Superfluid density and critical velocity near the fermionic Berezinskii-Kosterlitz-Thouless transition

Brendan Mulkerin

Theoretical Condensed Matter Physics
Swinburne University
Todays talk

• Thanks A/Profs. Xia-ji Liu, Chris Vale, Paul Dyke, Jia Wang, Lianyi He, and Hui Hu for our collaborative work
• Thanks to our PhD students Umberto Toniolo, Sebastian Schaffer, Xiao-Long Chen, and Christopher Hoegaard for all their hard work
• BEC-BCS crossover in two dimensions
• The equation of state: Breathing mode in 2D gases
• BKT transition
Why strongly interacting Fermi gases?

Strongly interacting Fermi gases with balanced populations very difficult to solve

- Strongly correlated Fermi systems are a playground for many-body physics
- They are stable on long timescales and for strong interactions

**Figure:** Xia-Ji Liu Physics Reports 524 (2), 37-83.

- They play a fundamental role in very different areas or physics
- Lower dimensions increase the fluctuations, quantum effects are larger
Two dimensional BCS-BEC crossover

2D scattering always allows a bound state and is energy dependent,

\[ f(q) = \frac{4\pi}{\ln \left( \frac{1}{a_{2D}^2} q^2 \right) + i\pi}, \quad \varepsilon_B = \frac{\hbar^2}{ma_{2d}^2} \]

No unitary regime but interactions can be changed from the BEC - BCS side through

\[ \eta = \ln \left( k_F a_{2D} \right) = -\frac{1}{2} \ln \left( \frac{2E_F}{\varepsilon_B} \right) \]

BCS side: weakly interacting pairs  

BEC side: Tightly bound bosonic molecules

Fluctuations in 2D are larger:

This prevents long-range order [Mermin-Wagner-Hohenberg]
Experimental progress

Experimentalists can directly measure the equation of state (E.O.S) and thermodynamic properties of the gas.
Equation of state

The equation of state shows the non-trivial E.O.S. even in the normal state

Figure: Fenech et al PRL 116 045302 (2016) (Top) and Boettecher et al PRL 116 045303 (2016) (Bottom).

We can use the E.O.S. to calculate the breathing mode anomaly
Using the 2D equation of state we can explore the breathing mode anomaly:

**Delta** function \( V_{2D}(\mathbf{r} - \mathbf{r}') = g_{2D} \delta(\mathbf{r} - \mathbf{r}') \) interaction is the most important interaction in a two-component Fermi gas scales as \( \lambda^{-2} \) in 2D, regularisation destroys this scaling:

\[
g_{2D} \rightarrow \log(k_F a_{2D})
\]

Including a harmonic trap, \( H_{\text{trap}} = \frac{1}{2} m \omega^2 r^2 \), breaks the scale invariance,

\[
\mathbf{r} \rightarrow \lambda \mathbf{r}, \quad H_{\text{trap}} \rightarrow \lambda^2 H_{\text{trap}}
\]

However there is a hidden \( SO(2, 1) \) symmetry

This symmetry can excite a breathing mode, \( \omega_B = 2\omega \rightarrow \) the quantum anomaly will break this hidden symmetry
Scale Invariance and Viscosity of a Two-Dimensional Fermi Gas

Enrico Vogt, Michael Feld, Bernd Fröhlich, Daniel Pertot, Marco Koschorreck, * and Michael Köhl
Cavendish Laboratory, University of Cambridge, JJ Thomson Avenue, Cambridge CB3 0HE, United Kingdom
(Received 4 November 2011; published 17 February 2012)

We investigate collective excitations of a harmonically trapped two-dimensional Fermi gas from the collisionless (zero sound) to the hydrodynamic (first sound) regime. The breathing mode, which is sensitive to the equation of state, is observed with an undamped amplitude at a frequency 2 times the dipole mode frequency for a large range of interaction strengths and different temperatures. This provides evidence for a dynamical SO(2,1) scaling symmetry of the two-dimensional Fermi gas. Moreover, we investigate the quadrupole mode to measure the shear viscosity of the two-dimensional gas and study its temperature dependence.

DOI: 10.1103/PhysRevLett.108.070404

PACS numbers: 67.85.-d, 03.75.Ss

\[ \ln [k F a_{2D}] \]

\[ \delta \omega / \omega_0 \]

\[ T = 0 \text{ polytropic fit} \]
\[ T = 0 \text{ BEC} \]
\[ T = 0 \text{ BCS} \]
\[ T = 0.42 T_F \text{ Vogt et al. [30]} \]
Theoretical results

Using the hydrodynamic formalism and equation of state:

\[ \mu(r) = \mu - V_{\text{trap}}(r), \]
\[ n(r) \lambda^2 = f_n \left( \frac{\mu}{k_B T} \right), \quad \frac{P(r)}{k_B T} = f_p \left( \frac{\mu}{k_B T} \right), \quad \frac{df_p(x)}{dx} = f_n(x) \]

\[ S^{(2)} = \frac{1}{2} \int d\mathbf{r} \left[ \omega^2 \rho_0 \mathbf{u} - (\nabla \rho_0 \cdot \mathbf{u}) \left( \frac{\nabla V_{\text{trap}}}{M} \cdot \mathbf{u} \right) + 2 \left( \rho_0 \frac{\nabla V_{\text{trap}}}{M} \cdot \mathbf{u} \right) (\nabla \cdot \mathbf{u}) - \rho_0 \left( \frac{\partial P}{\partial \rho} \right) \bar{s} (\nabla \cdot \mathbf{u})^2 \right] \]

No significant result at finite temperature → we do see damping
Theoretical results

Using the hydrodynamic formalism and equation of state:

Equation of state and the local density approximation, $\mu(r) = \mu - V_{\text{trap}}(r)$,

$$n(r)\lambda^2 = f_n\left(\frac{\mu}{k_B T}\right), \quad P(r)\lambda^2 = f_p\left(\frac{\mu}{k_B T}\right), \quad \frac{df_p(x)}{dx} = f_n(x)$$

$$S^{(2)} = \frac{1}{2} \int d\mathbf{r} \left[ \omega^2 \rho_0 \mathbf{u} - (\nabla \rho_0 \cdot \mathbf{u}) \left( \frac{\nabla V_{\text{trap}}}{M} \cdot \mathbf{u} \right) + 2 \left( \rho_0 \frac{\nabla V_{\text{trap}}}{M} \cdot \mathbf{u} \right) (\nabla \cdot \mathbf{u}) - \rho_0 \left( \frac{\partial P}{\partial \rho} \right)_s (\nabla \cdot \mathbf{u})^2 \right]$$

No significant result at finite temperature → we do see damping
Theoretical results

Using the hydrodynamic formalism and equation of state:

\[
\mu(r) = \mu - V_{\text{trap}}(r),
\]

\[
n(r) \lambda^2 = f_n \left( \frac{\mu}{k_B T} \right), \quad P(r) \lambda^2 = f_p \left( \frac{\mu}{k_B T} \right), \quad \frac{dp}{dx} = f_n(x)
\]

\[
S^{(2)} = \frac{1}{2} \int d\mathbf{r} \left[ \omega^2 \rho_0 \mathbf{u} - (\nabla \rho_0 \cdot \mathbf{u}) \left( \frac{\nabla V_{\text{trap}}}{M} \cdot \mathbf{u} \right) + 2 \left( \rho_0 \frac{\nabla V_{\text{trap}}}{M} \cdot \mathbf{u} \right) (\nabla \cdot \mathbf{u}) - \rho_0 \left( \frac{\partial P}{\partial \rho} \right) \bar{s} (\nabla \cdot \mathbf{u})^2 \right]
\]

There is a strange behaviour of the breathing mode in the high temperature regime.

**Figure:** The breathing mode anomaly for \( T/T_F = 0.8 \)
Strongly interacting Fermi gases with balanced populations, well studied, very difficult to solve.

To begin with we have the thermodynamic potential found through the partition function

$$\Omega = -\beta^{-1} \ln Z,$$

where the partition function and action are given by,

$$Z = \int \mathcal{D}[\psi, \bar{\psi}] e^{-S[\psi, \bar{\psi}]} \quad \text{and} \quad S = \int_0^{\hbar \beta} d\tau \left[ \int dr \sum_\sigma \bar{\psi}_\sigma(x) \partial_\tau \psi_\sigma(x) + H \right],$$

and the action defined by a Hamiltonian is

$$S = \int_0^{\hbar \beta} d\tau \left[ \int dr \sum_\sigma \bar{\psi}_\sigma(x) \partial_\tau \psi_\sigma(x) + H \right],$$

Decouple through the Hubbard-Stratonovich transformation:

$$S_{\text{eff}} [\Delta, \Delta^*] = \int dx \left[ \frac{|\Delta(x)|^2}{U_0} - \text{Tr} \ln [-G^{-1}] \right].$$

This is true for general dimension, where $\int dx = \int d^d r d\tau$ and $U_0$ is regularised appropriately.
Strongly interacting Fermi gases with balanced populations, well studied, very difficult to solve.

Expand the thermodynamic potential by taking the Bose field $\Delta(r,t)$ about its saddle point $\Delta_0$,

$$\Delta(r,t) = \Delta_0 + \varphi(r,t),$$

The action is expanded in order of $\Delta_0$ and the thermodynamic potential is

$$\Omega = \Omega_{MF} + \Omega_{GF}.$$ 

Extend to the general case condensed pairs flow with a wavevector $Q$: $\Delta e^{iQ \cdot r}$

In this case, the mean-field thermodynamic potential is given by

$$\Omega_{MF}(Q) = \frac{\Delta^2}{U} + \sum_k \left[ \tilde{\xi}_k - E_k - \frac{2}{\beta} \ln \left( 1 + e^{\beta E_k^+} \right) \right],$$

and the mean-field gap equation,

$$\sum_k \left[ \frac{1 - 2f\left(E_k^+\right)}{2E_k} - \frac{1}{\hbar^2 k^2/M + \epsilon_B} \right] = 0.$$

The thermodynamic potential for gaussian pair fluctuations (GPF) is:

$$\Omega_{\text{GF}}(Q) = k_B T \sum_{Q \equiv (q, i\nu_l)} S(Q) e^{i\nu_l 0^+} ,$$

$$S(Q) = \frac{1}{2} \ln \left[ 1 - \frac{M_{12}^2(Q)}{M_{11}(Q) M_{11}(-Q)} \right] + \ln M_{11}(Q) ,$$

See our recent paper PRA 96 053608 (2017) for the long definitions of $M_{11}$ and $M_{12}$

This is difficult to solve below $T_c$, the Matsubara summation is tricky, instead we use:

$$\frac{1}{\beta} \sum_{|l| > l_0} S_\eta(q, i\nu_l) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \frac{\text{Im} S_\eta(q, \omega + i\gamma)}{e^{\beta\omega} + 1}$$

where $S_\eta(q, i\nu_l) \equiv S(q, i\nu_l)e^{i\nu_l \eta}$ and $\gamma = (2l_0 + 1)\pi/\beta$ for arbitrary positive integer $l_0$
To illustrate the importance of our full treatment of the GPF we find the E.O.S.

Figure: Comparing the results to experiment and Luttinger-Ward $T$-matrix theory PRA 96 053608 (2017)

The GPF underestimates the pressure but has a superfluid order parameter
The superfluid density can be found by adding a twist, giving the density to be:

\[ n_s = \frac{4m}{\hbar^2} \left[ \frac{\partial^2 \Omega(Q)}{\partial Q^2} \right]_{Q=0} \]

We can now look at the strongly correlated BCS side in 2D.

**Figure:** The behaviour of the order parameter and superfluid fraction in 2D for \( \varepsilon_B/\varepsilon_F = 0.1 \)

There is a region where \( n_s = 0 \) and \( \Delta_{\text{GPF}} > 0 \)
The Kosterlitz-Thouless criterion defines the BKT transition temperature:

\[ k_B T_{\text{BKT}} = \frac{\pi}{2} \frac{\hbar^2}{4m} n_s(T) \]

**Figure:** The order parameter and superfluid fraction in 2D for \( \varepsilon_B/\varepsilon_F = 0.1 \) and other theoretical attempts
Figure: The superfluid fraction normalised by an ideal gas
Superfluid density in 2D

Figure: The superfluid fraction normalised by an ideal gas

Figure: Critical chemical potential as a function of interaction strength for Swinburne (squares) and Heidelberg (circles)

We can define a critical chemical potential, which can be measured directly in experiment

Critical chemical potential $\rightarrow$ critical radius: $\mu_c = \mu - V(r_c)$
Unambiguously find the BKT transition

An unambiguous method find the BKT transition: use the LDA $\mu_g = \mu - V(r)$

Figure: The critical velocity $v_c = \hbar Q / (2m)$ as a function of dimensionless chemical potential
• The breathing mode is damped as a function of temperature and is significant in the high temperature regime
• We have explicitly included pairing fluctuations in the calculation of the superfluid density
• Through stirring the gas we can find an unambiguous method to measure the BKT transition
Thank you for your attention today