

Witnessing metrologically useful multiparticle entanglement

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Why multipartite entanglement and metrology are important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- We should tell
 - How entangled the state is
 - What the state is good for, etc.

1 Introduction and motivation

2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion

3 Detecting metrologically useful entanglement

- Basics of quantum metrology
- Witnessing metrological usefulness
- Metrology with measuring $\langle J_z \rangle$
- Metrology with measuring $\langle J_z^2 \rangle$
- Metrology with measuring any operator

Entanglement

A state is **(fully) separable** if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes \dots \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is **entangled** (Werner, 1989).

k -producibility/ k -entanglement

A pure state is **k -producible** if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \dots$$

where $|\Phi_i\rangle$ are states of at most k qubits.

A mixed state is k -producible, if it is a mixture of k -producible pure states.

[e.g., Gühne, GT, NJP 2005.]

- If a state is not k -producible, then it is at least **$(k + 1)$ -particle entangled**.



two-producible



three-producible

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Many-particle systems for $j=1/2$

- For spin- $\frac{1}{2}$ particles, we can measure the **collective angular momentum operators**:

$$J_I := \frac{1}{2} \sum_{k=1}^N \sigma_I^{(k)},$$

where $I = x, y, z$ and $\sigma_I^{(k)}$ are Pauli spin matrices.

- We can also measure the **variances**

$$(\Delta J_I)^2 := \langle J_I^2 \rangle - \langle J_I \rangle^2.$$

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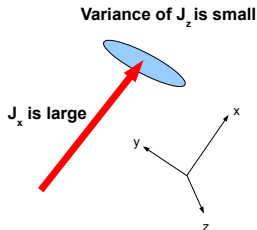
The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

If $\xi_s^2 < 1$ then the state is entangled. [Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:



Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- A full set of generalized spin squeezing criteria is known for the case above.

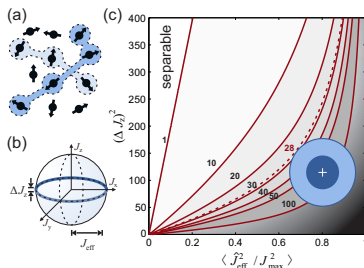
[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

[Higher spins: G. Vitagliano, P. Hyllus, I. Egusquiza, GT, Phys. Rev. Lett. 2011]

[Experiments with singlets: Behbood *et al.*, Phys. Rev. Lett. 2014;
GT, Mitchell, New. J. Phys. 2010.]

Multipartite entanglement detection with spin squeezing (only **two** criteria!)

- Original spin-squeezing method
[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001);
experimental test: C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, Nature 464, 1165 (2010).]
- Generalized method. BEC, 8000 particles.
28-particle entanglement is detected.



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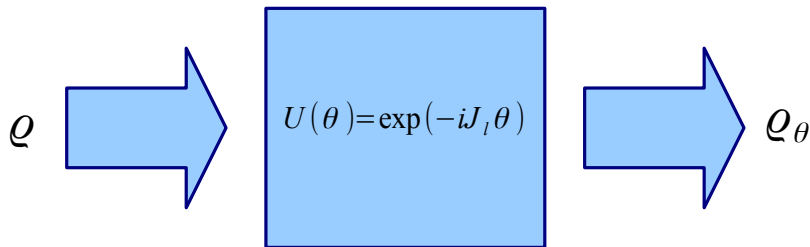
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Our main goals

- Detect **metrologically useful multipartite entanglement**, not just entanglement in general.
- Detect multipartite entanglement in the vicinity of **various** states.

Quantum metrology

- Fundamental task in metrology with a **linear interferometer**



- We have to estimate θ in the dynamics

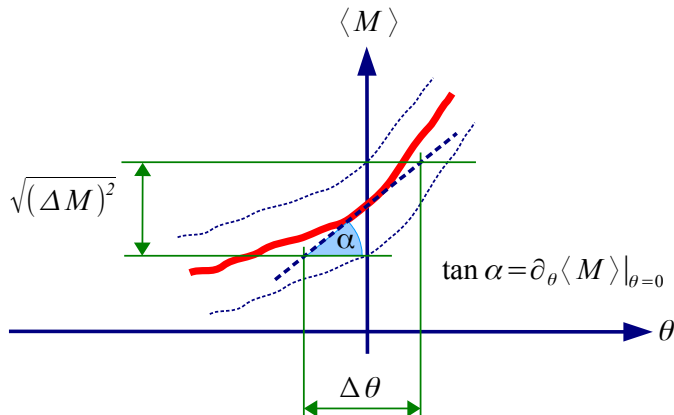
$$U = \exp(-iJ_I\theta)$$

where $I \in \{x, y, z\}$.

Precision of parameter estimation

- Measure an operator M to get the estimate θ . The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \quad \frac{1}{(\Delta\theta)^2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the **quantum Fisher information**.

- The quantum Fisher information is given by an explicit formula for ϱ and A .

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_I] \leq N.$$

[Pezze, Smerzi, PRL 2009; Hyllus, Gühne, Smerzi, PRA 2010]

- For states with at most k -particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_I] \leq kN.$$

[Hyllus *et al.*, PRA 2012; GT, PRA 2012].

- If a state violates the above inequality then it has $(k + 1)$ -particle **metrologically useful entanglement**.

Metrological precision vs. entanglement

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{N}.$$

[Pezze, Smerzi, PRL 2009; Hyllus, Gühne, Smerzi, PRA 2010]

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Witnessing metrological usefulness

- Direct measurement of the sensitivity
 - Measure $(\Delta\theta)^2$.
 - Obtain bound on F_Q and multipartite entanglement, $F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2}$.
 - Experimentally challenging, since **we need quantum dynamics**.
 - The precision is affected by the **noise during the dynamics**.

[Experiments in cold atoms by the groups of M. Oberthaler, C. Klempt; photonic experiments of the Weinfurter group.]

- Witnessing (our choice)
 - Estimate how good the precision were, **if we did the metrological process**.
 - Assume a perfect metrological process. **Characterizes the state only**.

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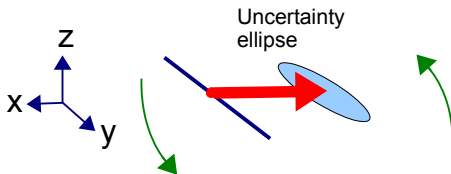
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Metrology with spin-squeezed states

- Pezze-Smerzi bound

$$(\Delta\theta)^2 = \frac{(\Delta J_z)^2}{|\partial_\theta \langle J_z \rangle|^2} = \frac{(\Delta J_z)^2}{\langle J_x \rangle^2} = \frac{\xi_s^2}{N}.$$

- We measure $\langle J_z \rangle$.



[Pezze, Smerzi, PRL 2009.]

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Metrology with Dicke states

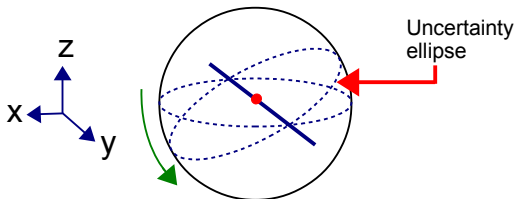
- For Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ . (We cannot measure first moments, since they are zero.)



Formula for maximal precision II

Maximal precision with a closed formula

$$(\Delta\theta)_{\text{opt}}^2 = \frac{2\sqrt{(\Delta J_z^2)^2(\Delta J_x^2)^2 + 4\langle J_x^2 \rangle - 3\langle J_y^2 \rangle - 2\langle J_z^2 \rangle(1 + \langle J_x^2 \rangle) + 6\langle J_z J_x^2 J_z \rangle}}{4(\langle J_x^2 \rangle - \langle J_z^2 \rangle)^2}.$$

- Collective observables, like in the spin-squeezing criterion.
- Metrological usefulness can be verified **without carrying out the metrological task**.
- Tested on experimental data.

[Apellaniz, Lücke, Peise, Klempt, GT, NJP 2015.]

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Large step: we do not assume any metrological scheme

- We would like to know how good a state is for quantum metrology.
- We allow any operator to be measured for parameter estimation.
- Thus, we need to witness the quantum Fisher information.

Measure the quantum Fisher information

- We would like to measure the quantum Fisher information.
- Related problem: For systems in thermal equilibrium

$$F_Q(T) = \frac{4}{\pi} \int_0^\infty d\omega \tanh\left(\frac{\omega}{2T}\right) \chi''(\omega, T)$$

Needs measuring the dynamic susceptibility χ'' as a function of ω .
[Hauke *et al.*, Nat. Phys. 12, 778 \(2016\).](#)

- We have systems **not in thermal equilibrium**, and can measure only a **few** operators.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \psi_k} \sum_k p_k (\Delta A)_k^2,$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Legendre transform

- Optimal linear lower bound on a convex function $g(\varrho)$ based on an operator expectation value $w = \langle W \rangle_{\varrho} = \text{Tr}(W\varrho)$

$$g(\varrho) \geq rw - \text{const.},$$

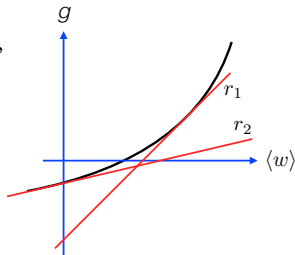
where $w = \text{Tr}(\varrho W)$.

- For every slope r there is a “const.”
- Textbooks say

$$g(\varrho) \geq \mathcal{B}(w) := rw - \hat{g}(rW),$$

where $\hat{g}$ is the Legendre transform

$$\hat{g}(W) = \sup_{\varrho} [\langle W \rangle_{\varrho} - g(\varrho)].$$



Legendre transform II

- Bound is best if we optimize over r as

$$g(\varrho) \geq \mathcal{B}(w) := \sup_r [rw - \hat{g}(rW)],$$

where again $w = \text{Tr}(\varrho W)$.

- F_Q is the convex roof of the variance. Hence, it is sufficient to carry out an optimization over pure states

$$\hat{g}(W) = \sup_{\Psi} [\langle W \rangle_{\Psi} - g(\Psi)].$$

- Similar simplification has been used for entanglement measures.

[Gühne, Reimpell, Werner, PRL 2007; Eisert, Brandao, Audenaert, NJP 2007.]

Legendre transform III

- For our case, the Legendre transform is

$$\hat{\mathcal{F}}_Q(W) = \sup_{\Psi} [\langle W - 4J_I^2 \rangle_{\Psi} + 4 \langle J_I \rangle_{\Psi}^2].$$

- With further simplifications, **an optimization over a single real variable** is needed

$$\hat{\mathcal{F}}_Q(W) = \sup_{\mu} \left\{ \lambda_{\max} \left[W - 4(J_I - \mu)^2 \right] \right\}.$$

Legendre transform IV

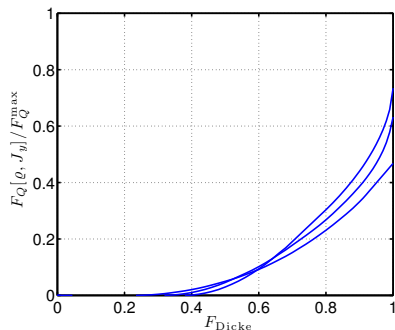
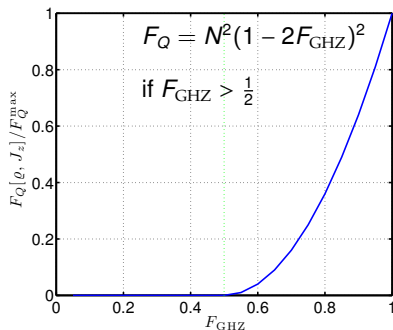
Big surprise

The quantum Fisher information is the **ideal quantity** for using the Legendre transform technique.

Witnessing the quantum Fisher information based on the fidelity

- Let us bound the quantum Fisher information based on some measurements. First, consider small systems.

[See also Augusiak *et al.*, 1506.08837.]



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, PRA 95, 032330 (2017).]

Bounding the qFi based on collective measurements

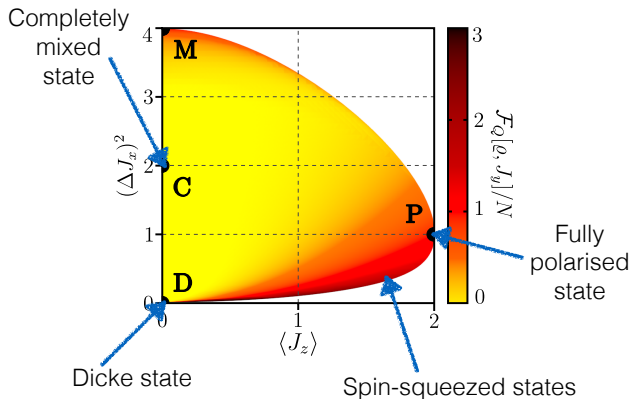
Bound for the quantum Fisher information for spin squeezed states
(Pezze-Smerzi bound)

$$F_Q[\varrho, J_y] \geq \frac{\langle J_z \rangle^2}{(\Delta J_x)^2}.$$

[Pezze, Smerzi, PRL 2009.]

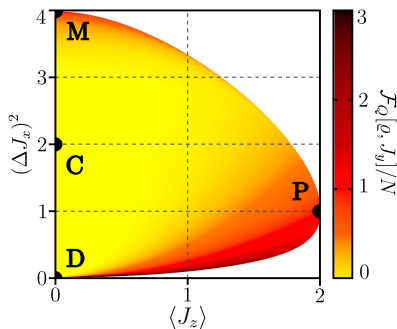
Bounding the qFi based on collective measurements II

- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles



Bounding the qFi based on collective measurements III

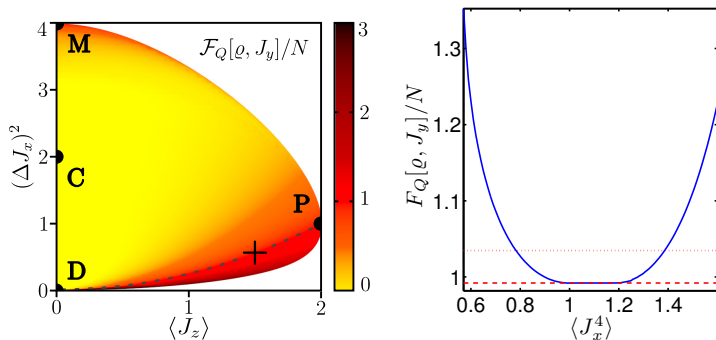
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On the bottom part of the figure [$(\Delta J_x)^2 < 1$] the bound is very close to the Pezze-Smerzi bound!

Bounding the qFi based on collective measurements IV

- The bound can be obtained if additional expectation value, i.e., $\langle J_x^2 \rangle$ is measured, or we assume symmetry:



[Apellaniz, Kleinman, Gühne, GT, PRA 95, 032330 (2017).]

Spin squeezing experiment

- Experiment with $N = 2300$ atoms,

$$\xi_s^2 = -8.2\text{dB} = 10^{-8.2/10} = 0.1514.$$

[Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 2010.]

- The Pezze-Smerzi bound is:

$$\frac{\mathcal{F}_Q[\varrho_N, J_y]}{N} \geq \frac{1}{\xi_s^2} = 6.605.$$

- We get the same value for our method!

[Pezze, Smerzi, PRL 2009]

- Similar calculations for Dicke state experiments!

[Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt, PRL 2014.]

- Lower bound on the quantum Fisher information with the variance and the purity

$$(\Delta J_I)^2 - \frac{1}{4} F_Q[\varrho, J_I] \leq \frac{N^2}{2} [1 - \text{Tr}(\varrho^2)].$$

[G. Tóth, arXiv:1701.07461.]

Summary

- We discussed a **very flexible** method to detect multipartite entanglement and metrological usefulness.
- We can choose a set of operators and the method gives an optimal lower bound on F_Q .

Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015);

Apellaniz, Kleinmann, Gühne, GT, Phys. Rev. A 95, 032330 (2017),
Editors' Suggestion.

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