Topology of density matrices and their detection

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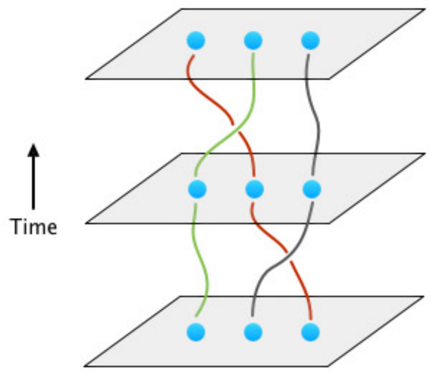
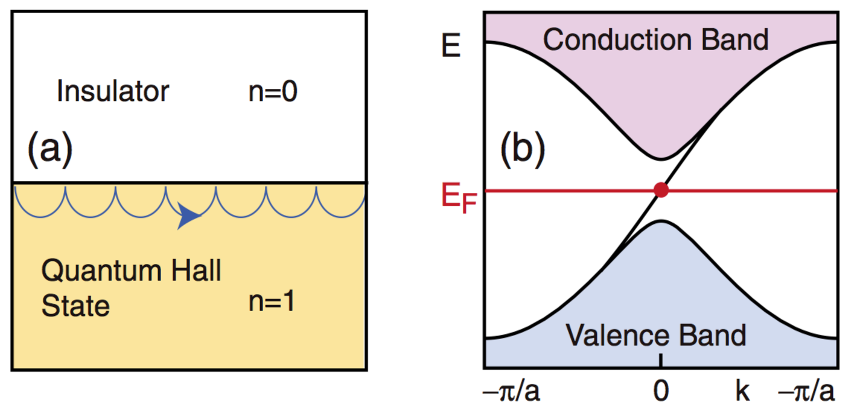
Adv. School & Workshop on Quantum Science & Technolog, 11.09.2017

picture: wikipedia

exotic quantum states

topological protection

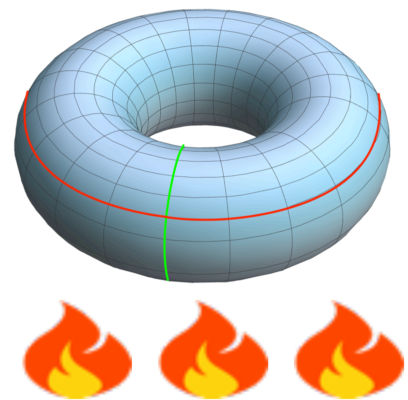




Abelian & non-Abelian anyons

protected edge states & edge transport

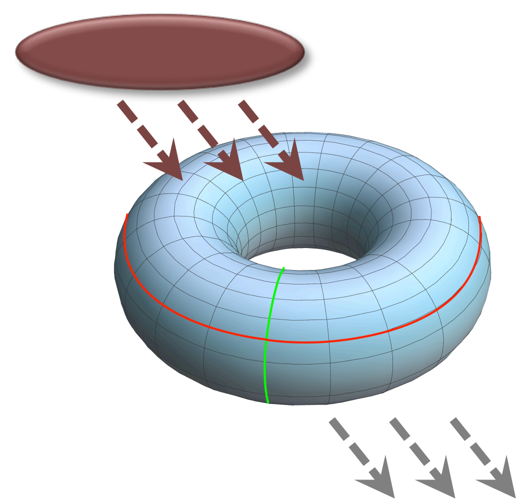
- **topology at finite T: what is left ??**



- **topology in non-equilibrium driven, open systems??**

- topology at finite T: what is left ??

- **topology in non-equilibrium driven, open systems??**



the 10-fold way

- **Topological insulators of non-interacting fermions**

$$H = \sum_{ij} h_{ij}\, \hat{c}_i^{(\dagger)} \hat{c}_j$$

$\mathcal{T}$ $\qquad$ $\mathcal{C}$ $\qquad$ $\mathcal{S} = \mathcal{T}\mathcal{C}$

$$U_T^\dagger h^* U_T = +h \qquad U_C^\dagger h^* U_C = -h$$

Cartan label	T	C	S	Hamiltonian
A (unitary)	0	0	0	$\mathrm{U}(N)$
AI (orthogonal)	+1	0	0	$\mathrm{U}(N)/\mathrm{O}(N)$
AII (symplectic)	−1	0	0	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$
AIII (ch. unit.)	0	0	1	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$
BDI (ch. orth.)	+1	+1	1	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$
CII (ch. sympl.)	−1	−1	1	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$
D (BdG)	0	+1	0	$\mathrm{SO}(2N)$
C (BdG)	0	−1	0	$\mathrm{Sp}(2N)$
DIII (BdG)	−1	+1	1	$\mathrm{SO}(2N)/\mathrm{U}(N)$
CI (BdG)	+1	−1	1	$\mathrm{Sp}(2N)/\mathrm{U}(N)$

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mixed states of non-interacting fermions

- **Non-interacting (Gaussian) open systems**

density matrix

$$\rho \sim \exp\Big\{-\sum_{ij} \hat{c}_i^\dagger G_{ij} c_j\Big\}$$

single-particle correlations

$$[\tanh(G/2)]_{ij} = \langle [\hat{c}_i^\dagger, \hat{c}_j] \rangle$$

$$H = \sum_{ij} h_{ij}\, \hat{c}_i^{(\dagger)} \hat{c}_j \qquad L_j \sim \alpha\, \hat{c}_j^\dagger + \beta\, \hat{c}_j$$

outline

- topological invariants & polarization
- topological pumps
- Ensemble Geometric Phase
- detecting polarization & realizing the effective Hamiltonian

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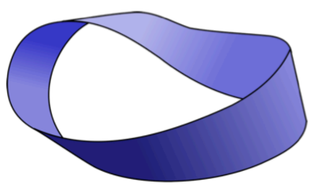
- **detecting polarization & realizing the effective Hamiltonian**

topological invariants & polarization



topological invariants: geometric phases

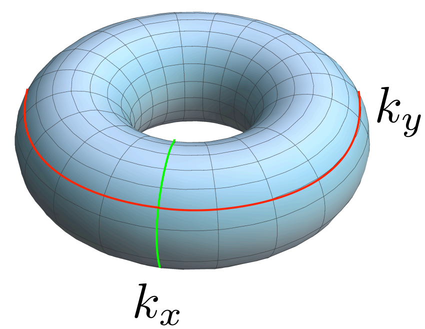
- **Zak (Berry) phase**

$$\phi_{\mathrm{Zak}} = \int_{-\pi/a}^{\pi/a} dk \, \langle u_k | i\partial_k | u_k \rangle$$

- **1D: winding number**

$$\hat{H} = \hat{H}(\lambda)$$

$$\nu = \frac{1}{2\pi} \oint d\lambda \, \frac{\partial \phi_{\mathrm{Zak}}}{\partial \lambda}$$



- **2D: Chern number**

$$C = \frac{i}{2\pi} \iint_{\mathrm{BZ}} \mathrm{d}^2 k \sum_{\alpha} \Big\{ \langle \partial_{k_y} u_k^{\alpha} | \partial_{k_x} u_k^{\alpha} \rangle - \langle \partial_{k_x} u_k^{\alpha} | \partial_{k_y} u_k^{\alpha} \rangle \Big\}$$

topological phases & quantized bulk transport

Thouless, Kohmoto, Nightingale, den Nijs (TKNN) PRL (1982)

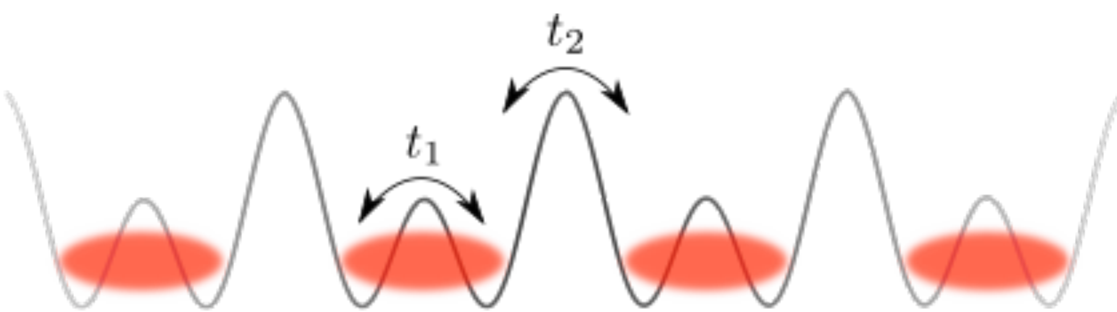
$$\Delta n = \frac{1}{2\pi} \oint d\lambda \, \frac{\partial \phi_{\mathrm{Zak}}}{\partial \lambda}$$

symmetry-protected topological order

- **Su-Shrieffer-Heeger model**

$$\hat{\mathcal{H}}_{\mathrm{SSH}} = -t_1 \sum_{j:\mathrm{even}} a^\dagger_{j+1} a_j - t_2 \sum_{j:\mathrm{odd}} a^\dagger_{j+1} a_j + \mathrm{h.a.}$$

chiral symmetry $\left\{\hat{\mathcal{H}}, \hat{\Sigma}_z\right\} = 0$



Su, Schrieffer, Heeger. Phys. Rev. Lett. (1979)

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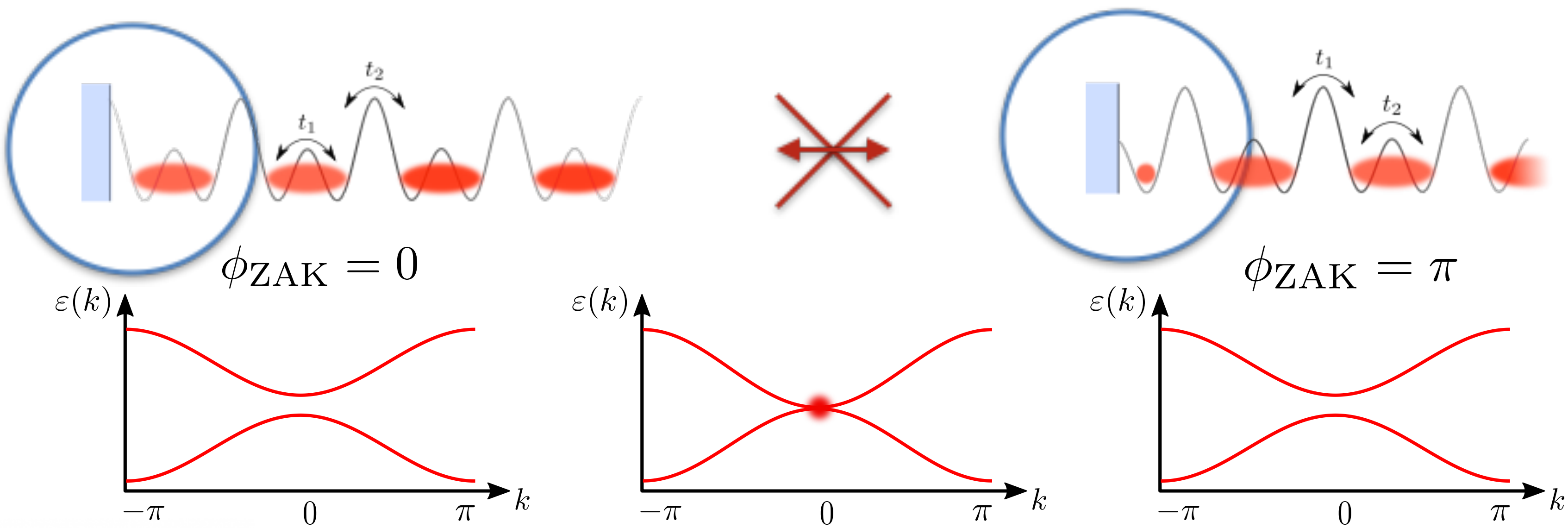
topological phases

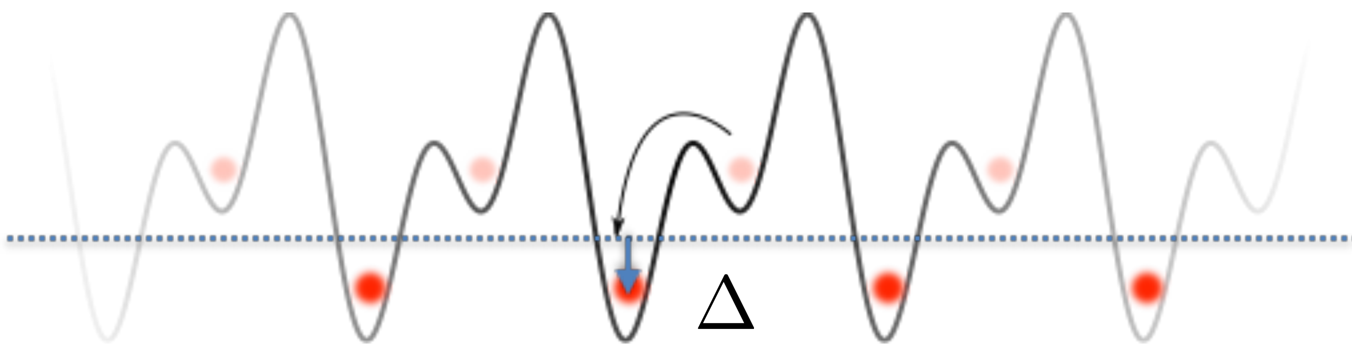


$t_1 < t_2$ $t_1 = t_2$ $t_1 > t_2$

charge pumps (Thouless)

- **Rice-Mele model**

$$\hat{\mathcal{H}}_{\mathrm{RM}} = \mathcal{H}_{\mathrm{SSH}} + \Delta \sum_j (-1)^j a_j^\dagger a_j \qquad \left\{\hat{\mathcal{H}}, \hat{\Sigma}_z\right\} \neq 0$$



Rice & Mele, Phys. Rev. Lett. (1982)

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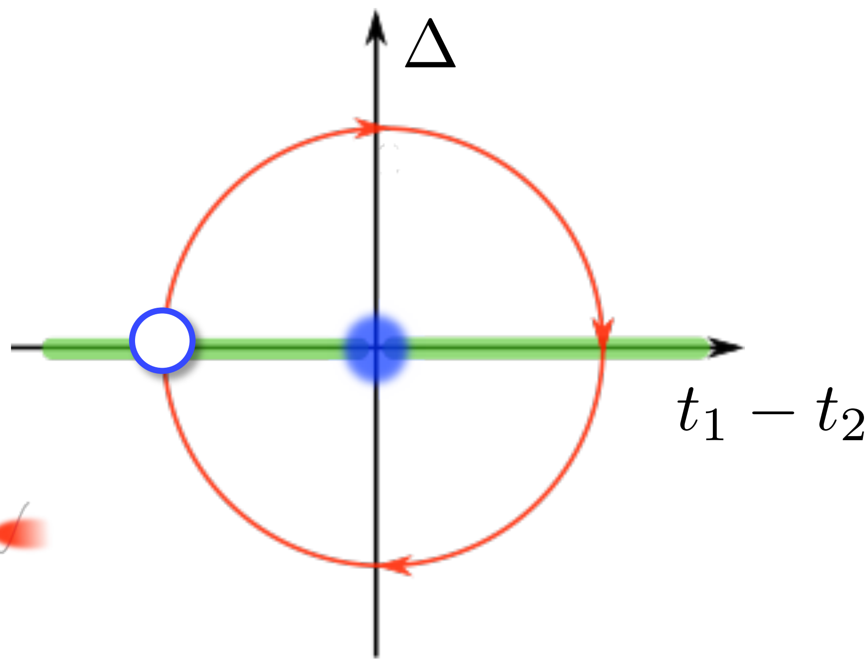

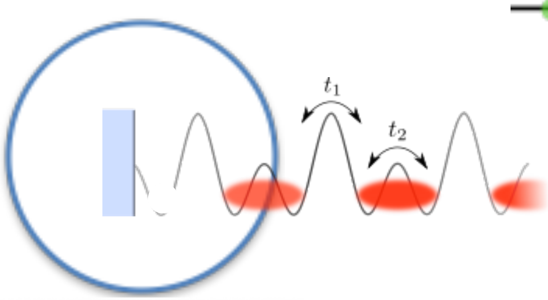

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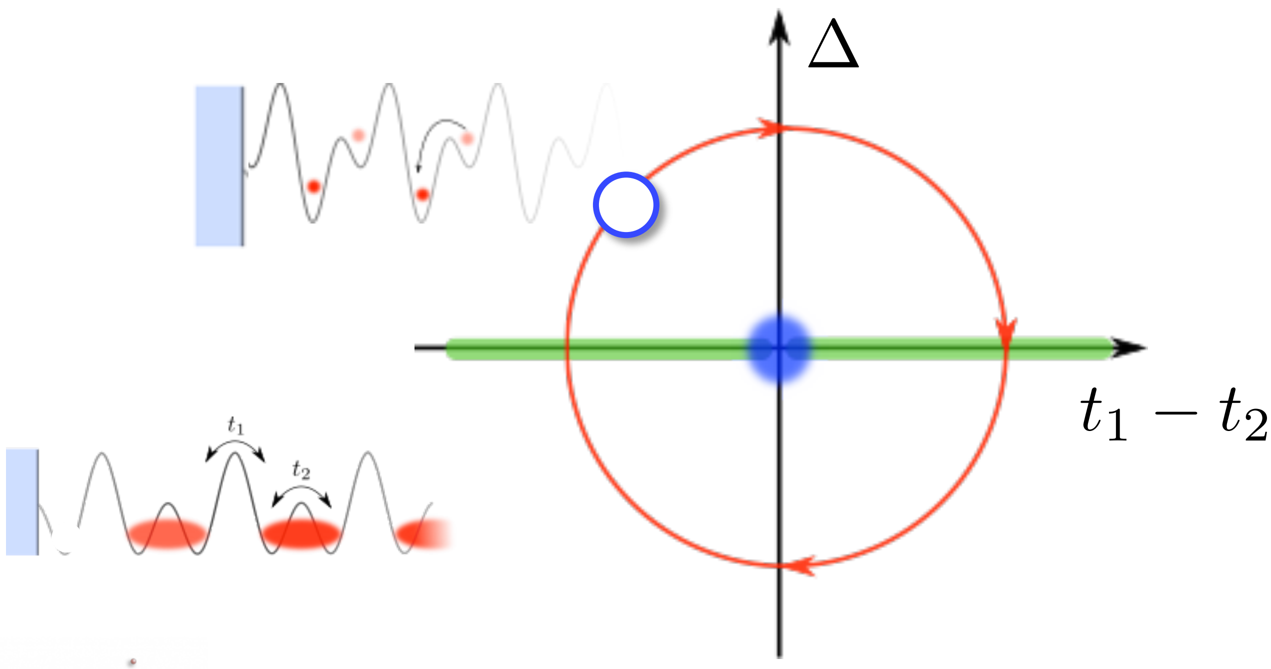

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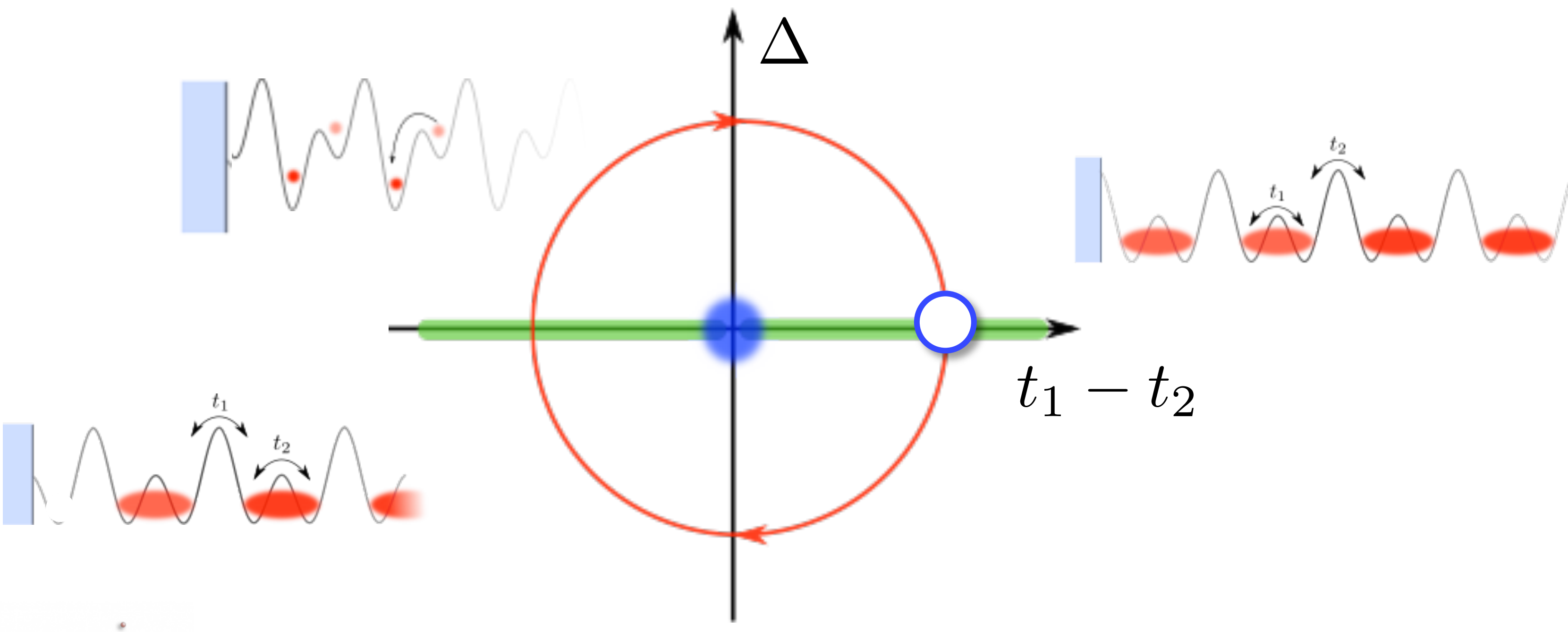

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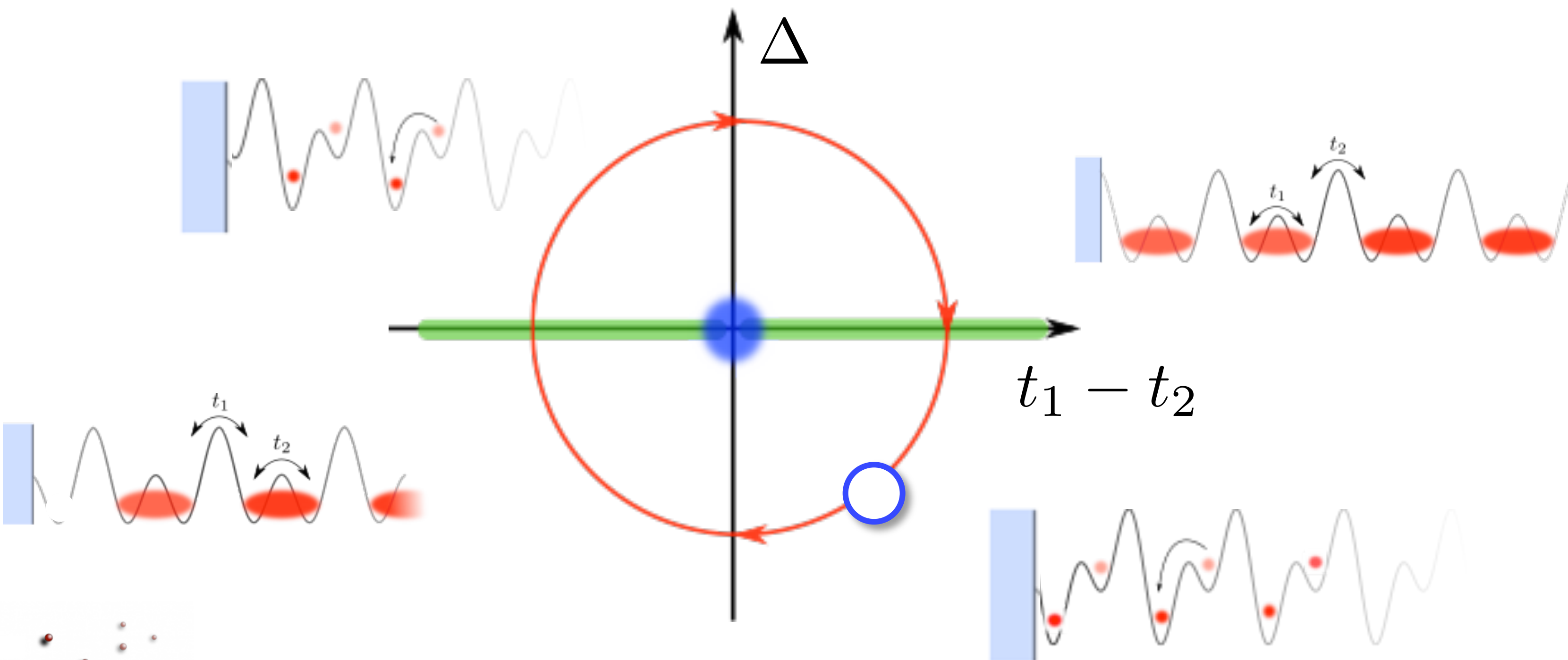

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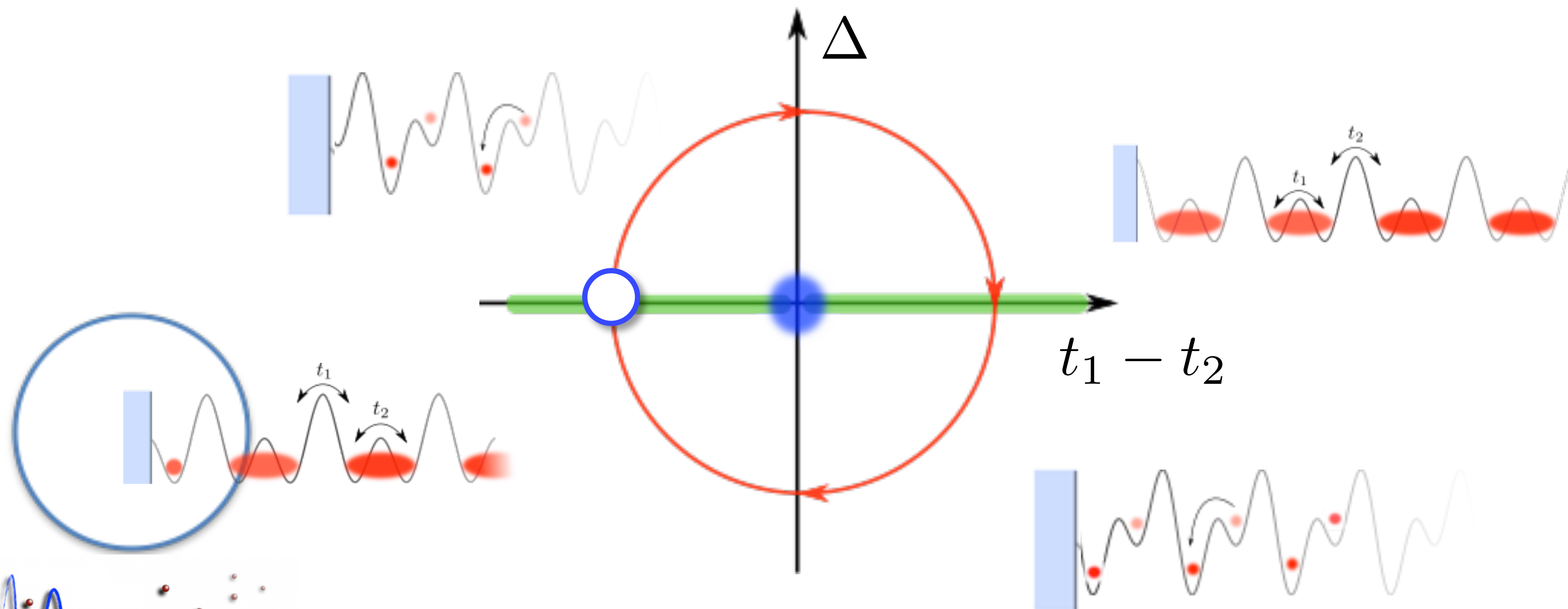


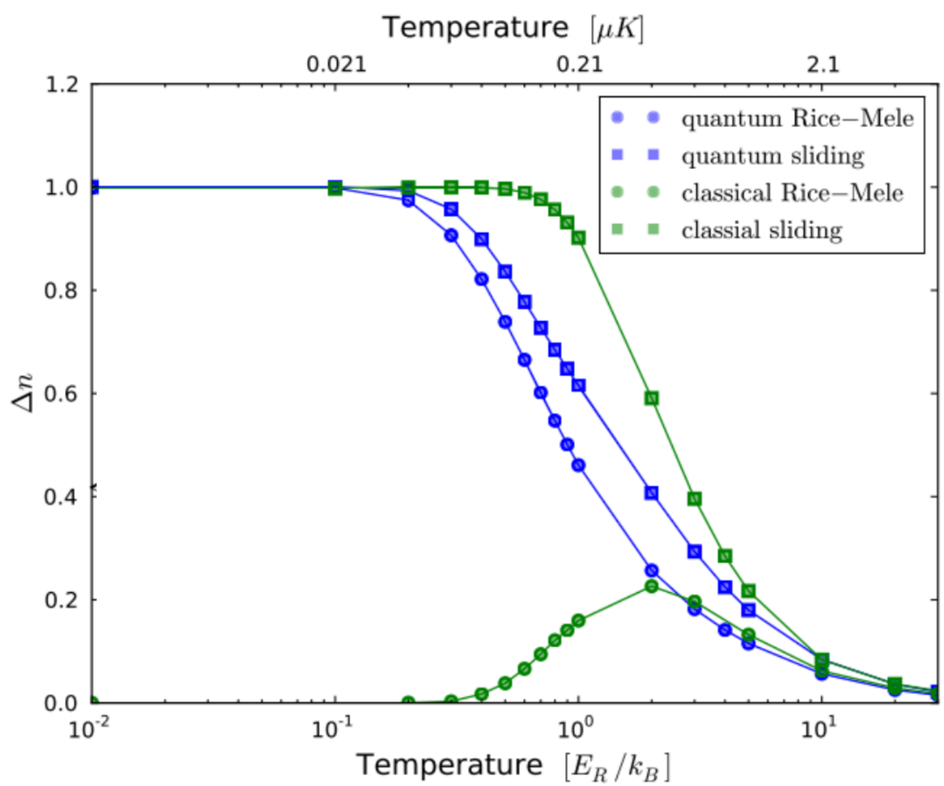
geometric phase

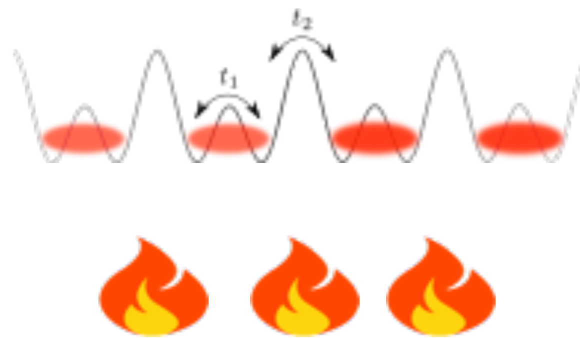
topological pumps

$$\Delta\phi_{\mathrm{Zak}} \longleftrightarrow \Delta n$$

mixed states

charge pumps at finite T





particle transport no longer quantized

Wang, Troyer, Dai, Phys. Rev. Lett. **111**, 026802 (2013)

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geometric phase

$$\Delta\phi_{\mathrm{Zak}}$$

mixed states

topological pumps

$$\Delta n$$

polarization to quantify topology

geometric phase $\Delta\phi_{\mathrm{Zak}}$ ↔ **topological pumps** Δn

$$\Delta P$$

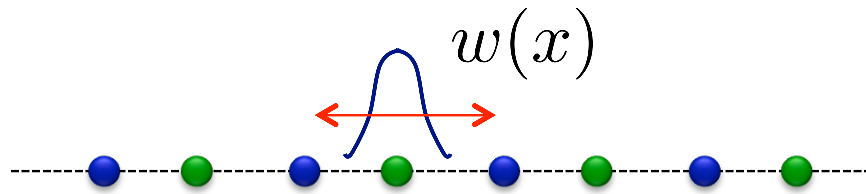
polarization

polarization

King-Smith, Vanderbildt PRB (1983)

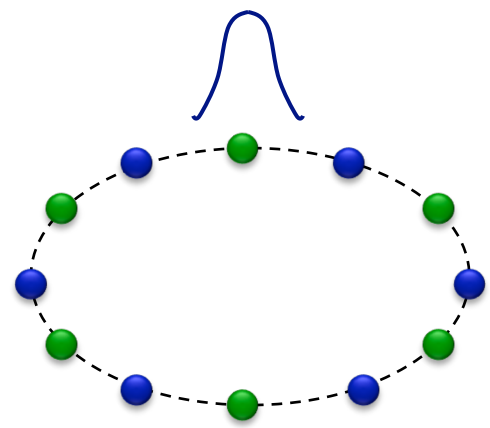
$$\Delta\phi_{\mathrm{Zak}} = \frac{2\pi}{a}\Delta P$$

$$P = \int dx\, w^*(x)\, x\, w(x)$$



R. Resta PRL **80**, 1800 (1998)

$$P = \frac{1}{2\pi}\mathrm{Im}\ln\left\langle \exp\left\{ i\frac{2\pi}{L}\hat{X}\right\}\right\rangle$$



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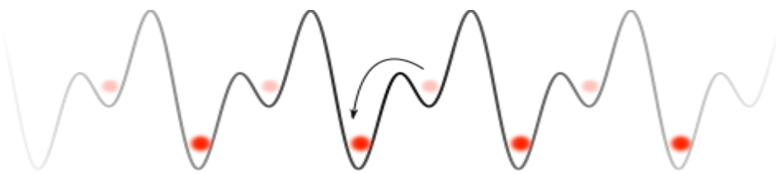
topological pumps



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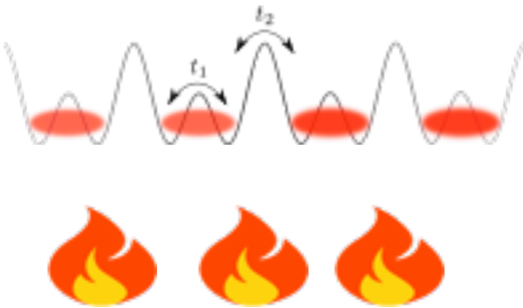
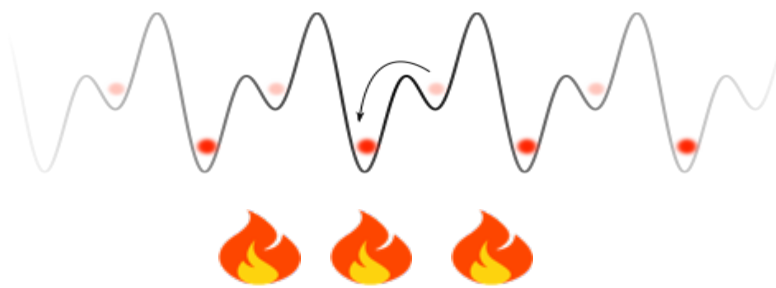
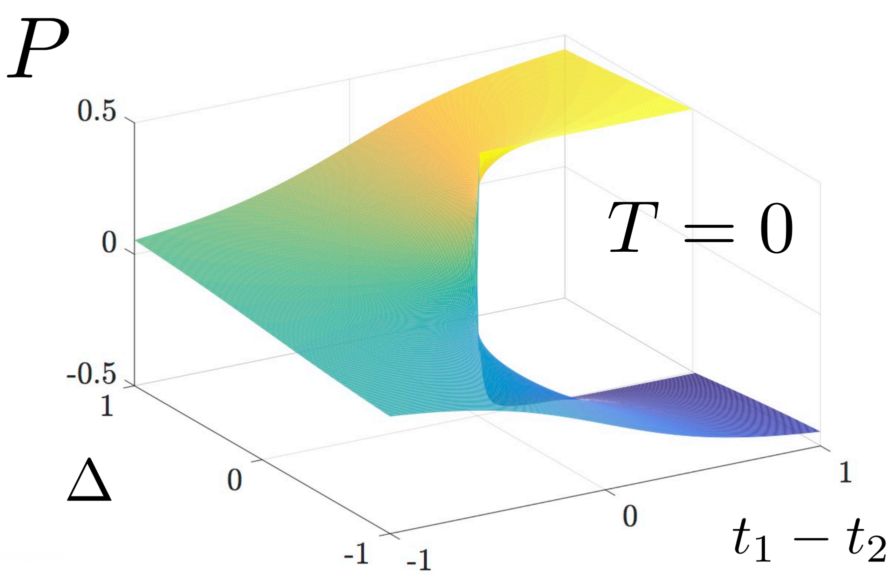
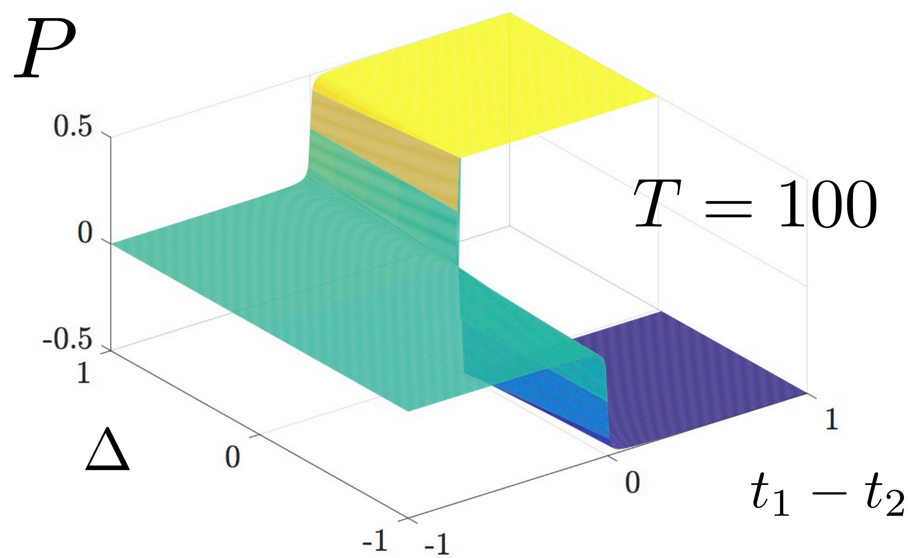
I. finite-temperature Rice-Mele model



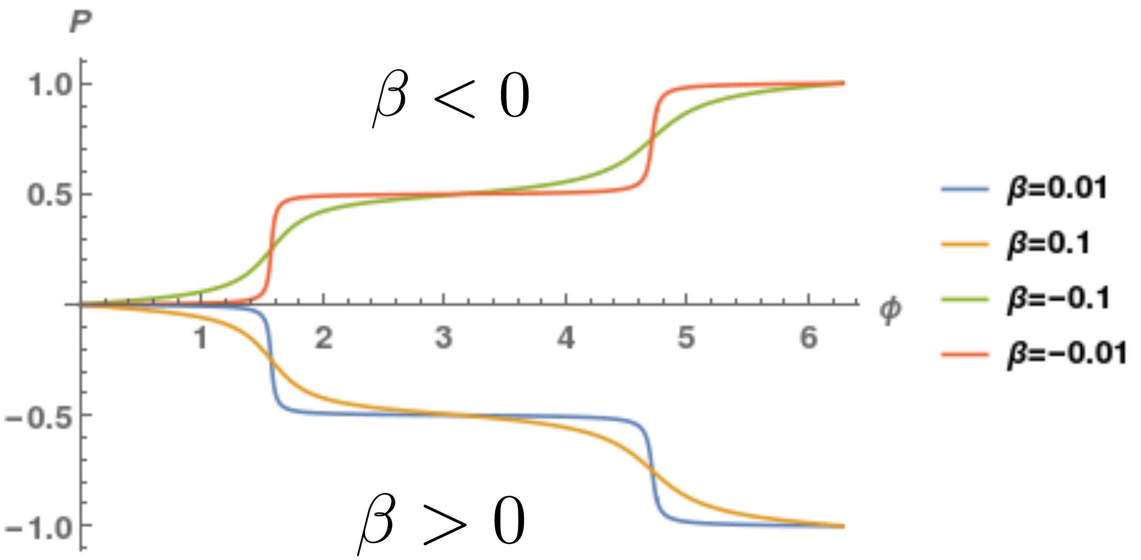
finite-T Rice-Mele

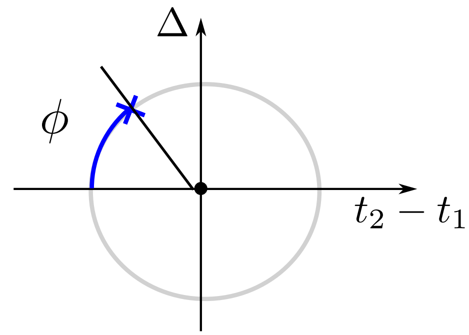






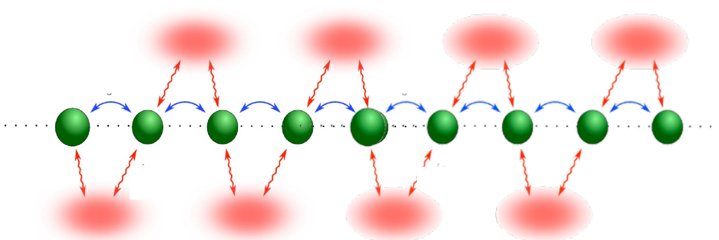
finite-T Rice-Mele





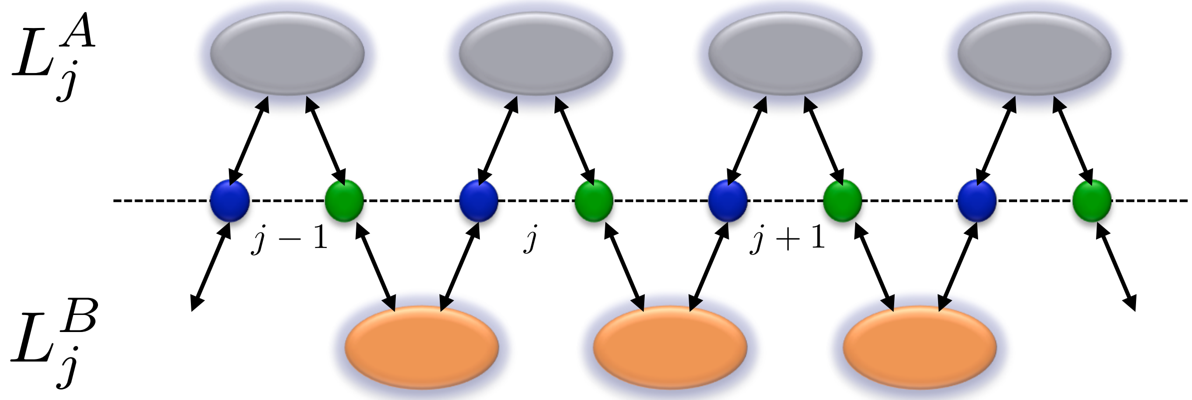
- polarization winding remains non-trivial for all T
- changes sign at T = ∞, i.e. going to negative T

II. reservoir-induced topological pump



D. Linzner, L. Wawer, F. Grusdt, M. F., PRB (R) **94**, 201105 (2016)

model



$$\dot{\rho} = \mathcal{L}\rho = \sum_{j,\mu} \left(2L_j^\mu \rho L_j^{\mu\dagger} - L_j^{\mu\dagger} L_j^\mu \rho - \rho L_j^{\mu\dagger} L_j^\mu \right)$$

Lindblad generators

$$L_j^A = \sqrt{1+\varepsilon} \left[(1-\lambda) \left(\hat{c}_{L,j} + \hat{c}_{R,j}^\dagger \right) + (1+\lambda) \left(\hat{c}_{L,j}^\dagger - \hat{c}_{R,j} \right) \right]$$

$$L_j^B = \sqrt{1-\varepsilon} \left[(1-\lambda) \left(\hat{c}_{L,j+1} + \hat{c}_{R,j}^\dagger \right) + (1+\lambda) \left(\hat{c}_{L,j+1}^\dagger - \hat{c}_{R,j} \right) \right]$$

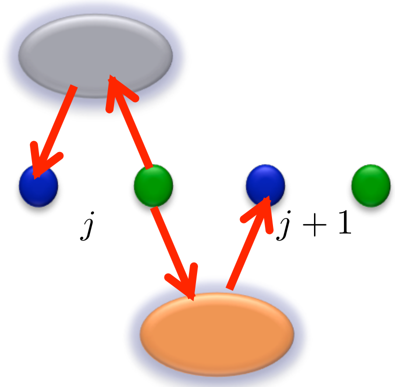
model

- **action of Lindblad generators**

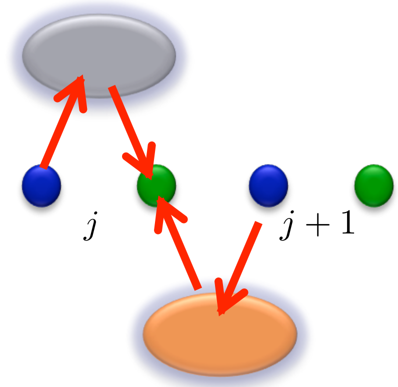
$$L_j^A = \sqrt{1+\varepsilon}\left[(1-\lambda)\left(\hat{c}_{L,j}+\hat{c}^\dagger_{R,j}\right)+(1+\lambda)\left(\hat{c}^\dagger_{L,j}-\hat{c}_{R,j}\right)\right]$$

$$L_j^B = \sqrt{1-\varepsilon}\left[(1-\lambda)\left(\hat{c}_{L,j+1}+\hat{c}^\dagger_{R,j}\right)+(1+\lambda)\left(\hat{c}^\dagger_{L,j+1}-\hat{c}_{R,j}\right)\right]$$

$\lambda = +1$

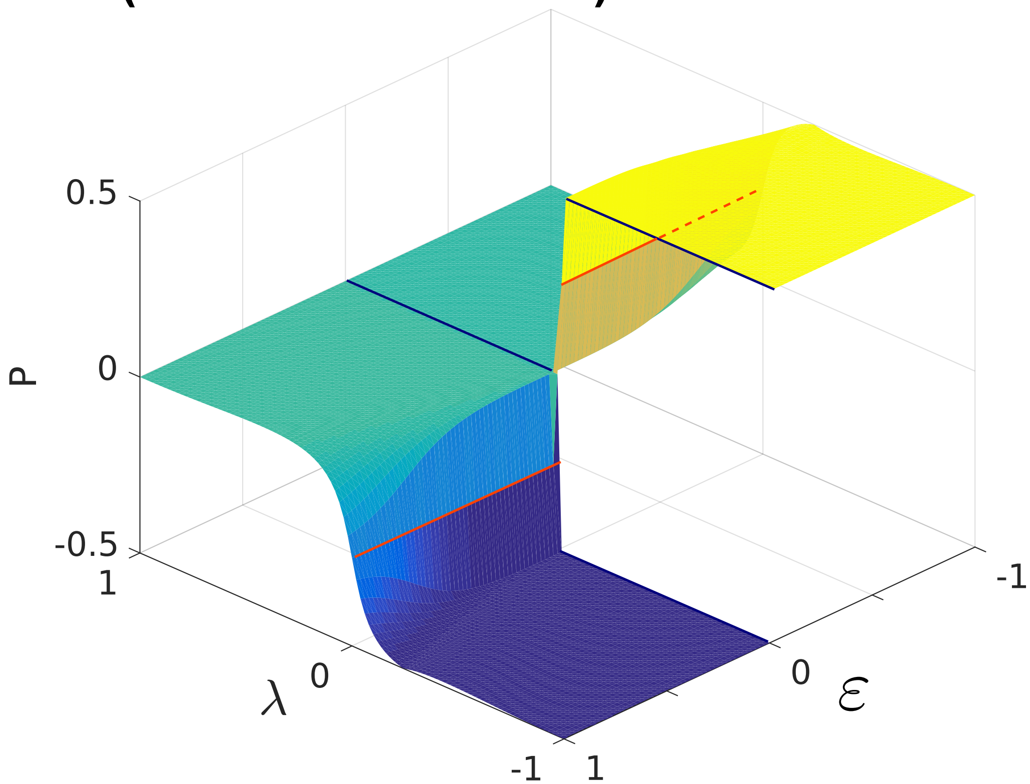


$\lambda = -1$



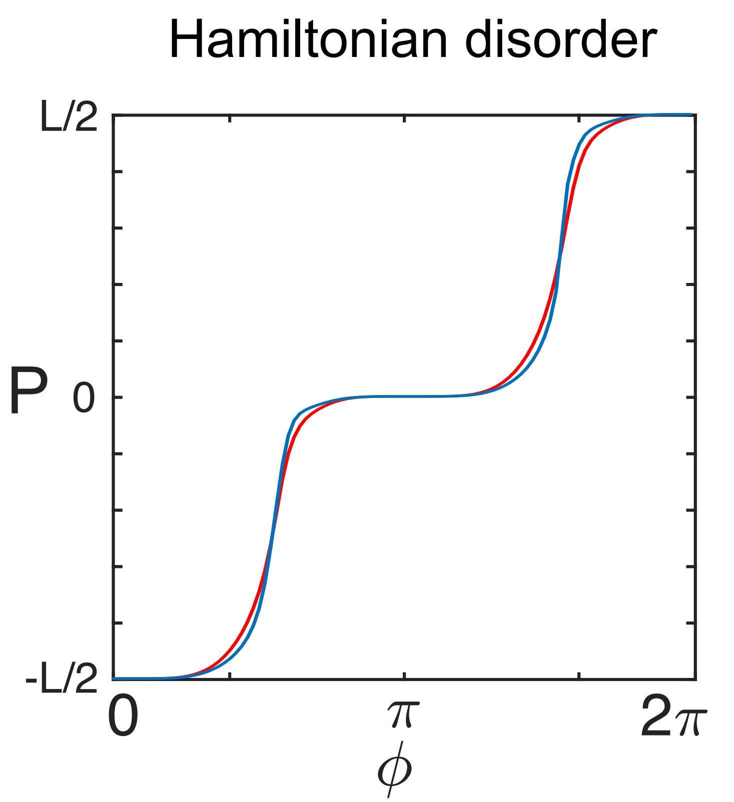
steady-state Thouless pump

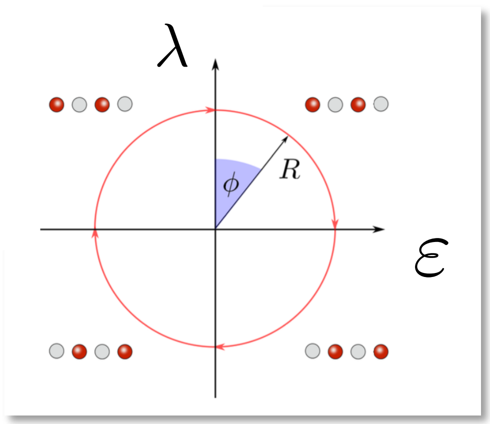
- **Polarization (TEBD simulations)**



winding of P → topological invariant !

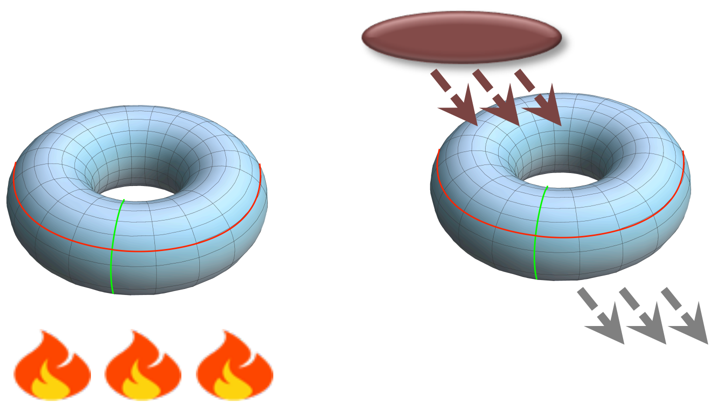
robustness





robust to disorder and losses

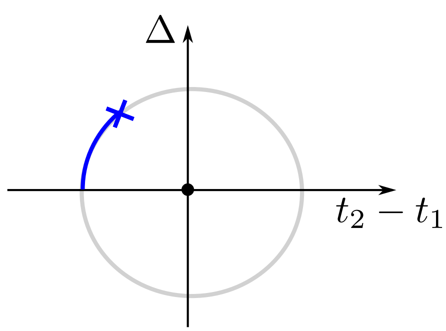
Ensemble Geometric Phase

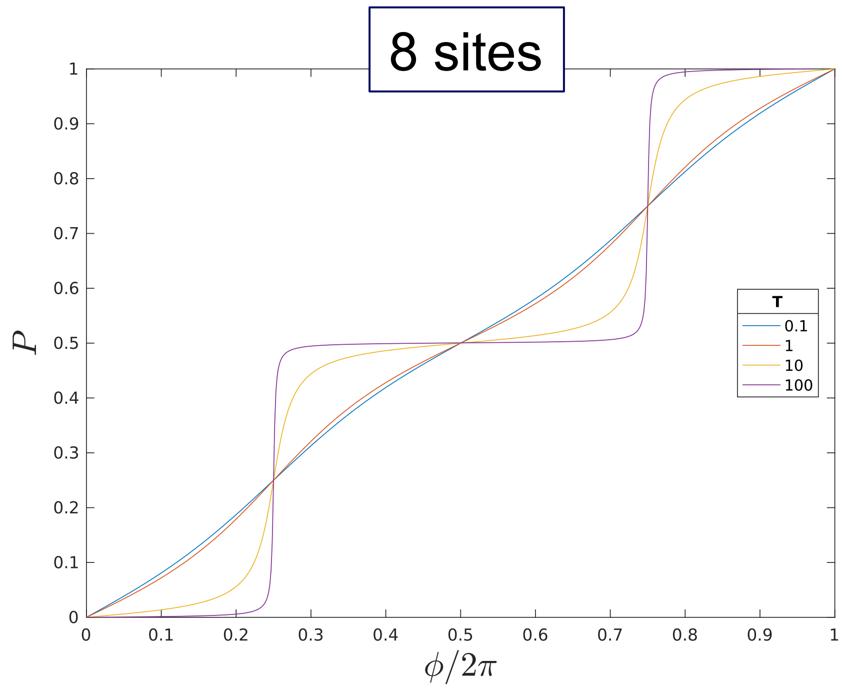


ensemble geometric phase

Bardyn, Wawer, Altland, Fleischhauer, Diehl (arxiv: 1706.02741)

- **finite-T Rice Mele model (Thouless charge pump)**





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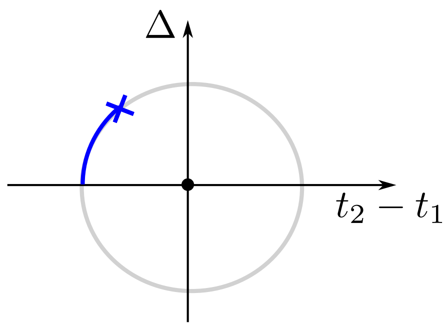

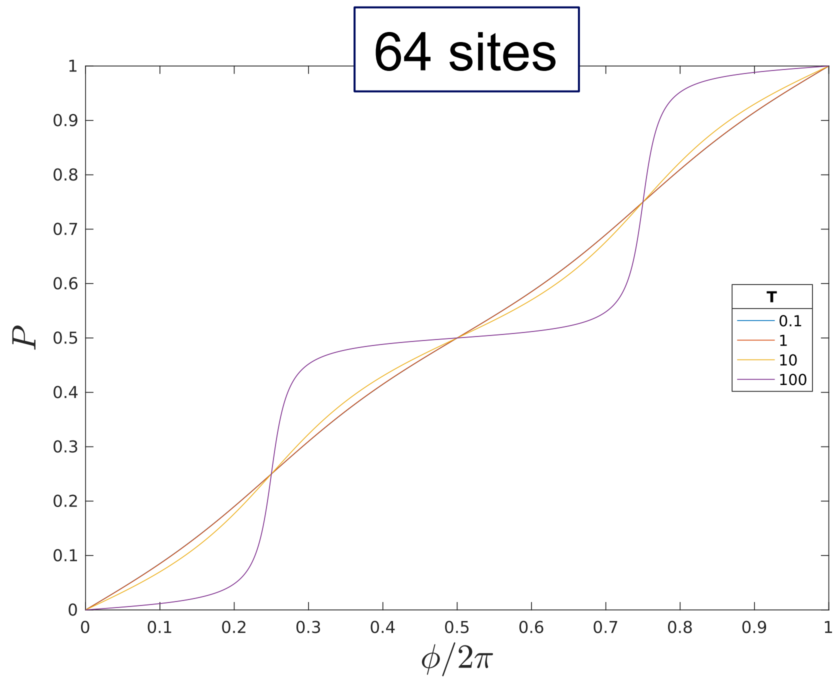

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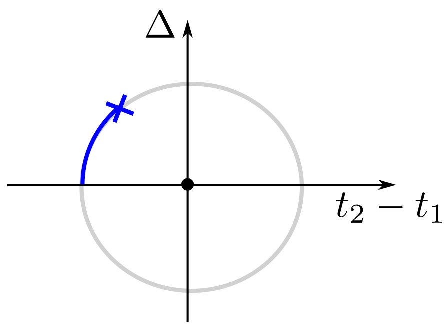

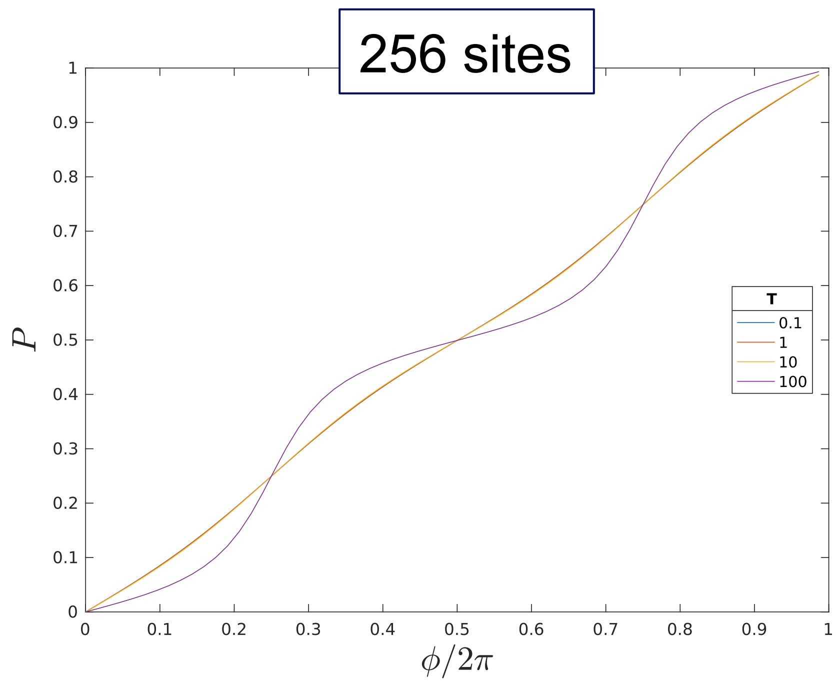

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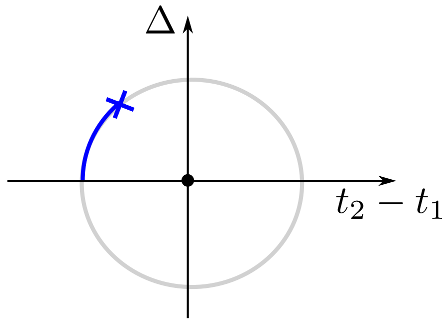
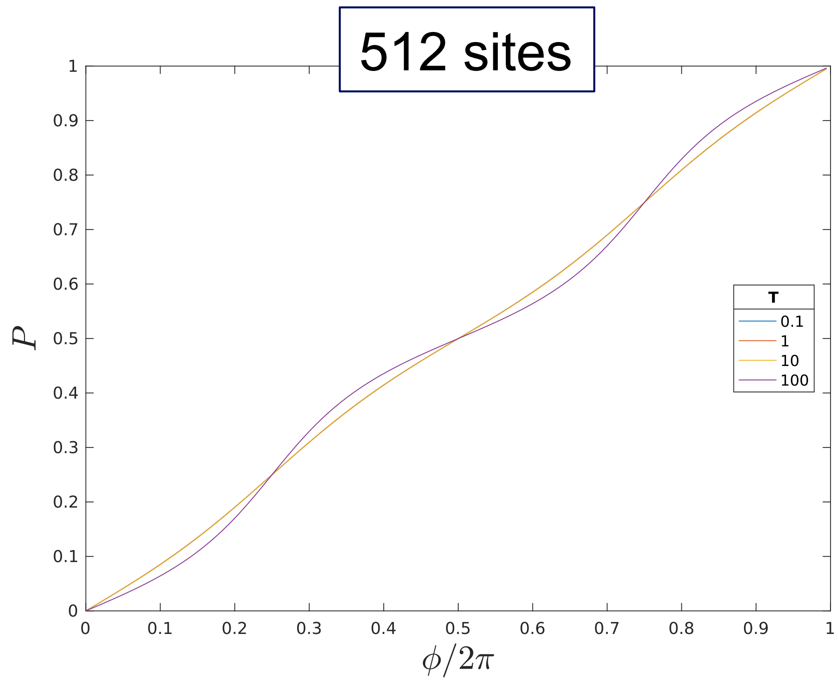




Ensemble Geometric Phase

Bardyn, Wawer, Altland, Fleischhauer, Diehl (arxiv: 1706.02741)

$$P(\rho_{\rm ss}) = P\big(|\psi\rangle\langle\psi|\big) + \mathcal{O}\left(L^{-1}\right)$$

$|\psi\rangle$ ground state of $$H_{\rm eff} = \sum_{ij} G_{ij}\hat{c}_i^\dagger \hat{c}_j$$

$$\Delta P(\rho_{\rm ss}) = \oint d\lambda \, \frac{\partial}{\partial\lambda} P(\rho_{\rm ss}) = \Delta P(|\psi\rangle\langle\psi|)$$

$$\Delta\phi_{\rm EGP} = \frac{2\pi}{a}\Delta P$$ = Zak phase of $|\psi\rangle$

topological classification

$$H_{\text{eff}} = \sum_{ij} G_{ij} \hat{c}_i^\dagger \hat{c}_j$$

- **symmetries of effective Hamiltonian classify topology**

  C.E. Bardyn, et al. New J. Phys (2013)

- **topological phase transitions**

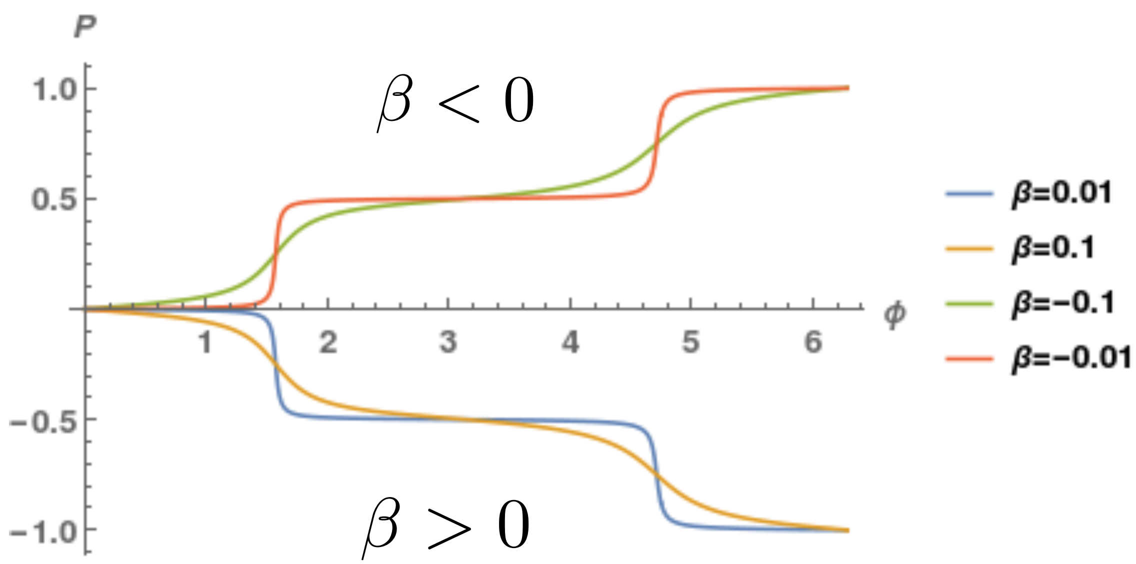
  (I) closing of the damping gap (criticality)

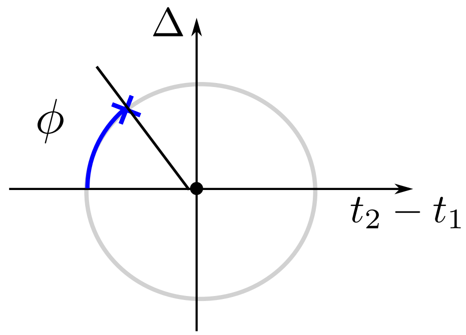
  (II) closing of the purity gap = gap of effective Hamiltonian

- **extention to interacting systems ?**

  see also: V. Gurarie, PRB (2011)

finite-T Rice-Mele

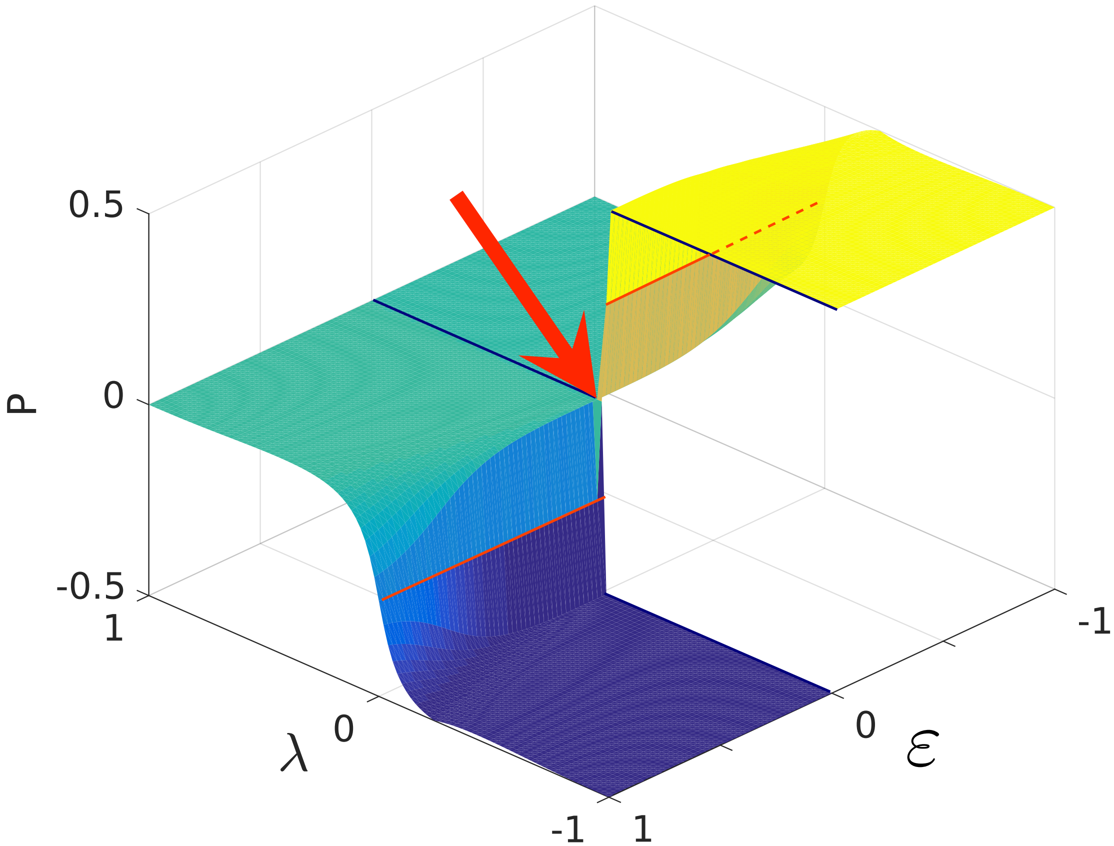




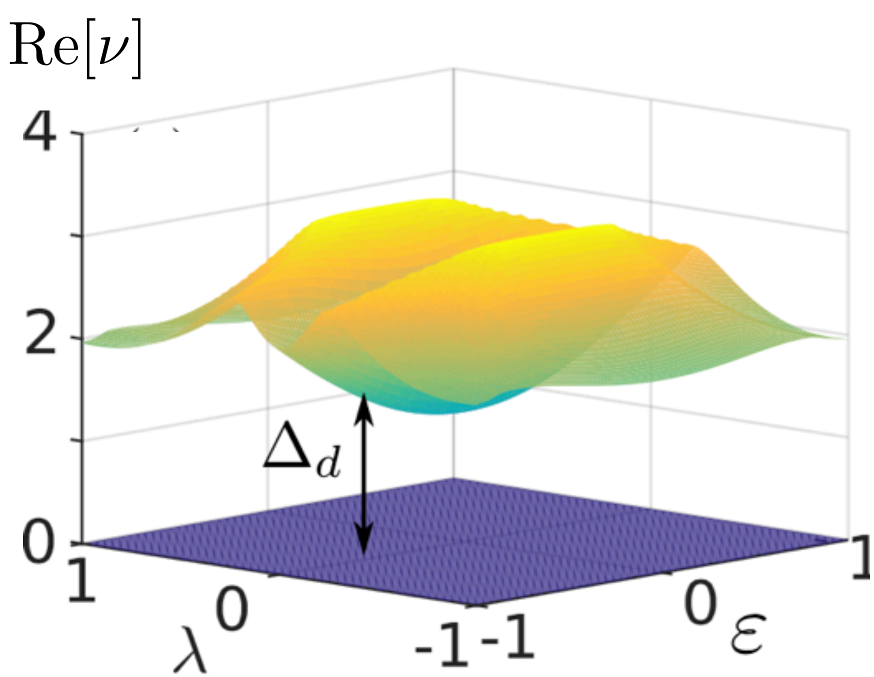
- thermal state:

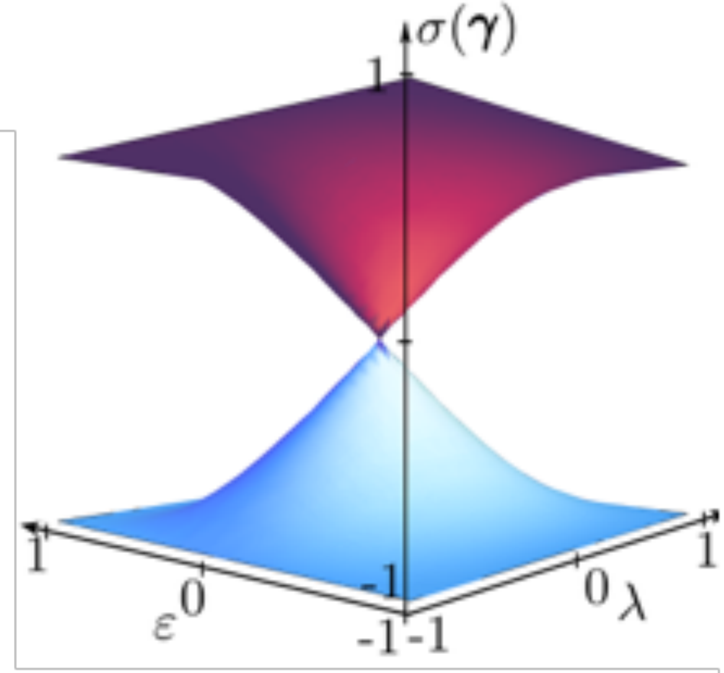
$$G_{ij} = \beta\, h_{ij}$$

reservoir-induced topological pump



reservoir-induced topological pump





damping spectrum

spectrum of G_{ij}

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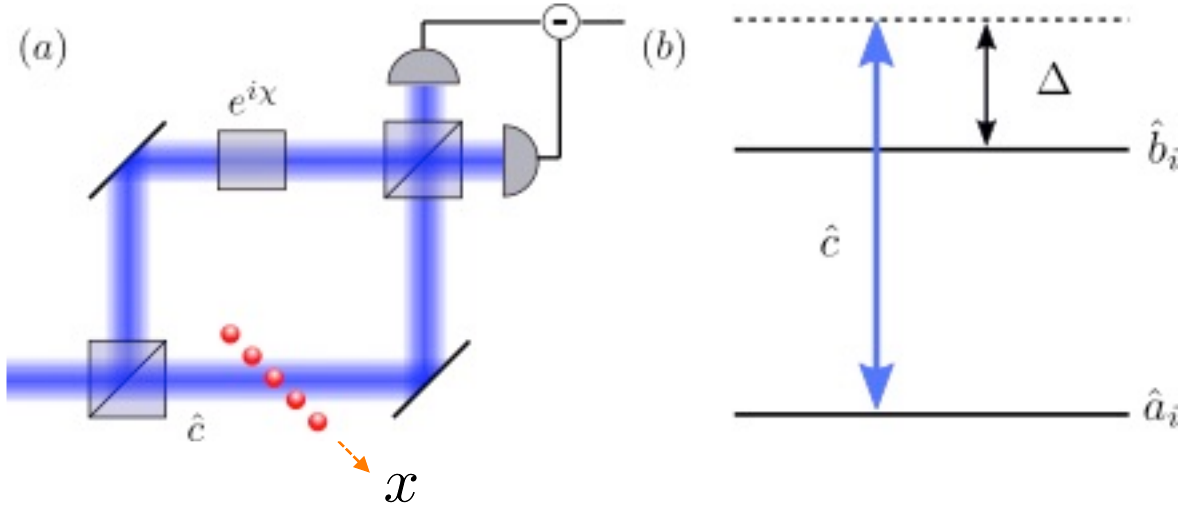
detecting polarization

detecting polarization

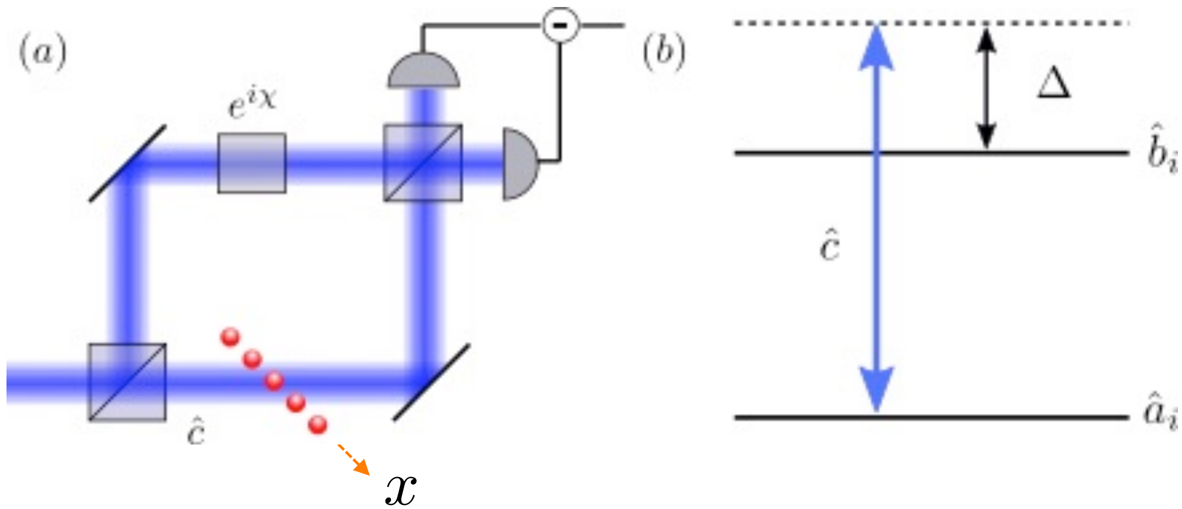
- **interferometer**



$$H_{\text{eff}} = \sum_j \frac{g^2}{\Delta} |f_j|^2 \, \hat{a}_j^\dagger \hat{a}_j \hat{c}^\dagger(z_j)\hat{c}(z_j) \sim \underbrace{\sum_j x_j \hat{a}_j^\dagger \hat{a}_j}_{\hat{X}} \hat{c}^\dagger(z_j)\hat{c}(z_j)$$

detecting polarization

- **interferometer**



$$\mathbf{U}_{\rm int} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \hat{T} + \begin{pmatrix} e^{i\chi} & 0 \\ 0 & 0 \end{pmatrix} \otimes 1 \qquad \hat{T} = \exp\left\{\frac{2\pi i}{L}\hat{X}\right\}$$

$$\Delta\hat{n} = \hat{n}_{\rm out}^{(+)} - \hat{n}_{\rm out}^{(-)} = |\langle\hat{T}\rangle| \cos\left[\chi - \arg\langle\hat{T}\rangle\right]$$

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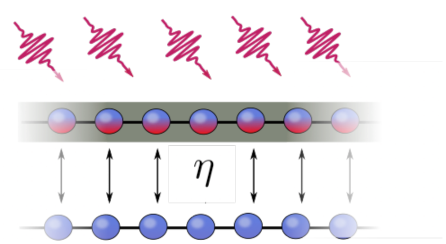
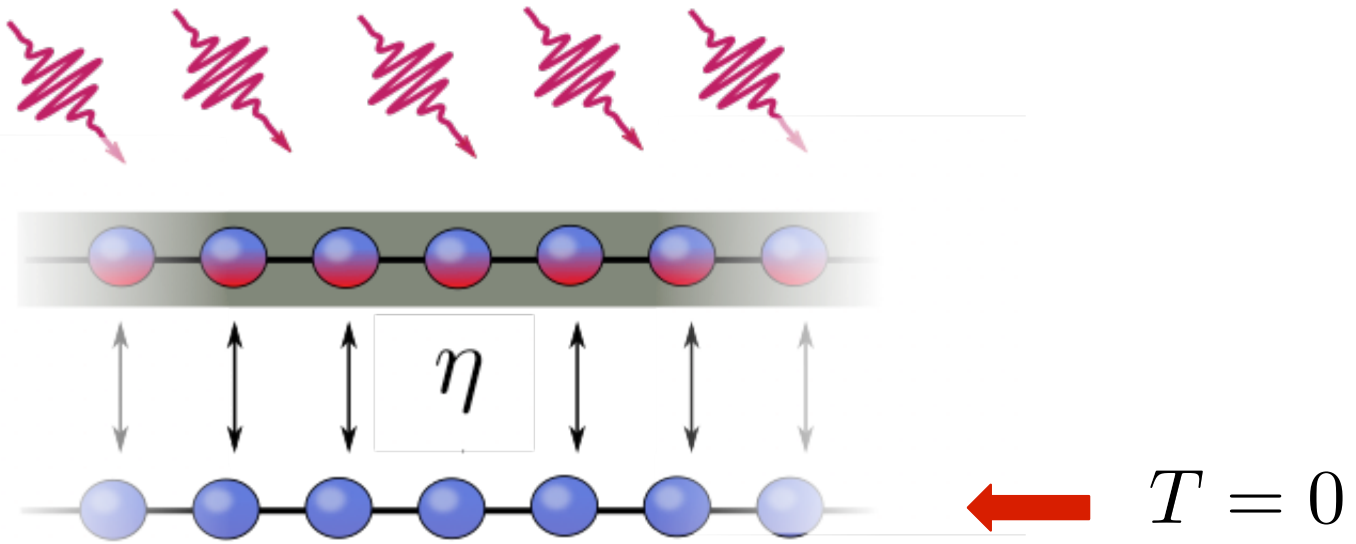
realizing the effective Hamiltonian: topology transfer

R. Li, M. Fleischhauer (in progress)



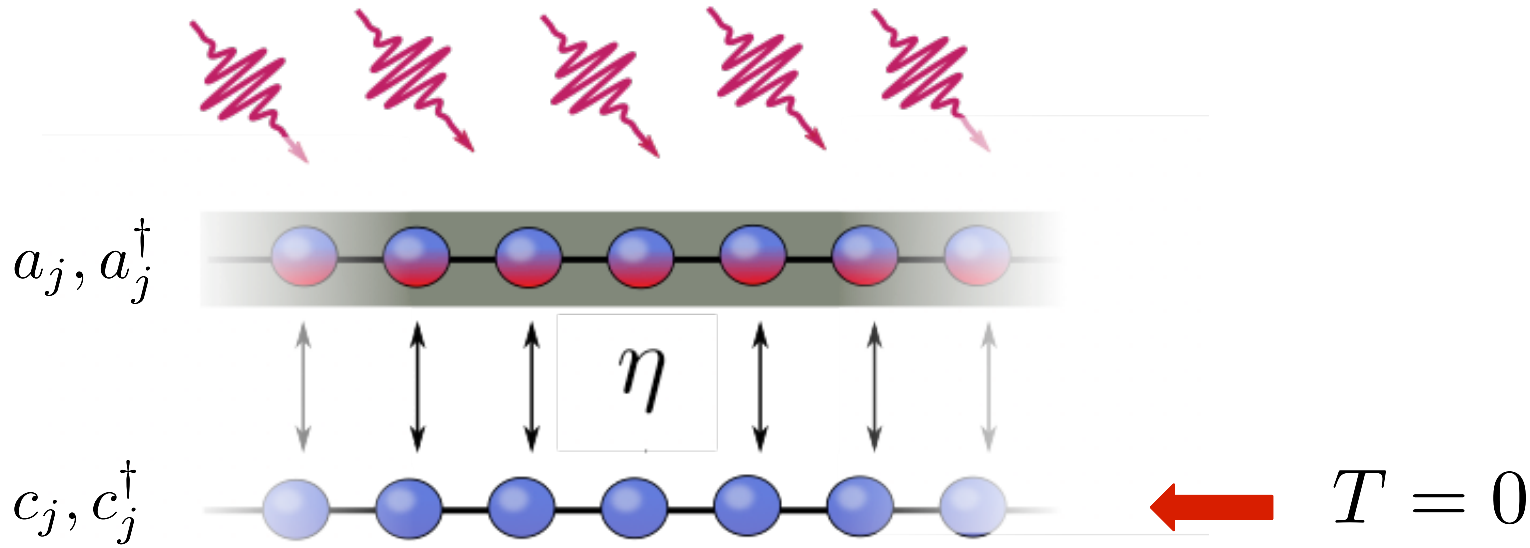
coupling to auxiliary system

- coupling of open (finite-T) system to closed fermion system at T = 0



coupling to auxiliary system

- coupling of open (finite-T) system to closed fermion system at T = 0



$$H = -\eta \sum_{k,\alpha,\alpha'} c^\dagger_{\alpha k} c_{\alpha' k} \, a^\dagger_{\alpha k} a_{\alpha' k}$$

induced Thouless pump

- **mean-field limit** $a_n^\dagger a_m \rightarrow \langle a_n^\dagger a_m \rangle$

$$H = -\eta \sum_{k,\alpha,\alpha'} c_{\alpha k}^\dagger c_{\alpha' k}\, a_{\alpha k}^\dagger a_{\alpha' k}$$

$$h_{ij}^{\mathrm{aux}} \sim G_{ij}(k)$$

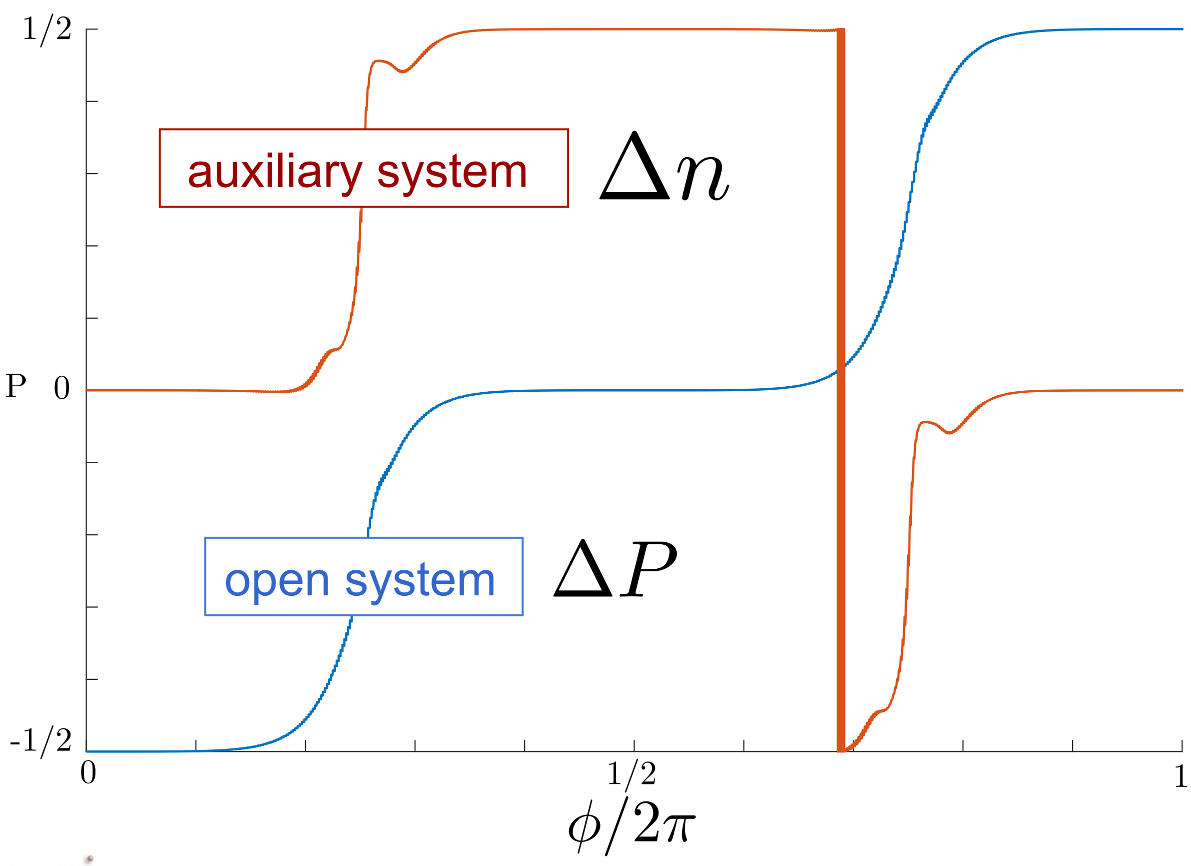
winding of P_{open} → quantized particle transport in auxiliary system

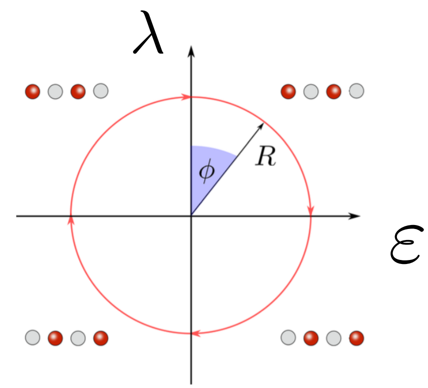
$$\oint d\phi \frac{\partial P}{\partial \phi}\Big|_{\mathrm{open}} = \oint d\phi \frac{\partial P}{\partial \phi}\Big|_{\mathrm{aux}} = \Delta n_{\mathrm{aux}}$$

$T = 0\,(!)$

induced Thouless pump

- **reservoir-induced topological pump (numerics)**





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summary

- topological classification of Gaussian systems

$$H_{\text{eff}} = \sum_{ij} G_{ij} \hat{c}_i^\dagger \hat{c}_j$$

- topological invariant: Ensemble Geometric Phase = many-particle polarization

$$\Delta\phi_{\text{EGP}} = \frac{2\pi}{a} \Delta P$$

- Detection of polarization & realization of effective Hamiltonian via topology transfer

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Alex Altland (Cologne)

SFB TR 158

Deutsche Forschungsgemeinschaft DFG

OPTIMAS light-spin-matter

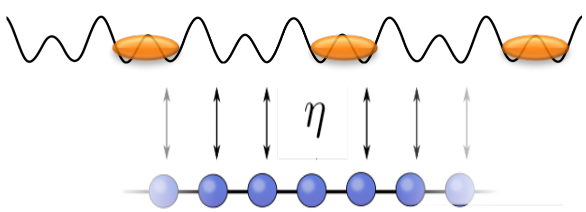
previous contributors: Grusdt (now Harvard),

Technische Universität Kaiserslautern

Thanks !

Technische Universität Kaiserslautern

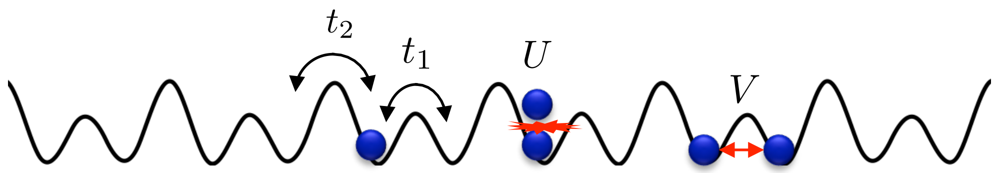
topology transfer from interacting to non-interacting systems



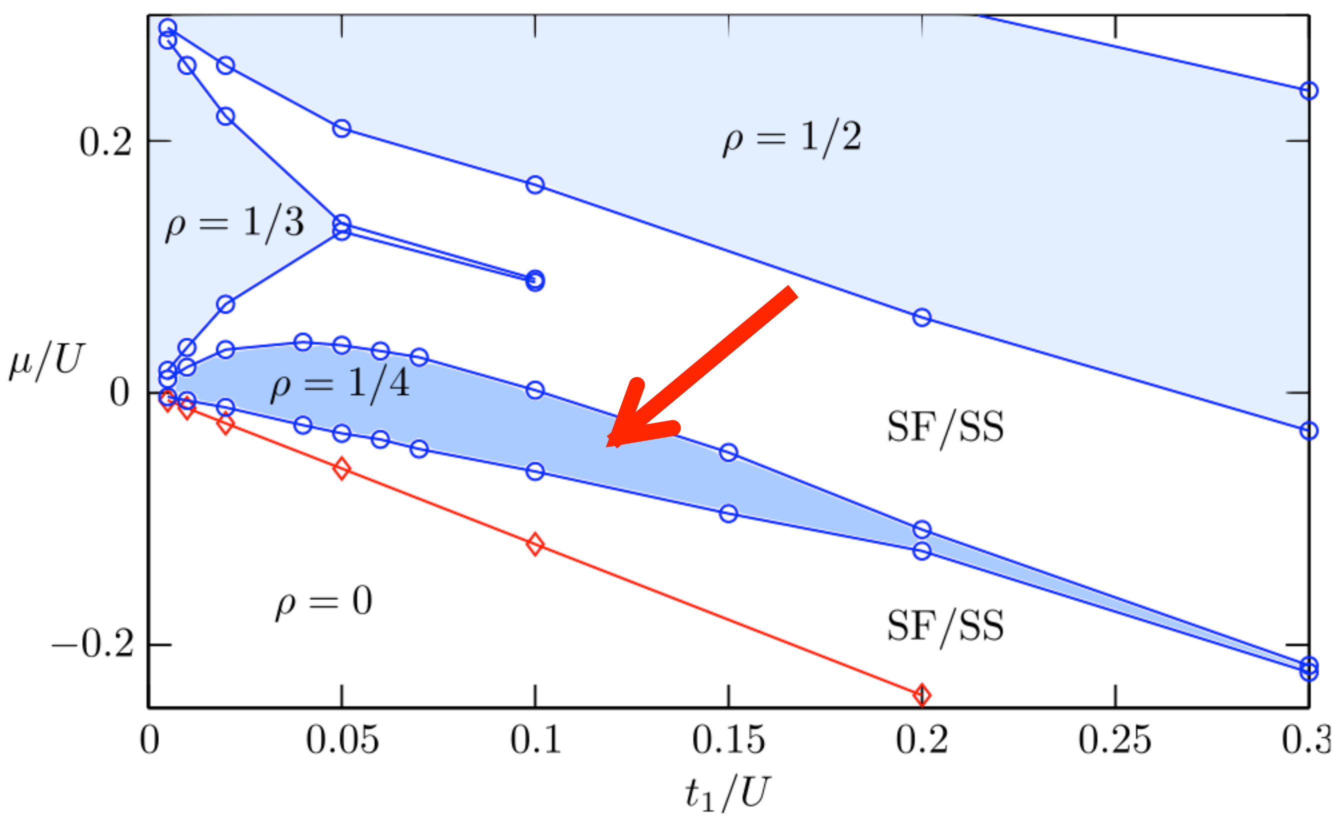
R. Li, D. Linzner, M. Fleischhauer (in progress)

extended superlattice BHM

- **model**

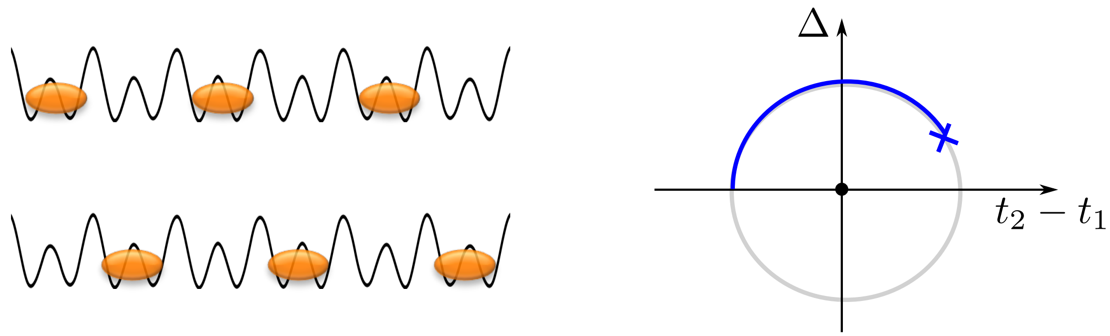


$$H = -t_1 \sum_{\text{odd}} \hat{a}_i^\dagger \hat{a}_{i+1} - t_2 \sum_{\text{even}} \hat{a}_i^\dagger \hat{a}_{i+1} + \frac{U}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i + V \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_i \hat{a}_j^\dagger \hat{a}_j$$

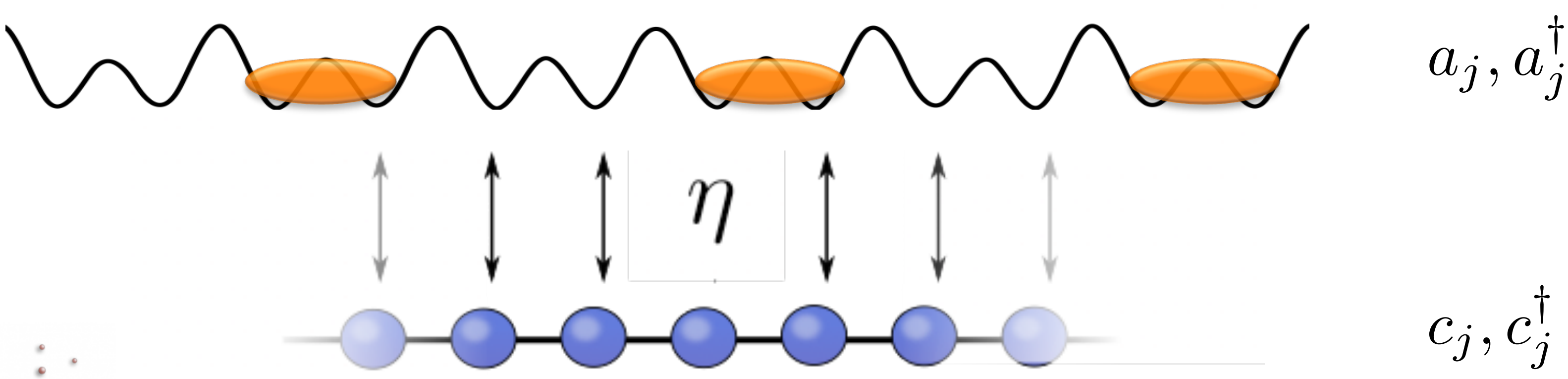


topology transfer

- **fractional charge transport**

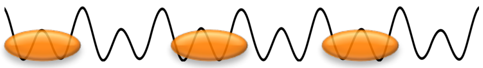


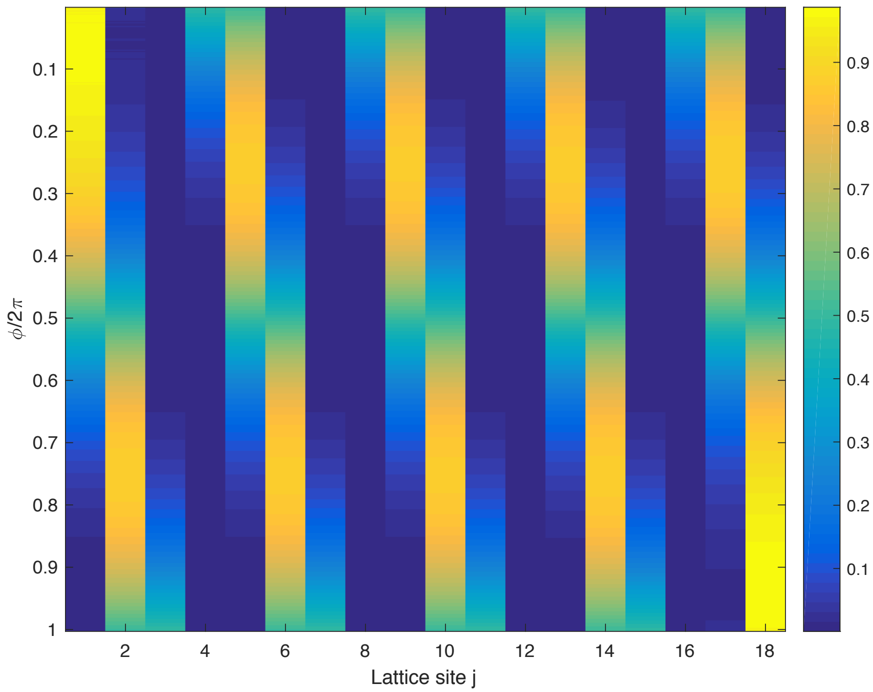
- **coupling to auxiliary fermion chain**

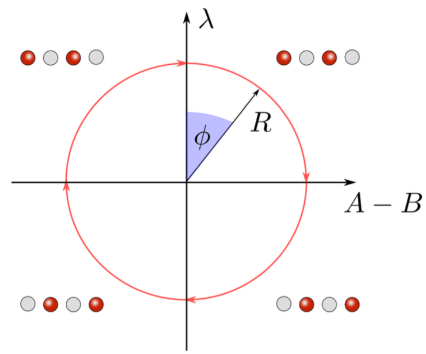


topology transfer

- **charge transport in boson system**

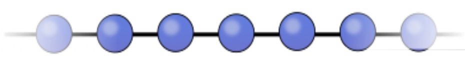


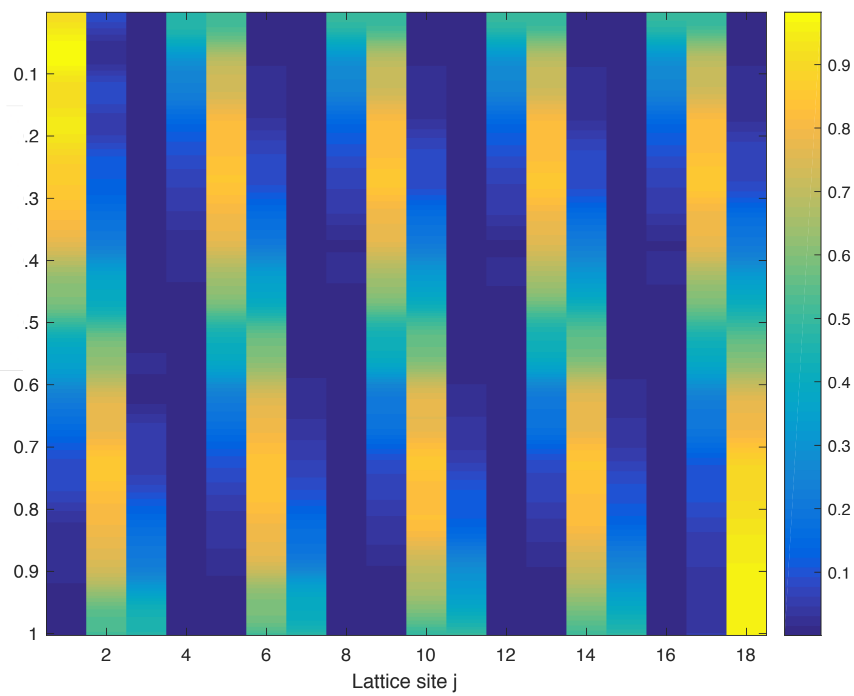


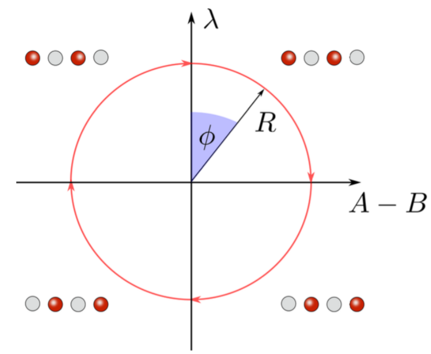


topology transfer

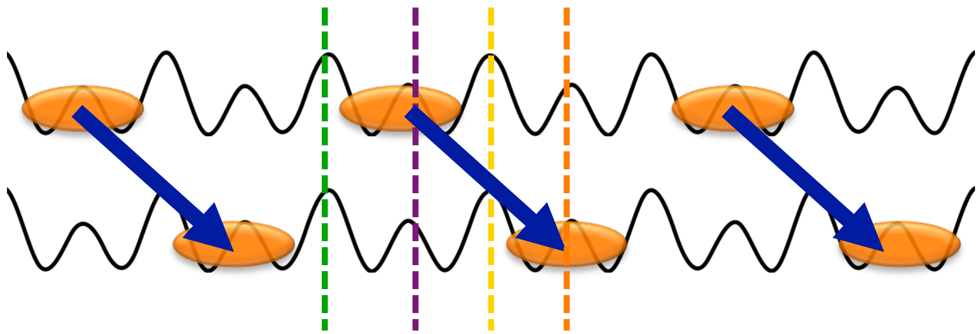
- **charge transport in fermion system**

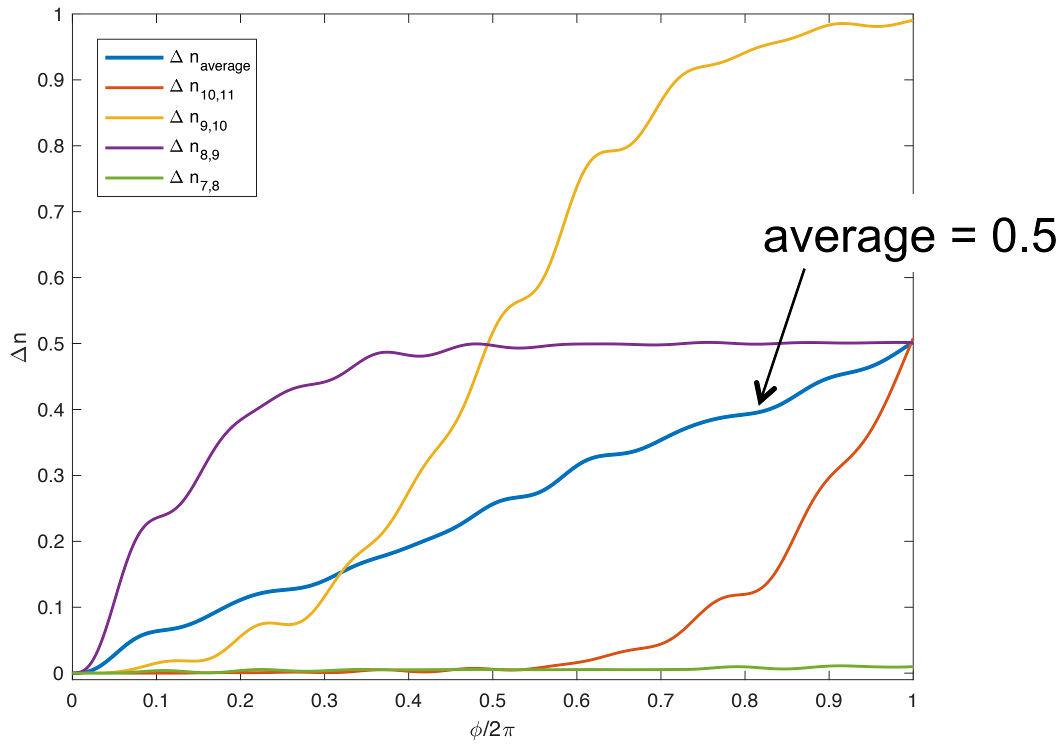




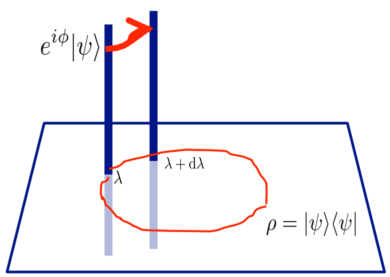


transported charge



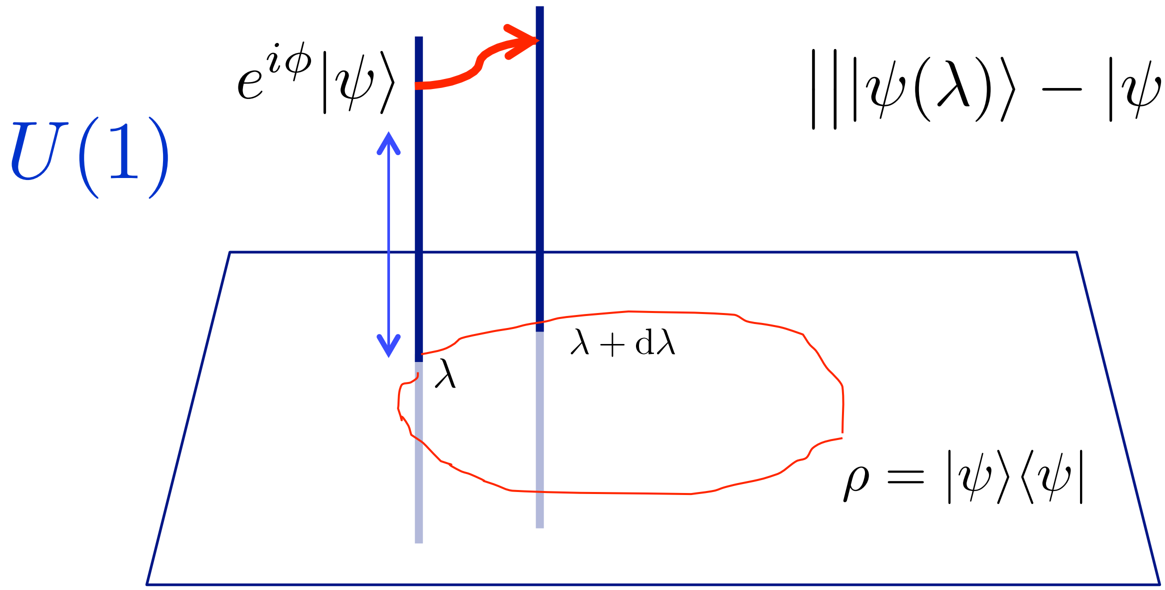


geometric phase & parallel transport



Berry phase and parallel transport

- **Berry parallel transport**



$$\big|\big||\psi(\lambda)\rangle - |\psi(\lambda + \mathrm{d}\lambda)\rangle\big|\big| = \min$$

Berry (Zak) phase: picked up at parallel transport cycle

$$\phi_{\mathrm{Zak}} = \int_{-\pi/a}^{\pi/a} dk \, \langle u_k | i\partial_k | u_k \rangle$$

geometric phases for density matrices

- **Uhlmann connection**

$$\rho = w\, w^\dagger$$

gauge degree of freedom: *U(n)*

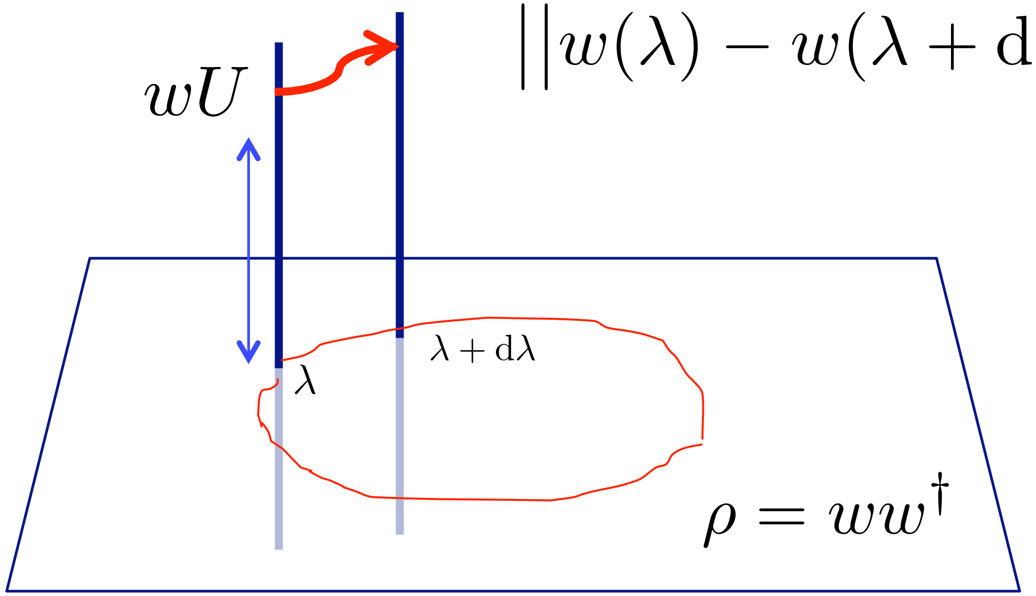
$$w \to w\, U \qquad\qquad w^\dagger \to U^\dagger\, w^\dagger$$

Berry for mixed states: Uhlmann phase

- **Uhlmann parallel transport**

$$U(n)$$

$$\big|\big|w(\lambda) - w(\lambda + \mathrm{d}\lambda)\big|\big| = \min$$



U(1) **Uhlmann phase**

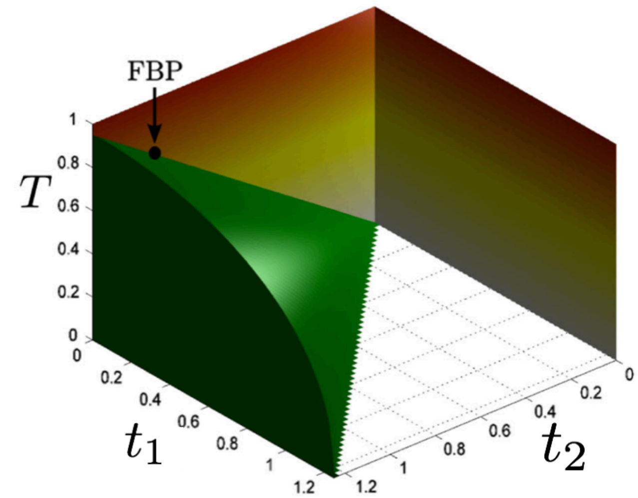
$$e^{i\phi} = \oint \mathrm{d}\lambda \; \mathrm{Tr}\big[w\partial_{\lambda}w^{\dagger}\big]$$

O. Viyuela, et al. Phys. Rev. Lett. (2014)

Z. Huang, D. P. Arovas, Phys. Rev. Lett. (2014)

Berry for mixed states: Uhlmann phase

- **Rice-Mele model at finite T**



Viyuela, Rivas, Martin-Delgado
Phys.Rev.Lett. (2014)

- **2D Chern insulator at finite T**

$$H(k) = \sum_j d^j(k)\,\hat{\sigma}_j$$

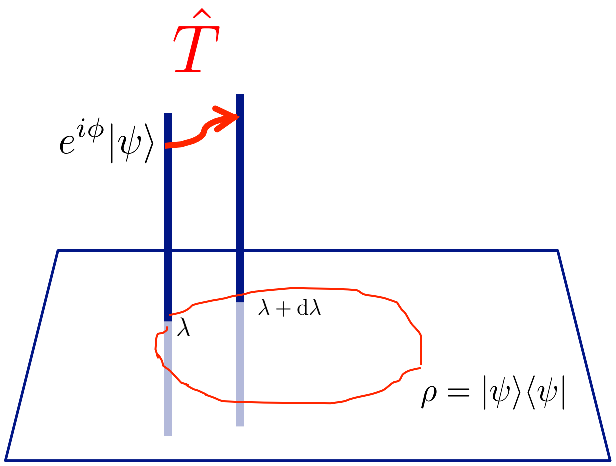
$$d^1 = \sin(k_x) \quad d^2 = 3\sin(k_y) \quad d^3 = 1 - \cos(k_x) - \cos(k_y)$$

$$C = \frac{1}{2\pi}\int dk_y \left(\frac{\partial\phi(k_y)}{\partial k_y}\right) \neq C' = \frac{1}{2\pi}\int dk_x \left(\frac{\partial\phi(k_x)}{\partial k_x}\right)$$

Budich, Diehl Phys.Rev. B (2015)

parallel transport & momentum shift

- **Berry parallel transport**



$$\phi_{\mathrm{Zak}} = \int_{-\pi/a}^{\pi/a} dk \, \langle u_k | i\partial_k | u_k \rangle$$

$$\phi_{\mathrm{Zak}} = \mathrm{Im} \ \ln \prod_j \langle u(k_j) | u(k_{j+1}) \rangle$$

$$\langle u(k_j) | u(k_{j+1}) \rangle = \langle u(k_j) | T(\Delta k) | u(k_j) \rangle$$

momentum shift operator

$$T(q) = \exp\left\{ iq\hat{X} \right\}$$