{b}^\dagger_{n,k} \hat{b}_{n,k}.$$

$\hat{b}_{n,k}$ : annihilation operator for Bloch state $(n, k)$.

Wannier basis :

$$H = -\sum_{n,i,j} J_n(i-j) \hat{a}^\dagger_{n,i} \hat{a}_{n,j},$$

$\hat{a}_{n,i}$ : annihilation operator for Wannier state $w_n(x - x_i)$.

Tunneling energies :

$$J_n(i-j) = \int dx \; w_n^*(x - x_j) \left( \frac{\hbar^2}{2M} \Delta - V_{\text{lat}}(x) \right) w_n(x - x_i).$$

(also called hopping parameters)

$$H = -\sum_{n,i,j} J_n(i-j)\hat{a}^\dagger_{n,i}\hat{a}_{n,j},$$

Tunneling energies :

$$J_n(i-j) = \frac{1}{N_s}\sum_{q,q'\in BZ1} e^{-i(qx_i - q'x_j)} \underbrace{\int dx\; u^*_{n,q}(x)\left(\frac{\hbar^2}{2M}\Delta - V_{\text{lat}}(x)\right)u_{n,q'}(x)}_{=-\varepsilon_n(q)\delta_{n,n'}\delta_{q,q'}},$$

$$= -\frac{1}{N_s}\sum_{q\in BZ1}\varepsilon_n(q)e^{-iq\cdot(x_i-x_j)}.$$

$J_n(i-j)$ depend only on the relative distance $x_i - x_j$ between the two sites.

- On-site energy $(i=j)$: Mean energy of band $n$

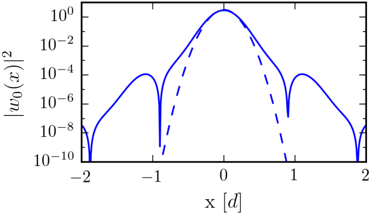
$$J_n(0) = -\frac{1}{N_s}\sum_{q\in BZ1}\varepsilon_n(q) = -\overline{E}_n$$

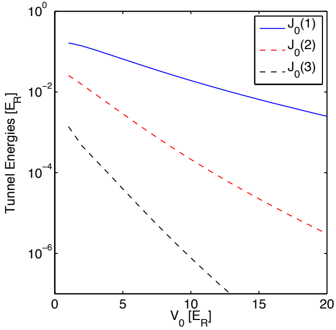
- Nearest-neighbor tunneling $(j = i \pm 1)$:

$$J_n(1) = -\frac{1}{N_s}\sum_{q\in BZ1}\varepsilon_n(q)e^{iqx} = -J_n$$

For deep lattices (roughly $V_0 \gg 5E_R$), the tunneling energies fall off very quickly with distance:

Wannier function for $V_0 = 10\,E_R$:




Two useful approximations :

- Tight-binding approximation : keep only the lowest terms
- Single-band approximation : keep only the lowest band–drop band index and let $J_0(1) \equiv J$

$$\hat{H}_{TB} = \overline{E}_0 \sum_i \hat{a}_i^\dagger \hat{a}_i - J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j,$$

Cubic lattice potential :

$$V_{\mathrm{lat}} = V_0 \sum_{\alpha} \sin(k_\alpha x_\alpha)^2$$

Dispersion relation :

$$\epsilon_{\mathbf{n}}(\mathbf{q}) = \sum_{\alpha=x,y,z} \epsilon_{n_\alpha}(q_\alpha),$$

- $\epsilon_n(q)$ : 1d dispersion relation,
- $\mathbf{n}$ : a triplet of integers indexing the various bands
- $\mathbf{q}$ : quasi momentum $\in$ BZ1 : $]-\pi/d, \pi/d]^3$.

Bloch functions :

$$\phi_{\mathbf{n},\mathbf{q}}(\mathbf{r}) = e^{i\mathbf{q}\cdot\mathbf{r}} u_{n_x,q_x}(x) u_{n_y,q_y}(y) u_{n_z,q_z}(z).$$

Wannier functions :

$$W_{\boldsymbol{n}}(\mathbf{r}-\mathbf{r}_{\boldsymbol{n}}) = w_{n_x}(x-n_x d_x) w_{n_y}(y-n_y d_y) w_{n_z}(z-n_z d_z).$$

Basic Hamiltonian for bosons interacting *via* short-range forces :

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int},$$

$$\hat{H}_0 = \int d\mathbf{r} \;\; \hat{\Psi}^\dagger(\mathbf{r})\Big[ -\frac{\hbar^2}{2M}\Delta + V_{\rm lat}(\mathbf{r})\Big]\hat{\Psi}(\mathbf{r}),$$

$$\hat{H}_{\rm int} = \frac{g}{2}\int d^{(3)}\mathbf{r} \;\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}).$$

- $\hat{\Psi}(\mathbf{r})$ : field operator annihilating a boson a position $\mathbf{r}$,
- $V_{\rm lat}(\mathbf{r})$ : lattice potential,
- $g = \frac{4\pi\hbar^2 a}{M}$ : coupling constant,
- scattering length $a>0$ : repulsive interactions.

- Not simpler in the Bloch basis.
- Can be drastically simplified in the Wannier basis

Basis of Wannier functions $W_\nu(\boldsymbol{r}-\boldsymbol{r}_i)$:

$$\hat{\Psi}(\mathbf{r}) = \sum_{\nu,i} W_\nu(\boldsymbol{r}-\boldsymbol{r}_i)\hat{a}_{\nu,i}.$$

- $\boldsymbol{r}_i$ : position of site $i$,
- $\nu$ : band index
- $\hat{a}_{\nu,i}$ : annihilation operator

Single-particle Hamiltonian :

- Tight-binding approximation : keep only the lowest terms
- Single-band approximation : keep only the lowest band–drop band index, $J_0(1) \equiv J$

$$\hat{H}_0 \longrightarrow \hat{H}_{TB} = -\sum_{\langle i,j\rangle} J\hat{a}_i^\dagger \hat{a}_j$$

Basis of Wannier functions $W_\nu(\boldsymbol{r}-\boldsymbol{r}_i)$:

$$\hat{\Psi}(\mathbf{r}) = \sum_{\nu,i} W_\nu(\boldsymbol{r}-\boldsymbol{r}_i)\hat{a}_{\nu,i}.$$

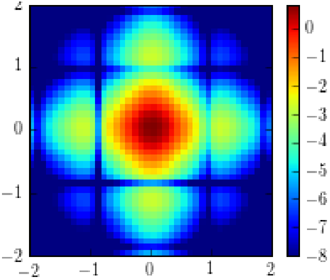
- $\boldsymbol{r}_i$ : position of site $i$,
- $\nu$ : band index
- $\hat{a}_{\nu,i}$ : annihilation operator

Interaction Hamiltonian :

$$\hat{H}_{\rm int} \longrightarrow \hat{H}_{\rm int} = \frac{1}{2}\sum_{ijkl} U_{ijkl}\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l$$

$$U_{ijkl} = g\int d\mathbf{r}\; W^*(\boldsymbol{r}-\boldsymbol{r}_i)W^*(\boldsymbol{r}-\boldsymbol{r}_j)W(\boldsymbol{r}-\boldsymbol{r}_k)W(\boldsymbol{r}-\boldsymbol{r}_l)$$

$\log(|W_(x,y,0)|^2)$ for $V_0 = 5E_R$:




In the tight binding regime, strong localization of Wannier function $W(\boldsymbol{r}-\boldsymbol{r}_i)$ around $\boldsymbol{r}_i$.

On-site interactions ($i=j=k=l$) are strongly dominant:

$$\hat{H}_{\rm int} \approx \frac{1}{2}\sum_i U_{iiii}\hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i + \cdots$$

1. Single band approximation
2. Tight-binding approximation
3. On-site interactions

Bose-Hubbard model :

$$H_{\mathrm{BH}} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i \left( \hat{n}_i - 1 \right).$$

$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ : operator counting the number of particles at site $i$.

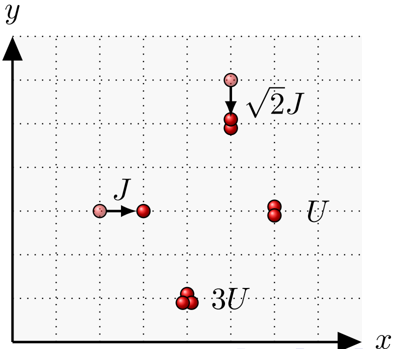
- Tunneling energy :

$$J = \frac{\max \varepsilon(\boldsymbol{q}) - \min \varepsilon(\boldsymbol{q})}{2z}$$

$z = 6$ : number of nearest neighbors

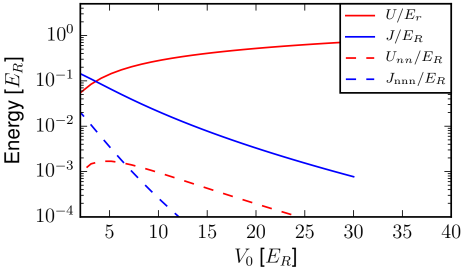
- On-site interaction energy :

$$U = g \int d\mathbf{r} \; w(\mathbf{r})^4.$$



# Parameters of the Bose Hubbard model

Calculation for $^{87}$Rb atoms [a=5.5 nm] in a lattice at $\lambda_L = 820$ nm:




Harmonic oscillator approximation :

$$\frac{\Delta_{\rm band}}{E_R} \approx \frac{\hbar\omega_{\rm lat}}{E_R} = \sqrt{\frac{2V_0}{E_R}}, \quad \frac{U}{E_R} \approx \sqrt{\frac{8}{\pi}} k_L a \left(\frac{V_0}{E_R}\right)^{3/4}.$$

1. Single band approximation :
   - $V_0 \gg E_R$
   - $U \ll \Delta_{\rm band}$ : $k_L a \ll \left(\frac{E_R}{V_0}\right)^{1/4}$
2. Tight-binding approximation : $V_0 \gg 5E_R$
3. On-site interactions : $V_0 \gg E_R$

BEC in the lowest energy Bloch state $\boldsymbol{q} = 0$ :

$$|\Psi\rangle_N = \frac{1}{\sqrt{N!}}\left(\hat{b}^\dagger_{\mathbf{q}=0}\right)^N|\varnothing\rangle = \frac{1}{\sqrt{N!}}\left(\frac{1}{\sqrt{N_s}}\sum_i \hat{a}_i^\dagger\right)^N|\varnothing\rangle$$

- Fixed number of particles $N$: canonical ensemble

Probability to find $n_i$ atoms at one given site $i$ :

$$p(n_i) \approx e^{-\overline{n}}\frac{\overline{n}^{n_i}}{n_i!} + \mathcal{O}\left(\frac{1}{N}, \frac{1}{N_s}\right)$$

Poisson statistics, mean $\overline{n}$, standard deviation $\sim \sqrt{\overline{n}}$

In the thermodynamic limit $N \to \infty, N_s \to \infty$, one finds the same result as for a coherent state with the same average number of particles $N$:

$$|\Psi\rangle_{\mathrm{coh}} = \mathcal{N}e^{\sqrt{N}\hat{b}^\dagger_{\mathbf{q}=0}}|\varnothing\rangle = \prod_i\left(\mathcal{N}_i e^{\sqrt{\overline{n}}\hat{a}_i^\dagger}|\varnothing\rangle\right)$$

- Fluctuating number of particles $N$: grand canonical ensemble
  $H_{\mathrm{BH}} \to G = H_{\mathrm{BH}} - \mu N$

Coherent state wavefunction in the grand canonical ensemble :

$$|\Psi\rangle_{\mathrm{coh}} = \prod_i |\alpha_i\rangle, \qquad |\alpha_i\rangle = \mathcal{N}_i \sum_{n_i=0}^{\infty} \frac{\alpha_i^{n_i}}{\sqrt{n_i!}} |n_i\rangle_i, \qquad \hat{a}_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$$

with $\{\alpha_i\}_{i=1,\cdots,N_s}$ the variational parameters.
One can relate the presence of the condensate to a non-zero expectation value of the matter wave field $\alpha_i = \langle \hat{a}_i \rangle$, playing the role of an order parameter :

- Condensate wavefunction : $\alpha_i = \langle \hat{a}_i \rangle = \sqrt{\frac{N}{N_s}} e^{i\phi}$
- Mean density : $\overline{n} = |\alpha_i|^2 =$ condensate density

Spontaneous symmetry breaking point of view.

Starting point to formulate a Gross-Pitaevskii (weakly interacting) theory :

variational ansatz with self-consistent $\alpha_i$ determined by the total (single-particle + interaction)Hamiltonian.

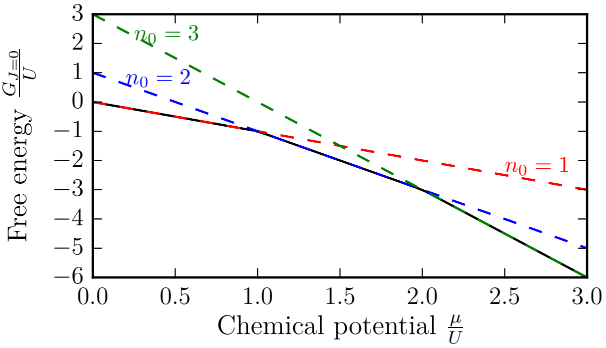
"Adiabatic continuation" from the ideal Bose gas.

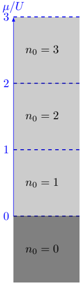
# "Atomic" limit $J = 0$

Lattice $\equiv$ many independent trapping wells

Many-body wavefunction : product state running over all lattice sites $|\Psi\rangle = \prod_i |n_0\rangle_i$
Free energy for one well: $G_{J=0} = \frac{U}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i$

$\langle n_i \rangle = n_0 = \mathrm{int}(\mu/U) + 1$: integer filling that minimizes $\mathcal{H}_{\mathrm{int}}$.




- $\mu/U = p$ integer : $n_0 = p$ and $n_0 = p + 1$ degenerate
  - on-site wave function = any superposition of the two Fock states.
  - density $\langle n_i \rangle \in [p, p+1]$.

# Gutzwiller ansatz for the ground state

Variational wavefunction :

$$|\Psi\rangle_{\text{Gutzwiller}} = \prod_i |\phi_i\rangle,$$

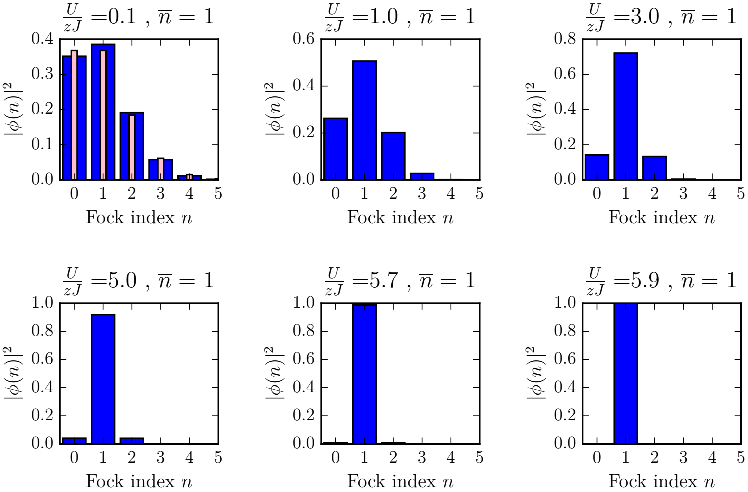
$$|\phi_i\rangle = \sum_{n_i=0}^{\infty} c(n_i)|n_i\rangle_i.$$

- Correct in both limits $J \to 0$ and $U \to 0$,
- Minimize $\langle\Psi|H_{\text{BH}} - \mu\hat{N}|\Psi\rangle$ with respect to $\{c(n_i)\}$ with the constraint $\sum_{n=0}^{\infty}|c(n)|^2 = 1$ for a given $\mu$,
- Equivalently : minimize $\langle\Psi|H_{\text{BH}}|\Psi\rangle$ with respect to $\{c(n_i)\}$ with the constraints $\sum_{n=0}^{\infty}|c(n)|^2 = 1$, $\sum_{n=0}^{\infty} n|c(n)|^2 = \overline{n}$.

Uniform system : $|\phi\rangle_i$ identical for all sites $i$

Strong interactions $U \geq J$: on-site number fluctuations become costly.

The on-site statistics $p(n_i)$ evolves from a broad Poisson distribution to a peaked one around some integer $n_0$ closest to the average filling : number squeezing.



Variational wavefunction :

$$|\Psi\rangle_{\text{Gutzwiller}} = \prod_i |\phi_i\rangle,$$
$$|\phi_i\rangle = \sum_{n_i=0}^{\infty} c(n_i)|n_i\rangle_i.$$

Condensate fraction $f_c$: normalized population of the quasi-momentum state $\boldsymbol{q}=0$,

$$f_c = \frac{1}{N}\langle \hat{b}_{\boldsymbol{q}}^\dagger \hat{b}_{\boldsymbol{q}}\rangle_{\boldsymbol{q}=0} = \frac{1}{NN_s}\sum_{i,j}\langle \hat{a}_i^\dagger \hat{a}_j\rangle,$$

taken in the thermodynamic limit (TL) $N, N_s \to \infty$.

For the Gutzwiller state: $\langle \hat{a}_i^\dagger \hat{a}_i\rangle = \overline{n}$, or $\langle \hat{a}_i^\dagger \hat{a}_j\rangle = \alpha_i^* \alpha_j$ for $i \neq j$

$$f_c = \frac{1}{N_s} + \frac{|\alpha|^2}{\overline{n}}\left(1 - \frac{1}{N_s}\right) \underset{\text{TL}}{\longrightarrow} \frac{|\alpha|^2}{\overline{n}}$$

- Non-interacting/Gross-Pitaevskii case $(U \to 0)$ : $\alpha = \sqrt{\overline{n}} \implies f_c = 1$,
- Fock states with $n_0$ atoms per site $(J=0)$: $\alpha = 0 \implies f_c \underset{\text{TL}}{\longrightarrow} 0$.

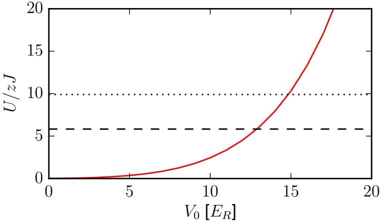
**Transition from a delocalized superfluid state to a localized Mott insulator state above a critical interaction strength $U_c$**

Superfluid:

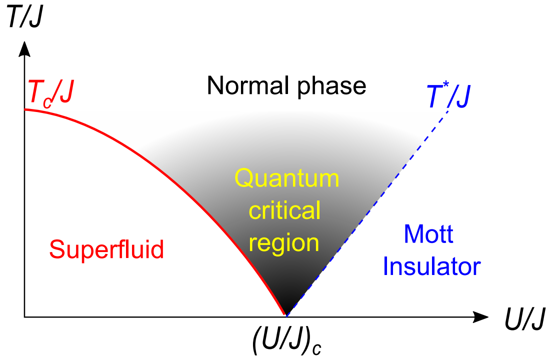
- non-zero condensed fraction $|\alpha|^2$
- on-site number fluctuations
- Gapless spectrum
- Long wavelength superfluid flow can carry mass across the lattice

Mott insulator :

- zero condensed fraction
- on-site occupation numbers pinned to the *same* integer value
- Energy gap $\sim U$ (far from transition)
- No flow possible unless one pays an extensive energy cost $\sim U$



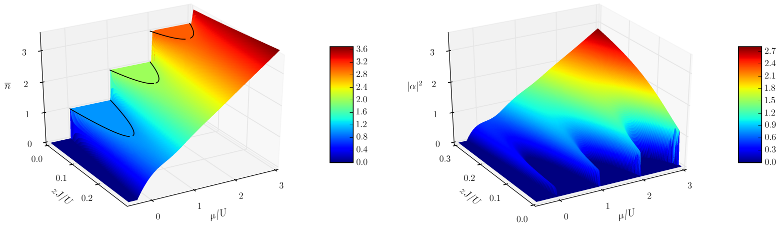
Mott transition : prototype of a quantum phase transition driven by two competing terms in the Hamiltonian




- Different from standard phase transition (competition between energy and entropy)
- Thermal crossover in the Mott insulator regime
- Strongly fluctuating quantum critical region

# Mean-field phase diagram

Generalization to incommensurate fillings:




Superfluid stable when

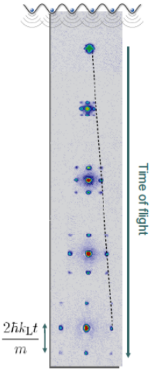
$$\mu_{n_0}^{(+)} \leq \mu \leq \mu_{n_0+1}^{(-)}.$$

$\mu_{n_0}^{(\pm)}$: upper/lower boundaries of the Mott region with occupation number $n_0$

$$\mu_{n_0}^{(\pm)} = U(n_0 - \frac{1}{2}) - \frac{zJ}{2} \pm \sqrt{U^2 - 2UzJ(2n_0+1) + (zJ)^2}$$

Evolution of field operator after suddenly switching off the lattice :

$$\hat{\psi}(\mathbf{r}, t=0) = \sum_i W(\mathbf{r}-\mathbf{r}_i)\hat{a}_i \rightarrow \hat{\psi}(\mathbf{k}) \propto \tilde{W}(\mathbf{K}) \sum_i e^{i\mathbf{K}\cdot\mathbf{r}_i}\hat{a}_i$$




Time of flight signal, far-field regime ($\mathbf{K} = \frac{M\mathbf{r}}{\hbar t}$) :

$$n_{\text{tof}}(\mathbf{K}) = \langle \hat{\psi}^\dagger(\mathbf{K})\hat{\psi}(\mathbf{K})\rangle \approx \mathcal{G}(\mathbf{K})\,\mathcal{S}(\mathbf{K})$$

- $\mathcal{G}(\mathbf{K}) = \left(\frac{M}{\hbar t}\right)^3 |\tilde{W}(\mathbf{K})|^2$

  smooth envelope
- $\mathcal{S}(\mathbf{K}) = \sum_{i,j} e^{i\mathbf{K}\cdot(\mathbf{r}_j - \cdot\mathbf{r}_i)}\langle \hat{a}_i^\dagger a_j\rangle$

  structure factor.

Key quantity : single-particle correlation function (also called $g^{(1)}(\boldsymbol{r}, \boldsymbol{r}')$)

$$\mathcal{C}(i,j) = \langle \hat{a}_i^\dagger a_j \rangle$$

Determines the structure factor and the interference pattern (or lack thereof)

# Time-of-flight interferences across the Mott transition

$$\mathcal{C}(i,j) = \langle \hat{a}_i^\dagger a_j \rangle \qquad \mathcal{S}(\mathbf{K}) = \sum_{i,j} e^{i\mathbf{K}\cdot(\mathbf{r}_j - \cdot \mathbf{r}_i)} \langle \hat{a}_i^\dagger a_j \rangle$$

Superfluid/BEC :

$$\mathcal{C}_{\mathrm{BEC}}(i,j) = \sqrt{\overline{n}_i \overline{n}_j}$$

$$\mathcal{S}_{\mathrm{SF}}(\mathbf{K}) \approx |\sum_i e^{i\mathbf{K}\cdot\mathbf{r}_i} \sqrt{\overline{n_i}}|^2,$$

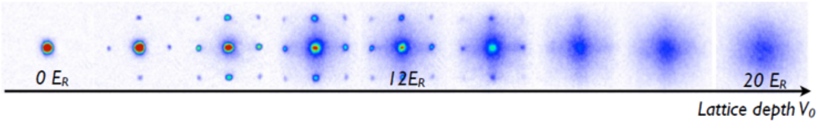
Bragg spots (height $\sim N_s^3$, width $\sim 1/N_s$)

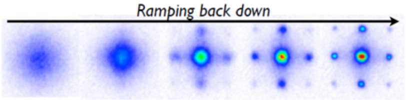
Mott insulator :

$$\mathcal{C}_{\mathrm{Mott}}(i,j) = n_0 \delta_{i,j}$$

$$\mathcal{S}_{\mathrm{Mott}}(\mathbf{K}) \approx N_s$$

Featureless

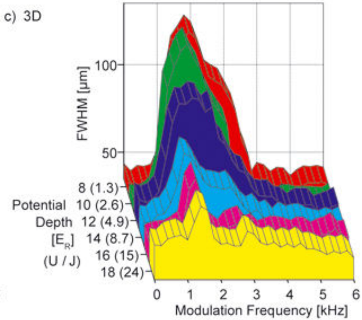


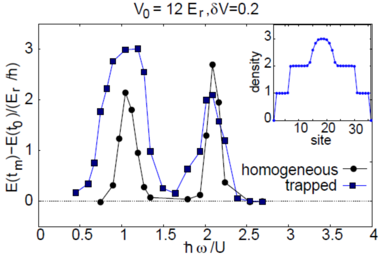

M. Greiner *et al.*, Nature 2002

# Probing the transition : lattice shaking

- Modulation of the lattice height : $V_0(t) = V_0 + \delta V_0 \cos(\omega_{\mathrm{mod}} t)$
- Main effect for deep lattices : $\delta \hat{V} = -\delta J \cos(\omega_{\mathrm{mod}} t) \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j$
- Superfluid regime : broad response at all frequencies
- Mott insulator regime : Coupling to particle-hole excitations $\Longrightarrow$ peaks at $\omega_{\mathrm{mod}} \approx \frac{U}{\hbar}$




Exp: Schori *et al.*, PRL 2004

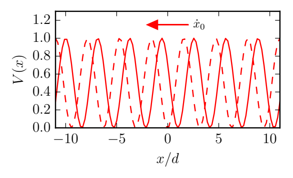



Th.: Kollath *et al.*, PRL 2006

Uniformly accelerated lattice : $V_{\rm lat}[x - x_0(t)]$ with $x_0 = -\frac{Ft^2}{2m}$

**Lab frame:**

$$H_{\rm lab} = \frac{p^2}{2m} + V_{\rm lat}[x - x_0(t)]$$

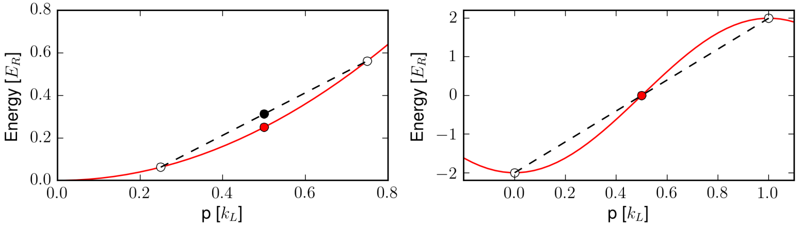



Non-interacting atoms undergo Bloch oscillations.

What happens with interactions ?

Collision of two atoms with momentum $\boldsymbol{p}_0$ :

$$2\boldsymbol{p}_0 = \boldsymbol{p}_1 + \boldsymbol{p}_2$$
$$2\varepsilon(\boldsymbol{p}_0) = \varepsilon(\boldsymbol{p}_1) + \varepsilon(\boldsymbol{p}_2)$$

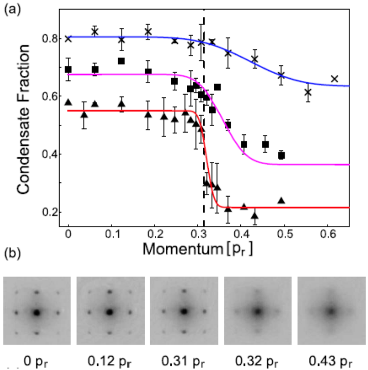


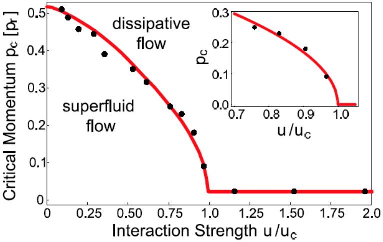

Because of the band structure, collisions redistributing quasi-momentum in the Brillouin zone are kinematically allowed in a lattice (for $|q| > \frac{\pi}{2d}$ in 1D).

This leads to a *dynamical instability* of wavepackets exceeding a certain critical velocity ($v_c = \frac{\hbar k_L}{M}$ in 1D).

MIT experiment [Mun *et al.*, PRL 2007]:

- moving lattice dragging the cloud along
- $p_r = \hbar k_L$: momentum unit
- cycle the lattice back and forth through the cloud (period 10 ms)



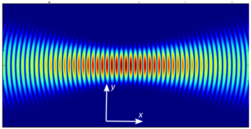

Critical point near $U/J \approx 34.2(2)$

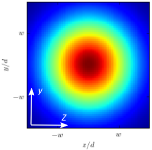
Mean field theory predicts $34.8$ , quantum Monte-Carlo $29.3$ : ?

# Non-uniform lattices : Harmonic trapping

Actual laser beams have Gaussian profile : Lattice potential of the form

$$V_{1\mathrm{D}} = -V_0 \cos^2(k_L x)\, e^{-2\frac{y^2+z^2}{w^2}}.$$




In the Wannier basis, additional potential energy term :

$$\delta V_{\mathrm{x}} \approx \sum_i \frac{1}{2} M\Omega^2 \left(y_i^2 + z_i^2\right) \hat{a}_i^\dagger \hat{a}_i,$$

with $\Omega^2 \approx \frac{8V_0}{Mw_x^2}\left(1 - \frac{k_L \sigma_w}{2}\right)$.

For a 3D lattice :

$$\delta V \approx \sum_i \underbrace{\frac{1}{2} M\Omega^2 \left(x_i^2 + y_i^2 + z_i^2\right)}_{V_h(\boldsymbol{r}_i)} \hat{a}_i^\dagger \hat{a}_i,$$

with $V_h$ a harmonic potential.

An insulator is incompressible :

Within a Mott lobe, changing the chemical potential does not change the density.

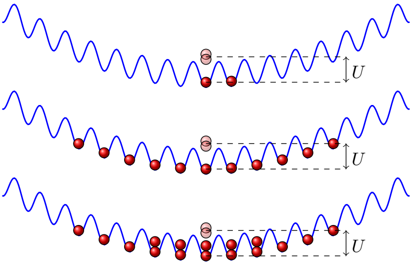
Consequence of the gap for producing particle/hole excitations, which vanishes at the phase boundaries.

Consequence : non-uniform density profile in a trap

Simple picture in 1D :

Filling the lattice atom by atom in the atomic limit ($J = 0$)

Formation of shells due to competition between interaction and potential energy

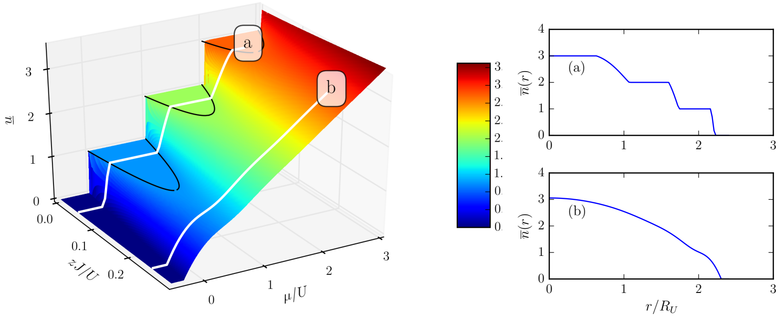


Local density approximation for a smooth potential :

$$\mu_{\rm loc}(\mathbf{r}) = \mu - V_{\rm h}(\mathbf{r}),$$

$\mu$ : global chemical potential fixed by constraining the total atom number to $N$

Density profile given by the equation of state $n[\mu]$ for the uniform system, evaluated at $\mu = \mu_{\rm loc}(\mathbf{r})$.




- **Mott insulator (a):** density changes abruptly; plateaux with uniform density
- **Superfluid (b):** density changes smoothly from the center of the cloud to its edge

Munich experimental setup (Sherson *et al.* 2010) :

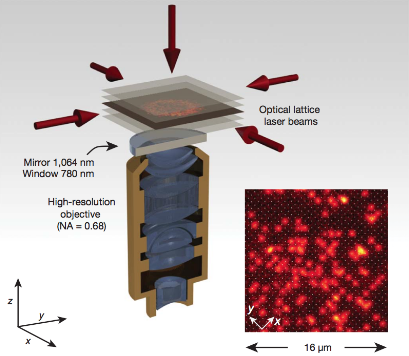
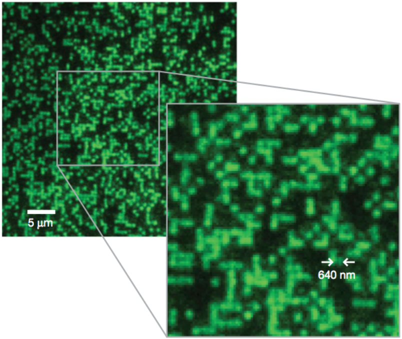
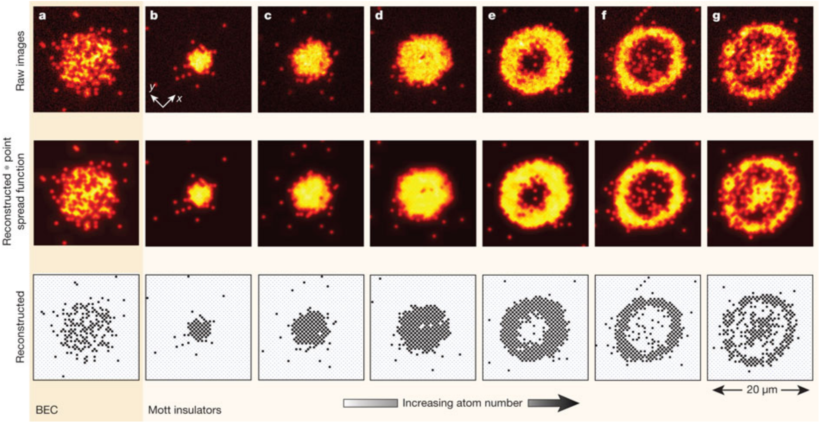



image of a dilute gas

image of a Bose-Einstein condensate in a 2D lattice [Bakr *et al.*, 2010]:

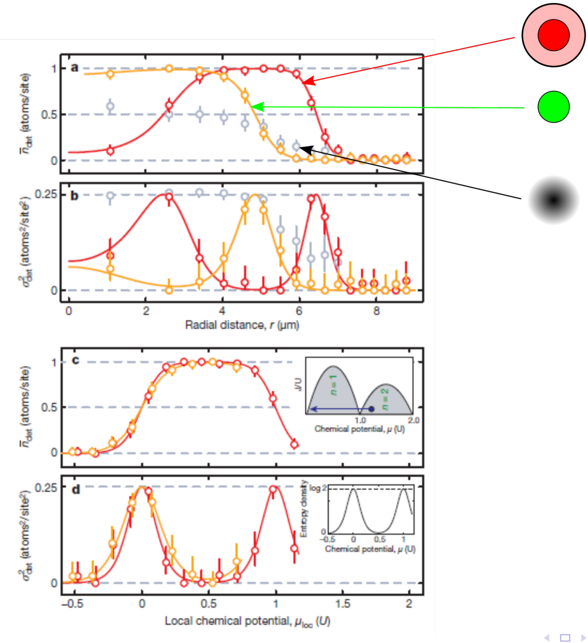


In-situ images of a BEC and of Mott insulators [Sherson *et al.*, Nature 2010]:




Total atom number (or chemical potential) increases from left to right.
Lowest row : reconstructed map of the atom positions, obtained by deconvolution of the raw images to remove the effect of finite imaging resolution.

# Mott shells and LDA



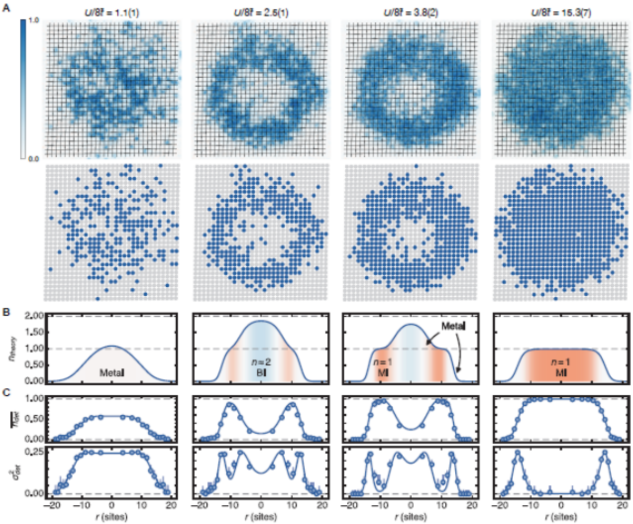

Mott insulator,
$n_0$=1 and $n_0$=2 shells

Mott insulator,
$n_0$=1 and $n_0$=2 shells

BEC
$n_{max}$~1

# A glance at fermions

Two-component fermions with repulsive interactions: Fermionic Mott insulator




[Greif *et al.*, arxiv1511.06366 (2015)]
Fermionic quantum gas microscopes also demonstrated in : Haller *et al.*, arxiv1503.02005 (2015); Cheuk *et al.*, arxiv1503.02648 (2015); Omran *et al.*, arxiv1510.04599 (2015).

Two-component repulsive Hubbard model :

- antiferromagnetic Néel phase below $S_c/Nk_B \lesssim 0.5$
- $d$-wave superconductors ? unknown but certainly much lower entropy.

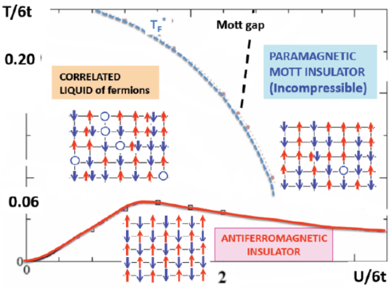



figure from [Georges & Giamarchi, arxiv1308.2684 (2014)]

- Experiments are actually not performed at temperatures low enough that one can safely take the limit $T = 0$. achieve $S/Nk_B \sim 1$,
- many interesting phases are awaiting below that scale
- limits of the current "cooling then adiabatic transfer" technique,

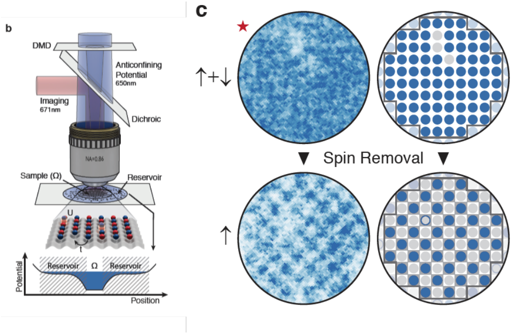
New methods to cool atoms directly in the lattice needed.

See reviews for a more detailed discussion :

[ McKay & DeMarco, , Rep. Prog. Phys. **74**, 0544401 (2011)],

[Georges & Giamarchi, arxiv1308.2684 (2014)]

Recent realization of magnetic ordering in M. Greiner's group (Harvard University)




[Marurenko *et al.*, Nature 2017]

- Spin-resolved quantum gas microscopy (right picture)
- Density of spin ↑ atoms displays a checkerboard pattern characteristic of Néel ordering.
- key experimental advance : using carefully shaped trap potential to create high- and low-entropy regions; Néel ordering occurs only in the latter.

The End