Disordered fermions in two dimensions: is Anderson insulating phase the only possibility?

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Outline

- Basic concepts on the Anderson localization
- QFT approach: derivation of the non linear $\sigma$-model and symmetry classifications
- (Anti)-localization effects for all symmetry classes
- Combined effects of disorder and interactions: not-universal behaviors and enhancement of critical temperatures.
Anderson localization

In the presence of strong enough disorder in $D > 2$ or for any amount of disorder in $D \leq 2$ a metal can turn into an insulator.

Interference effect ($\lambda_{DB} \simeq \ell$) $\Rightarrow$ localization of the wavefunctions

The probability to find the particle at point $C$ is:

$$|a_1|^2 + |a_2|^2 + 2 \text{Re}(a_1 a_2^*) = 4|a_1|$$

$\Rightarrow$ enhancement of probability to find a particle at $C$

$\Rightarrow$ reduction of probability to find it at $B$ (conductivity $\downarrow$)

Probability of self-intersection

$$\frac{\delta \sigma}{\sigma} \sim -\int_{\tau}^{\tau_{\varphi}} \frac{v \lambda^{d-1} dt}{(Dt)^{d/2}} \Rightarrow \left\{
\begin{array}{l}
\delta \sigma \propto -\left(\frac{1}{\ell} - \frac{1}{L_{\varphi}}\right), \\
\delta \sigma \propto -\log\left(\frac{L_{\varphi}}{\ell}\right), \\
\delta \sigma \propto -(L_{\varphi} - \ell),
\end{array}\right. \quad d = 3, 2, 1
$$

with $L_{\varphi} = \sqrt{D_{\tau_{\varphi}}}$ and $\tau_{\varphi} \sim T^{-1}$
Scaling theory of localization


Thouless idea: sample \((2L)^d\) made of cubes \(L^d\)

\[ \Rightarrow \text{an eigenstate for } (2L)^d \text{ is a mixture of e.s. of } L^d \text{ depending on overlap integrals and energy differences (as in perturbation theory)} \]

- energy differences \(\sim\) level spacing \(\delta W = (\nu_0 L^d)^{-1}\)
- overlap \(\sim\) bandwidth \(\delta E\) (if localized e.s. \(\delta E\) exp. small, otherwise \(\sim \hbar D/L^2\))

One parameter: \(\frac{\delta E}{\delta W}\) related to the conductance \(G\) (units of \(e^2/\hbar\))

- small disorder: \(G(L) = \sigma L^{d-2}\)
- strong disorder: \(G(L) \sim \exp(-L/\xi)\)
Scaling theory of localization

- strong disorder: $G(L) \sim \exp(-L/\xi)$

$$\beta(G) = \frac{d \log G}{d \log L} = \log \frac{G}{G_c}.$$

- small disorder: $G(L) \sim \sigma L^{d-2}$, expanding in $1/G$

$$\beta(G) = (d - 2) - \frac{a}{G}.$$

$$\Rightarrow \sigma(L) - \sigma_0 \propto \begin{cases} \left(\frac{1}{\ell} - \frac{1}{L}\right) & d=3 \text{ (metal)} \\ \log\left(\frac{L}{\ell}\right) & d=2 \text{ (insulator)} \\ (L - \ell) & d=1 \text{ (insulator)} \end{cases}$$
Hamiltonian with some random potential

\[ H = H_0 + V \]

Disorder variance \( \overline{V(r)V(r')} = w_0 \delta_{rr'} \)

Bare Green function

\[ G_0 = \]

In Born approximation, \( \Sigma = \)

Green function

\[ G^\pm(E, p) = (E - H_0(p) \pm i/2\tau) \]

Kubo formula for conductivity (paramagnetic part)

\[ \sigma(\omega) = \frac{e^2}{2\pi} \int d\varepsilon \frac{\partial n_\varepsilon}{\partial \varepsilon} Tr \left[ \hat{v} G^+_{\varepsilon+\omega} \hat{v} (G^+_{\varepsilon} - G^-_{\varepsilon}) \right] \approx \frac{e^2 \nu v_F^2}{d} \frac{\tau}{1 + i\omega\tau} \]

\[ \sigma_0 = \sigma(0) = \frac{e^2 \nu v_F^2 \tau}{d} \) (Drude conductivity) \]
The dc electrical conductivity can be written in terms of current-current or density-density correlation functions

\[ \sigma = i \lim_{\omega \to 0} \frac{1}{\omega} K_{ij}(0, \omega) \delta_{ij} = i \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} K_{00}(q, \omega) \]

Ladder summation (diffuson)

\[ D(q, \omega) = \frac{1}{2\pi \nu \tau^2} \frac{1}{Dq^2 - i\omega} \]

with \( D = v_F \ell / d = v_F^2 \tau / d \) (diffusion coefficient)

\[ K_{00}(q, \omega) = -e^2 \nu \frac{Dq^2}{Dq^2 - i\omega} \]

from which \( \sigma = \sigma_0 = e^2 \nu D \).
Diagrammatics: Weak Localization

Inclusion of crossing diagrams

\[ C(q, \omega) = \frac{1}{2\pi \nu \tau^2} \frac{1}{Dq^2 - i\omega} \]

Ladder summation in the particle-particle channel: cooperon

Since now \( q = p + p' \), the contribution to the current-current correlator

\[ \delta K^{ii}(0, \omega) = \frac{i\omega \sigma_0}{\nu \pi} \int dq \frac{1}{Dq^2 - i\omega} \]

The correction to the dc conductivity is

\[ \delta\sigma = -\frac{\sigma_0}{\nu \pi} \int dq \frac{1}{Dq^2 - i\omega} \propto \begin{cases} \left(\frac{1}{\ell} - \frac{1}{L}\right) & d = 3 \\ \log\left(\frac{L}{\ell}\right) & d = 2 \\ (L - \ell) & d = 1 \end{cases} \]
**Anderson insulator**

- **1D - 2D**: Weak localization is IR-divergent in 1D and 2D: 
  \[ \delta \sigma \sim \sigma_0 \] at a scale 

  \[ \xi \sim \pi \nu D, \quad \text{for 1D} \]

  \[ \xi \sim \ell \exp (\pi^2 \nu D), \quad \text{for 2D} \]

- **3D**: Localization only above a critical value of the disorder

Localisation length at criticality

\[ \xi \sim (\sigma_0 - \sigma_c)^{-\nu} \]

In the localized phase \( D(q, \omega) = C(q, \omega) \) becomes massive

\[ D(r, \omega) \sim \exp \left(-\frac{r}{\xi}\right) \]
Field theory approach: non-linear $\sigma$-model
(Wegner, ZPB (1979); Efetov, Larkin, Khmel’nitsky, JETP (1980))

- Write $G^\pm$ in terms of Grassmann variables with action

$$S = \int \bar{\Psi}(E - H_0 - V \pm i \omega)\Psi$$

- Average over disorder $V$ by replica method

$$S_{\text{eff}} = \int \bar{\Psi}(E - H_0 \pm i \omega)\Psi + w_0 \int (\bar{\Psi}\Psi)^2$$

- Hubbard Stratonovich transformation (auxiliary field $Q$)

- Integrating over fermionic fields $\Rightarrow S(Q)$

- Saddle point: $\frac{\delta S}{\delta Q} = 0 \Rightarrow Q_{sp}$

- Fluctuations around saddle point

- Gradient expansion $\Rightarrow$ N.L.$\sigma$ M.
Hubbard Stratonovich transformation

Integration over disorder ⇒ a quartic term in the action

\[ e^{-S_{\text{eff}}} = e^{-(S_0 + S_{\text{imp}})} \]

By Hubbard-Stratonovich decoupling,

\[ e^{-S_{\text{imp}}} = e^{w_0 \int (\bar{\Psi} \Psi)^2} = \int dQ \, e^{\int \frac{1}{2w_0} \text{Tr} [QQ^\dagger] - i \text{Tr} [\bar{\Psi}Q\Psi]} \]

For bipartite lattices the auxiliary field is not hermitian

\[ Q_j = Q_{0j} + i(-1)^j Q_{3j} \] (smooth and staggered components)

Integrating over \( \Psi \)

\[ S(Q) = \sum \frac{1}{2w_0} \text{Tr} \left[ Q^\dagger Q \right] - \frac{1}{2} \text{Tr} \ln \left( -H + iQ \right) \]

Saddle point: \( \frac{\delta S}{\delta Q} = 0 \rightarrow Q_{sp} = \sum \propto \tau^{-1} \)

the self-energy at the Born level, in the diagrammatics!
Transverse modes and symmetry classification

Quantum fluctuations around $Q_{sp}$ that leave $H$ invariant

$$Q = U^{-1}Q_{sp}U$$

$U \in G$ and $[U, Q_{sp}] \neq 0$

If $H$ subgroup of $G$ such that $h \in H$, $[h, Q_{sp}] = 0 \Rightarrow U \in G/H$ (Coset)

<table>
<thead>
<tr>
<th>Hamiltonian Class</th>
<th>RMT</th>
<th>$\mathcal{T}$</th>
<th>SU(2)</th>
<th>NL$\sigma$-model manifolds</th>
</tr>
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<tbody>
<tr>
<td><strong>Wigner-Dyson classes</strong></td>
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<tr>
<td>A</td>
<td>GUE</td>
<td>$-$ $+$</td>
<td>$U(2n)/U(n) \times U(n)$</td>
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<tr>
<td>AI</td>
<td>GOE</td>
<td>$+$ $+$</td>
<td>$Sp(4n)/Sp(2n) \times Sp(2n)$</td>
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<tr>
<td>AII</td>
<td>GSE</td>
<td>$+$ $-$</td>
<td>$O(2n)/O(n) \times O(n)$</td>
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<td><strong>Chiral classes</strong></td>
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<tr>
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<td>chGUE</td>
<td>$-$ $+$</td>
<td>$U(n)$</td>
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<tr>
<td>BDI</td>
<td>chGOE</td>
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<td>$U(4n)/Sp(2n)$</td>
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<tr>
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<td><strong>Bogoliubov-de Gennes</strong></td>
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<td>$-$ $+$</td>
<td>$Sp(2n)/U(2n)$</td>
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<tr>
<td>CI</td>
<td></td>
<td>$+$ $+$</td>
<td>$Sp(2n)$</td>
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<tr>
<td>D</td>
<td></td>
<td>$-$ $-$</td>
<td>$O(2n)/U(n)$</td>
<td></td>
</tr>
<tr>
<td>DIII</td>
<td></td>
<td>$+$ $-$</td>
<td>$O(n)$</td>
<td></td>
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</tbody>
</table>
Non linear $\sigma$-model

From the real part of $S(Q)$

$$Tr \ln (-H + iQ) + Tr \ln (-H - iQ^\dagger) =$$

$$= -Tr \ln (H^2 + Q_{sp}^2) - Tr \ln (1 + G_0 U),$$

where $G_0 = (H^2 + Q_{sp}^2)^{-1}$ and

$$U_{RR'} = iQ_R^\dagger H_{RR'} - iH_{RR'} Q_{R'} \approx -\vec{J} \cdot \vec{\nabla} Q$$

the current operator appears

$$J = -iH_{RR'} (R - R')$$

Expanding in $U_{RR'}$, the second term reads

$$Tr (G_0 U G_0 U) \simeq (JG_0 JG_0) Tr (\partial Q^\dagger \partial Q)$$

the factor $(JG_0 JG_0)$ is the Kubo formula for the conductivity!
Effective action (NLSM)

The final effective action in long wavelength limit

\[ S[Q] = \frac{\pi}{8} \sigma \int dR \ Tr \left( \nabla Q \nabla Q^\dagger \right) - 4\nu \ Tr(\hat{\omega} Q) \]

the bare \( \sigma = e^2 \nu D \) is the Drude conductivity!

Quantum corrections from Renormalization Group (RG) procedure:
Gaussian propagators = diffuson and cooperon in diagrammatics

\[ <QQ> = \frac{1}{2\pi\sigma} \int \frac{d^2q}{4\pi^2} \frac{1}{q^2 - i\omega} \equiv g \log(s) \]

where the effective coupling constant which controls the perturbative expansion is given by \( g = \frac{1}{2\pi^2\sigma} \) (the resistivity)

\[ \beta(g) = \frac{dg}{d \log s} \quad (s \text{ energy scaling factor}) \]

\( g \) is the running coupling constant.
RG of NLSMs (Wigner-Dyson classes) in $(2 + \epsilon)d$

Beta-functions by $\epsilon$-expansion

- **Class A** (unitary symmetry class, broken $T$)
  \[ \beta(g) = -\epsilon g + g^3/2 + 3g^5/8 + O(g^7) \]

- **Class AI** (orthogonal symmetry class, preserved $T$ and SU(2))
  \[ \beta(g) = -\epsilon g + g^2 + 3\zeta(3)g^5/4 + O(g^6) \]

- **Class AII** (simplectic symmetry class, preserved $T$, no SU(2))
  \[ \beta(g) = -\epsilon g - g^2 + 3\zeta(3)g^5/4 + O(g^6) \]

$(+g^2 \Rightarrow \text{weak localization}, -g^2 \Rightarrow \text{weak anti-localization})$

Anderson transitions ($\beta(g_c) = 0$)

- **3D** ($\epsilon = 1$). Example: class AI
  critical point: $g_c = \epsilon - 3\zeta(3)\epsilon^4/4 + O(\epsilon^5)$
  localization length exponent: $\nu = -1/\beta'(g_c) = \approx 1.7$
  (in good agreement with numerics, $\nu \approx 1.57$)

- **2D** for class AII
  critical point: $g_c = (4/3\zeta(3))^{1/3} \approx 1$

Metal-Insulator transition in 2D
Two-subattice models (Chiral classes)

(Gade, Wegner, NPB (1991))

The Hamiltonian is defined on a bipartite lattice

\[ H = - \sum_{\langle ij \rangle \sigma} t_{ij} e^{i \phi_{ij}} c_{i\sigma}^\dagger c_{j\sigma} - \sum_{i, \sigma} \mu c_{i\sigma}^\dagger c_{i\sigma} \]

- \[ t_{ij} = t_{ji} \] random hopping,
- \[ \phi_{ij} = -\phi_{ji}, \text{if } \neq 0, \text{breaks time reversal symmetry (T)} \]
- \[ \mu \neq 0 \text{ breaks sublattice symmetry (S)} \]

The effective action

\[
S[Q] = \frac{\pi}{16} \sigma \int dR \ Tr \left( \nabla Q \nabla Q^\dagger \right) - 4\nu Tr(\hat{\omega}Q) \\
- \frac{\pi}{8} \Pi \int dR \left[ Tr \left( Q^\dagger(R)\vec{\nabla}Q(R) \right) \right]^2
\]

(for \( \mu \neq 0 \Rightarrow \Pi = 0 \))
Results with and without sublattice symmetry in 2D

<table>
<thead>
<tr>
<th>μ ≠ 0, φ_{ij} = 0</th>
<th>Sp(4n)/Sp(2n) × Sp(2n)</th>
<th>Al</th>
<th>g^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ ≠ 0, φ_{ij} ≠ 0</td>
<td>U(4n)/U(2n) × U(2n)</td>
<td>A</td>
<td>O(g^3)</td>
</tr>
<tr>
<td>μ = 0, φ_{ij} = 0</td>
<td>U(8n)/Sp(4n)</td>
<td>BDI</td>
<td>0</td>
</tr>
<tr>
<td>μ = 0, φ_{ij} ≠ 0</td>
<td>U(4n) × U(4n)/U(4n)</td>
<td>AlIII</td>
<td>0</td>
</tr>
</tbody>
</table>

- without sublattice symmetry (μ ≠ 0):
  \[ \sigma = \sigma_0 - \frac{1}{2\pi^2} \log(\tau_\varphi/\tau) \]
  (insulator, like for the on-site disorder)

- with sublattice symmetry (μ = 0):
  \[ \sigma = \sigma_0 \]
  (conductor, Gade-Wegner criticality) at any order in g

\[ \beta(g) = 0 \] also for CII (Fabrizio, Dell'Anna, Castellani, PRL (2002))
Superconductors (Bogoliubov-de Gennes classes)

(Altland, Zirnbauer, PRB(1997))

For BdG Hamiltonians, since $U(1)$ is not preserved, charge diffusion is massive. The scaling parameter is the spin (or heat) conductivity:

- **Classes C and CI:** positive corrections $\beta(g) \sim g^2$
  $\Rightarrow$ weak localization

- **Classes D, DIII:** negative corrections $\beta(g) \sim -g^2$
  $\Rightarrow$ weak anti-localization (spin-metal - spin-insulator transition)

(Senthil, Fisher, Balents, Nayak, PRL (1998); Fabrizio, Dell’Anna, Castellani, PRL (2002))

Class C can be obtain also from random hopping Hamiltonian with magnetic impurities (Dell'Anna, AdP (2017))
Topological terms

For almost all classes (except for AI and BDI) in 2D the non-linear $\sigma$-model can be supplemented by a topological term:

- $\theta$-term for A, C, D (like the Pruisken term for the Integer Quantum Hall, with $\theta = \sigma_{ij}/8$) or All, CII

$$S_{\theta} = \theta \int dR Tr \epsilon_{\mu\nu} Q \partial_{\mu} Q \partial_{\nu} Q$$

- WZW-term for AllI, CI, DIII (chiral anomaly).

$$S_{\text{WZW}} = \frac{k}{24\pi} \int dR^2 \int_0^1 d\bar{R} Tr \epsilon_{\mu\nu\lambda} (Q^{-1} \partial_{\mu} Q)(Q^{-1} \partial_{\nu} Q)(Q^{-1} \partial_{\lambda} Q)$$

We can get WZW term taking the imaginary part of the action, left over in the $\sigma$-model derivation.

*(Dell’Anna, Fabrizio, Castellani, JSTAT (2007))*
Anderson criticality in 2D (summary)

- **Metal-Insulator transitions** breaking spin-rotation invariance: classes \( \text{All, D, DIII} \)

- **Gade-Wegner criticality**, line of fixed-points: \( \beta(g) = 0 \) for chiral classes: \( \text{AllI, BDI, CI} \)

- **Criticality from topological terms**
  - \( \theta \)-term: \( \mathbb{Z}_2 \) topology \((\theta = \pi)\) for classes \( \text{All and CI} \).
  - Two hypotheses: attractive fixed point to (i) finite or (ii) \( \infty \)-(ideal) conductivity \((\text{Ostrovsky, Gory, Mirlin, PRL (2007)})\)

- \( \theta \)-term: \( \mathbb{Z} \) topology for classes \( \text{A, C, D} \).
  - IQHE-like classes \( \Rightarrow \) fixed point between localized to localized

- WZW terms: Classes \( \text{AllI, CI, DIII} \).

Only one symmetry class \( \text{All} \) is always in the localized phase.
Interacting systems

(Altshuler, Aronov, SSC 1983; Finkel’stein, ZETF 1983; Castellani, Di Castro, PRB 1984)

3 scattering amplitudes (Finkel’stein, JETP 1984)

\[
\Gamma_s \text{ in p-h singlet channel: }
\]

\[
\Gamma_t \text{ in p-h triplet channel: }
\]

\[
\Gamma_c \text{ in p-p Cooper channel: }
\]

6 scattering amplitudes with chiral symmetry (Dell’Anna, NPB 2006)

\[
\Gamma_s^0 \quad \Gamma_t^0 \quad \Gamma_c^0 \quad \Gamma_s^3 \quad \Gamma_t^3 \quad \Gamma_c^3
\]

previous scattering terms

with \( k \to k + q_\pi \), where \( q_\pi = (\pi, \pi) \)
Lattice model with disorder and interactions

The interacting Hamiltonian is

\[
H = - \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} - \sum_{i, \sigma} \mu_i c_{i\sigma}^{\dagger} c_{i\sigma} + \frac{1}{2} \sum_{|k| < k_F} \sum_{p_1, p_2, \omega} \sum_{n, m}
\]

\[
\left\{ \Gamma^0_s c_{n}^{\dagger}(p_1) \sigma_0 c_{n+\omega}(p_1 + k) c_{m}^{\dagger}(p_2) \sigma_0 c_{m-\omega}(p_2 - k) \\
- \Gamma^0_t c_{n}^{\dagger}(p_1) \vec{\sigma} c_{n+\omega}(p_1 + k) c_{m}^{\dagger}(p_2) \vec{\sigma} c_{m-\omega}(p_2 - k) \\
+ \Gamma^0_c \sum_{\sigma \neq \sigma'} c_{n}^{\dagger \sigma}(p_1) c_{\omega-n}(k - p_1) c_{m}^{\sigma'}(p_2) c_{\omega-m}(k - p_2) \right\}
\]

\[(Finkel'stein, JETP 1984)\]
Lattice model with disorder and interactions

The interacting Hamiltonian is

\[ H = - \sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \frac{1}{2} \sum_{|k| \ll k_F} \sum_{p_1 p_2} \sum_{\omega_{nm}} \]

\[ \Gamma_s^0 c_n^\dagger(p_1) \sigma_0 c_{n+\omega}(p_1 + k) c_m^\dagger(p_2) \sigma_0 c_{m-\omega}(p_2 - k) \]

\[ -\Gamma_t^0 c_n^\dagger(p_1) \bar{\sigma} c_{n+\omega}(p_1 + k) c_m^\dagger(p_2) \bar{\sigma} c_{m-\omega}(p_2 - k) \]

\[ +\Gamma_c^0 \sum_{\sigma \neq \sigma'} c_n^\dagger \sigma(p_1) c_{\omega-n}(k - p_1) c_m^\dagger \sigma'(p_2) c_{\omega-m}(k - p_2) \]

\[ +\Gamma_s^3 c_n^\dagger(p_1) \sigma_0 c_{n+\omega}(p_1 + k + q_\pi) c_m^\dagger(p_2) \sigma_0 c_{m-\omega}(p_2 - k - q_\pi) \]

\[ -\Gamma_t^3 c_n^\dagger(p_1) \bar{\sigma} c_{n+\omega}(p_1 + k + q_\pi) c_m^\dagger(p_2) \bar{\sigma} c_{m-\omega}(p_2 - k - q_\pi) \]

\[ +\Gamma_c^3 \sum_{\sigma \neq \sigma'} c_n^\dagger \sigma(p_1) c_{\omega-n}(k - p_1 + q_\pi) c_m^\dagger \sigma'(p_2) c_{\omega-m}(k - p_2 + q_\pi) \]

(Dell’Anna, NPB 2006)
Interacting effective action
(Finkel’stein, JETP (1984); Dell’Anna, NPB (2006))

The corresponding effective action can be renormalized and reads

\[ S[Q] = S_{NLSM} \]

\[
\begin{align*}
- & \sum_{\alpha=0,3} \sum_{s=0,3} \sum_{\ell=0,3} \int' \text{Tr}(Q^{aa}_{n, n+\omega} \tau_\ell \sigma_0 \gamma_\alpha) \text{Tr}(Q^{aa}_{m+\omega, m} \tau_\ell \sigma_0 \gamma_\alpha) \\
+ & \sum_{\alpha=0,3} \sum_{t=0,3} \sum_{\ell=0,3} \int' \text{Tr}(Q^{aa}_{n, n+\omega} \bar{\sigma} \gamma_\alpha) \text{Tr}(Q^{aa}_{m+\omega, m} \bar{\sigma} \gamma_\alpha) \\
+ & \sum_{\alpha=0,3} \sum_{c=0,3} \sum_{\ell=1,2} \int' \text{Tr}(Q^{aa}_{n+\omega, -n} \tau_\ell \sigma_0 \gamma_\alpha) \text{Tr}(Q^{aa}_{m+\omega, -m} \tau_\ell \sigma_0 \gamma_\alpha)
\end{align*}
\]

\( \tau_i, \sigma_i, \gamma_i \) Pauli matrices in particle-hole, spin and sublattice spaces and \( \int' = \frac{\pi^2 \nu^2}{32} \int dR \sum_{nm\omega} \)
Results with interactions

Very rich and not universal behaviors of the $\beta$-functions, not uniquely determined by symmetry classes (Dell'Anna, AdP (2017))

- **Class A**
  - yes $S$, no $T$, SU(2) $\rightarrow$ U(1)
    Antiferromagnetic fluctuations induce by disorder
  - no $S$, no $T$, yes SU(2)
    RG $\rightarrow$ clean system with long-range interaction
  - no $S$, no $T$, no SU(2)
    Interaction is RG irrelevant, RG $\rightarrow$ free case

- **Class AIII**
  - yes $S$, no $T$, yes SU(2)
    Antiferromagnetic fluctuations induce by disorder
  - no $S$, no $T$, SU(2) $\rightarrow$ U(1)
    Localization (unlike free case), interactions $\rightarrow$ scale invariants

- **Class C**
  - yes $S$, no SU(2) (broken by magnetic impurities)
    Localization or Anti-localization, depending on the interaction
Results with interactions

Class AI and Class BDI

- **Far from instabilities**
  for $\Gamma^0_c > 0$, (and $\Gamma^s_3 > 0, \Gamma^t_3 < 0$ for BDI)

<table>
<thead>
<tr>
<th></th>
<th>No Interaction</th>
<th>Yes Interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AI</strong></td>
<td>Anderson Insulator</td>
<td>delocalization (Finkel’stein)</td>
</tr>
<tr>
<td><strong>BDI</strong></td>
<td>Metal (Gade-Wegner)</td>
<td>Anderson-Mott Insulator</td>
</tr>
</tbody>
</table>

- **Close to instabilities**
  - $\Gamma^0_c < 0$ can diverge under RG $\Rightarrow$ Superconductivity (SC)
  - $\Gamma^3_s < 0$ can diverge under RG $\Rightarrow$ Charge density wave (CDW)
  - $\Gamma^3_t > 0$ can diverge under RG $\Rightarrow$ Antiferromagnetism (AFM)

Since the dephasing time (time scale for the coherence to be destroyed by inelastic processes) is $\tau_\phi \sim T^{-1}$
$\Rightarrow$ temperature $T$ is the IR cutoff
*(Burmistrov, Gornyi, Mirlin, PRL 2012 (AI); Dell’Anna, PRB 2013 (BDI))*
Solving RG equations

\[ T_c \gg T_c^{BCS} \]
Enhancement of \( T_c \) for class BDI

Two counterintuitive results in the presence of disorder \((g_0)\) and interactions \((\gamma_0)\) (with \(\gamma_0 \ll g_0 \ll 1\)) in the presence of short-range repulsive interaction

- **Enhancement of superconductivity** by disorder

\[
T_c \sim (T_c^{BCS})^{-\frac{\gamma_0}{g_0}} \gg T_c^{BCS} \quad d = 2
\]

\[
T_c \sim (T_c^{BCS})^{1-g_0} \gg T_c^{BCS} \quad d = 3
\]

- **Antiferromagnetic fluctuations** driven by random hopping

\[
T_c \sim (T_c^N)^{-\frac{2\gamma_0}{3g_0}} \gg T_c^N \quad d = 2
\]

\[
T_c \sim (T_c^N)^{1-\frac{3g_0}{2}} \gg T_c^N \quad d = 3
\]

\((Dell'Anna, PRB (2013))\)

**Multifractal wavefunctions** \(\Rightarrow\) inhomogeneity of the pairing \(\Delta\)

\(\Rightarrow\) enhancement of \(T_c\) \((Feigelman, Ioffe, Kravtsov, Cuevas, AoP (2010))\)
Thank you for your attention