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Institute of Theoretical Physics

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A resource theory of superposition

Thomas Theurer

in cooperation with

Nathan Killoran, Dario Egloff and Martin B. Plenio

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ERC Synergy grant BioQ

QUCHIP Quantum Simulation on a Photonic Chip

Quantum resource theories

A fast quantum mechanical algorithm for database search

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600 Mountain Avenue
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Summary

An unsorted database contains N records, of which just one satisfies a particular property. The problem is to identify that one record. Any classical algorithm, deterministic or probabilistic, will clearly take $O(N)$ steps since on the average it will have to examine a large fraction of the N records. Quantum mechanical systems can do several operations simultaneously due to their wave

This paper applies quantum c mundane problem in information proce sents an algorithm that is significantly f classical algorithm can be. The problem an unsorted database containing N items out of which just one item satisfies a given condition - that one item has to be retrieved. Once an item is examined, it is possible to tell whether or not it satisfies the condition in

Algorithms for Quantum Computation: Discrete Logarithms and Factoring

Peter W. Shor
AT&T Bell Labs
Room 2D-149
600 Mountain Ave.
Murray Hill, NJ 07974, USA

Abstract

A computer is generally considered to be a universal computational device; i.e., it is believed able to simulate any physical computational device with a cost in computation time of at most a polynomial factor. It is not clear whether this is still true when quantum mechanics

[1, 2]. Although he did not ask whether quantum mechanics conferred extra power to computation, he did show that a Turing machine could be simulated by the reversible unitary evolution of a quantum process, which is a necessary prerequisite for quantum computation. Deutsch [9, 10] was the first to give an explicit model of quantum computation. He defined both quantum Turing machines and quantum

QUANTUM CRYPTOGRAPHY: PUBLIC KEY DISTRIBUTION AND COIN TOSSING

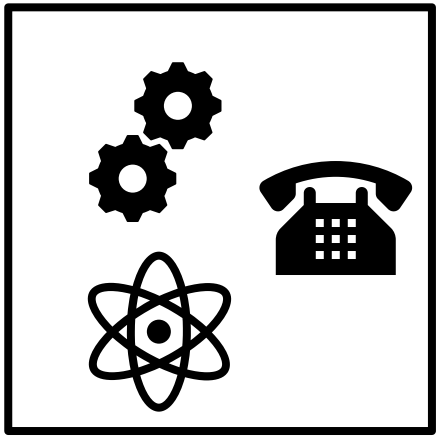
Charles H. Bennett (IBM Research, Yorktown Heights NY 10598 USA)
Gilles Brassard (dept. IRO, Univ. de Montreal, H3C 3J7 Canada)

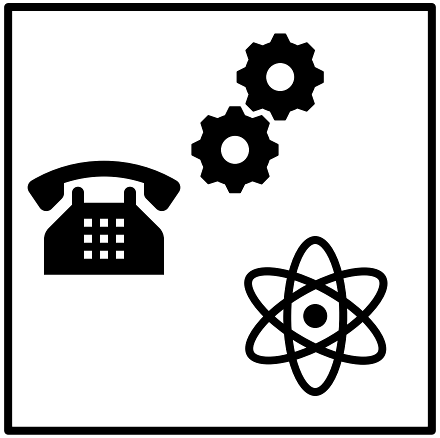
When elementary quantum systems, such as polarized photons, are used to transmit digital information, the uncertainty principle gives rise to novel cryptographic phenomena unachieveable with traditional transmission media, e.g. a communications channel on

principle impossible to counterfeit, and multiplexing two or three messages in such a way that reading one destroys the others. More recently [BBBW], quantum coding has been used in conjunction with public key cryptographic techniques to yield several

Entanglement as a resource theory

Restriction: Local operations and classical communication – physically motivated.





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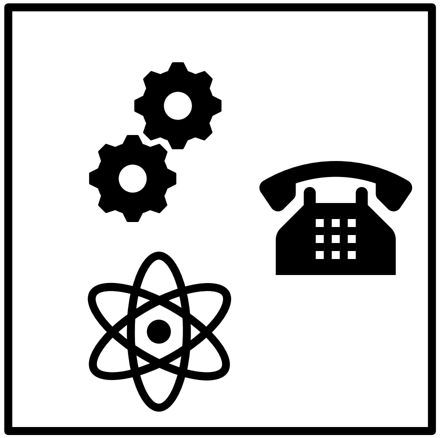

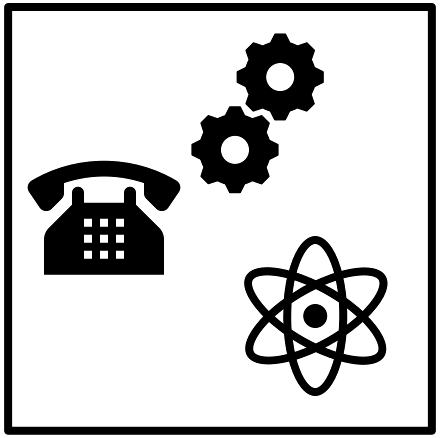

Entanglement cannot be created but allows for tasks otherwise forbidden.

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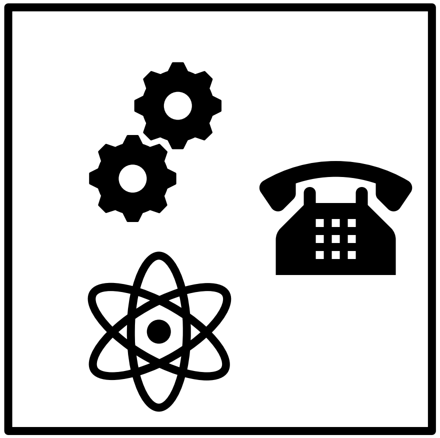
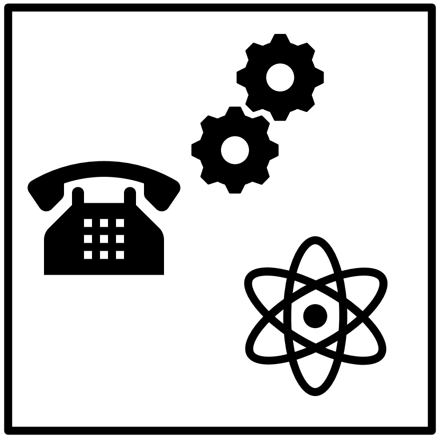




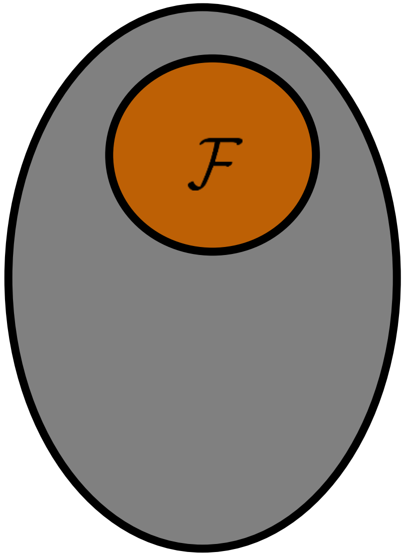
Entanglement cannot be created but allows for tasks otherwise forbidden.

Teleport the state of your system – allows to overcome the restriction

Quantum resource theories

- Main ingredients
  1. Free states ⟺ resource states
  2. Free operations

- Entanglement
  1. Separable states ⟺ entangled states
  2. LOCC



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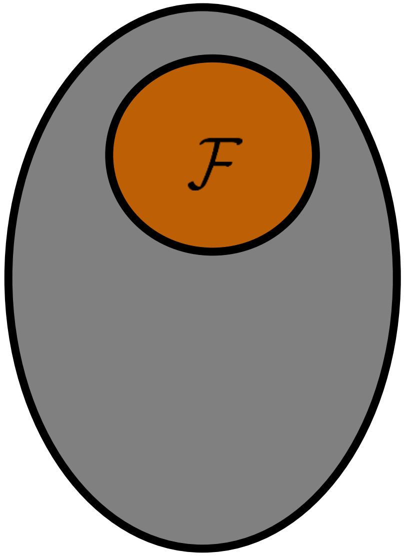
  - Operational advantage



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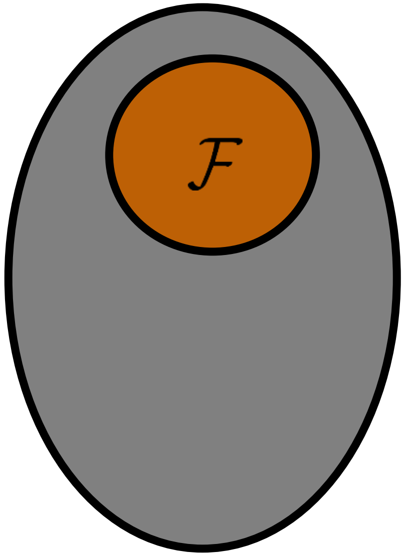
- Systematic investigation leads to a better usage in applications.



Relevance

[1] Killoran, Nathan, Frank ES Steinhoff, and Martin B. Plenio. "Converting Nonclassicality into Entanglement." *Physical review letters* 116.8 (2016): 080402.

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In principle, all aspects of quantum mechanics not present in classical physics can lead to operational advantages.
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- coherence,
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Investigate non-classicality in terms of superpositions[1].

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Agreement on the definition of entanglement – but how to define non-classicality?
- convert it to entanglement

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Alternative definitions possible!

Relevance

- Definition: Classical rank[1,2] (for an arbitrary set of free states):

$$r_C(|\psi\rangle) = \min \left\{ r \,\middle|\, |\psi\rangle = \sum_{j=1}^{r} \psi_j \, |c_j\rangle \right\}$$

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- Faithful conversions allow for the definition of the controversial notion of non-classicality based on the well-founded principles of entanglement.

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Relevance

Theorem: If the free states in a finite dimensional Hilbert space form a countable set, then linear independence of the free states is a necessary and sufficient condition for the existence of a faithful conversion operation.

Relevance

- Generalization of coherence theory[1,2,3].
    - Linear independence versus orthogonality.

[1] Aberg, J. (2006). Quantifying superposition. *arXiv preprint quant-ph/0612146*.
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- Generalization of coherence theory[1,2,3].
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- Example: Quantify non-classicality in the superposition of a finite number of optical coherent states.
  - Faithful conversion can be done using a beam splitter[4].

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- Starting point for more general resource theories.

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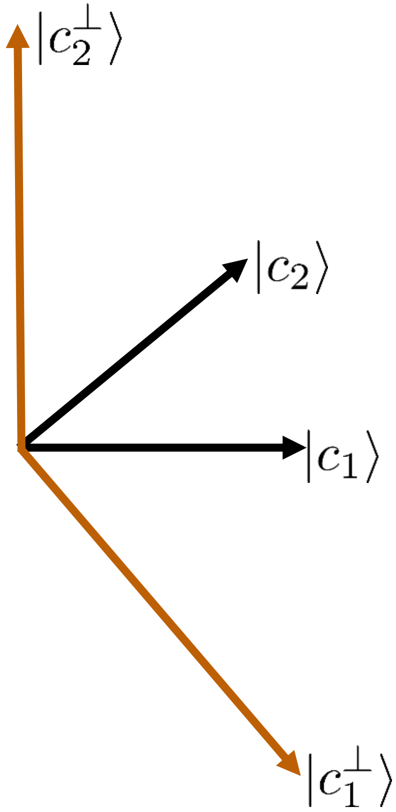
Free operations

Reciprocal vectors:

$$\delta_{i,j} = \langle c_i^{\perp} | c_j \rangle$$



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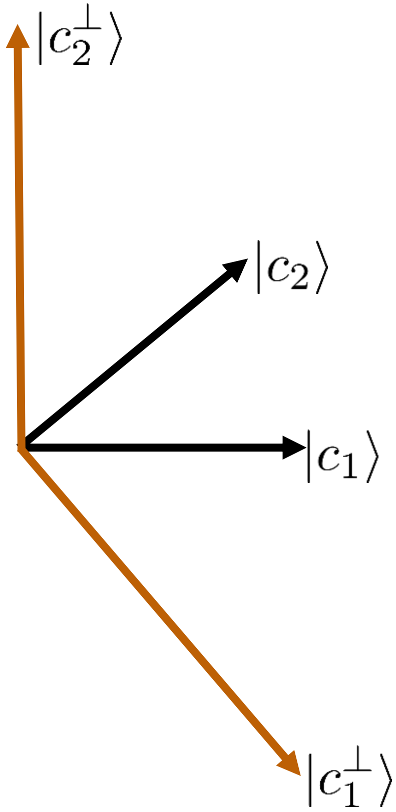

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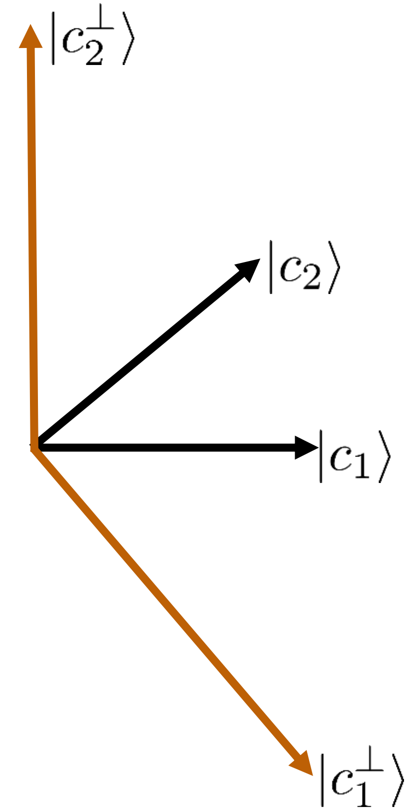
Free completion of maps:
- valid for coherence theory
- not possible in entanglement theory
- important for applications

$$\sum_m K_m^{\dagger} K_m \leq \mathbb{1} \Rightarrow \exists \text{ free } F_n : \sum_m K_m^{\dagger} K_m + \sum_n F_n^{\dagger} F_n = \mathbb{1}$$



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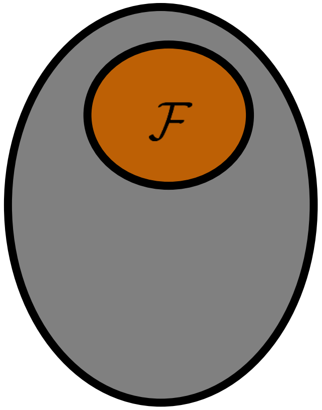

$$M(\rho) = 0 \iff \rho \in \mathcal{F}$$

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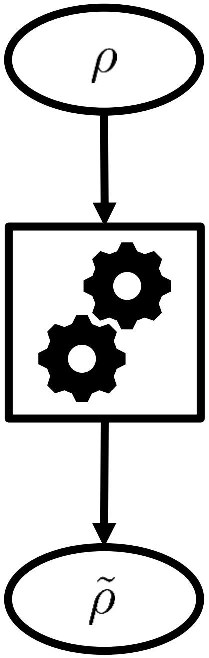

$$M(\rho) \geq M(\Phi(\rho)) \qquad \forall \Phi \in \mathcal{FO}$$

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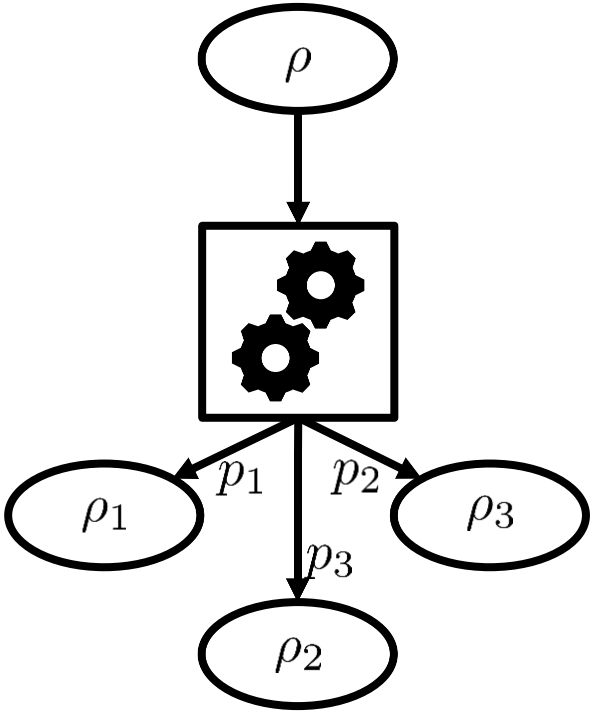

$$M(\rho) \geq \sum_n p_n M(\rho_n) \ : p_n = \mathrm{tr}(K_n \rho K_n^\dagger), \ \rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$$

$$\forall \{K_n\} : \sum_n K_n^\dagger K_n = \mathbb{1}, \ K_n \mathcal{F} K_n^\dagger \subset \mathcal{F}$$

Superposition measures

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- A function M mapping quantum states to the positive real numbers is called a superposition measure if it is
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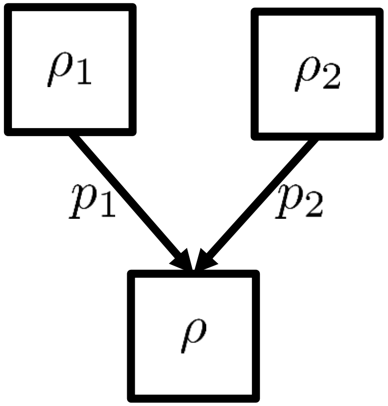
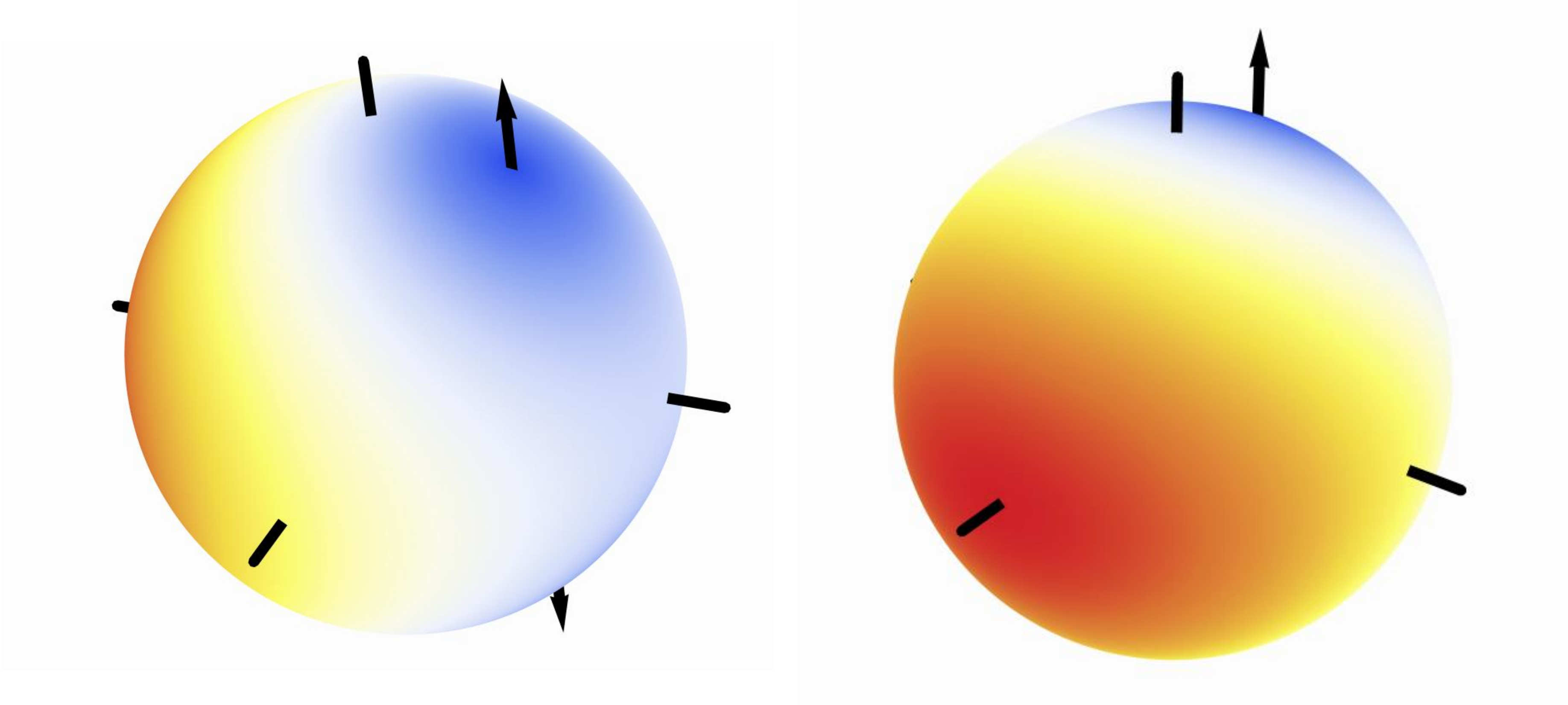
  4. Convex



$$\sum_n p_n M(\sigma_n) \geq M\left(\sum_n p_n \sigma_n\right) \qquad \forall \{\sigma_n\},\ p_n \geq 0,\ \sum_n p_n = 1$$

The l_1-measure of superposition

$$M_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| \qquad \text{for} \qquad \rho = \sum_{ij} \rho_{ij} \, |c_i\rangle \langle c_j|$$



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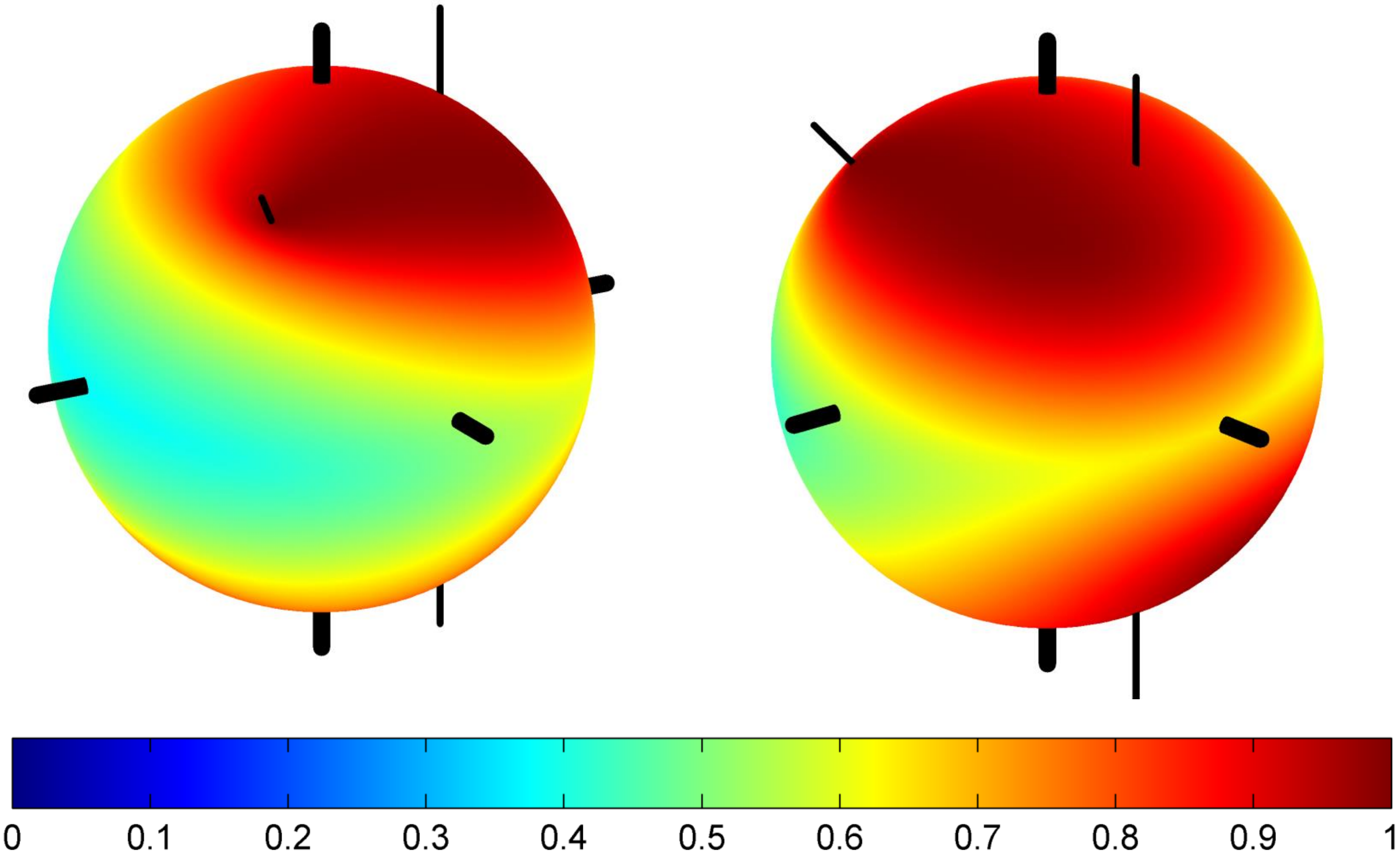
Highest probability is the solution of a semidefinite program.

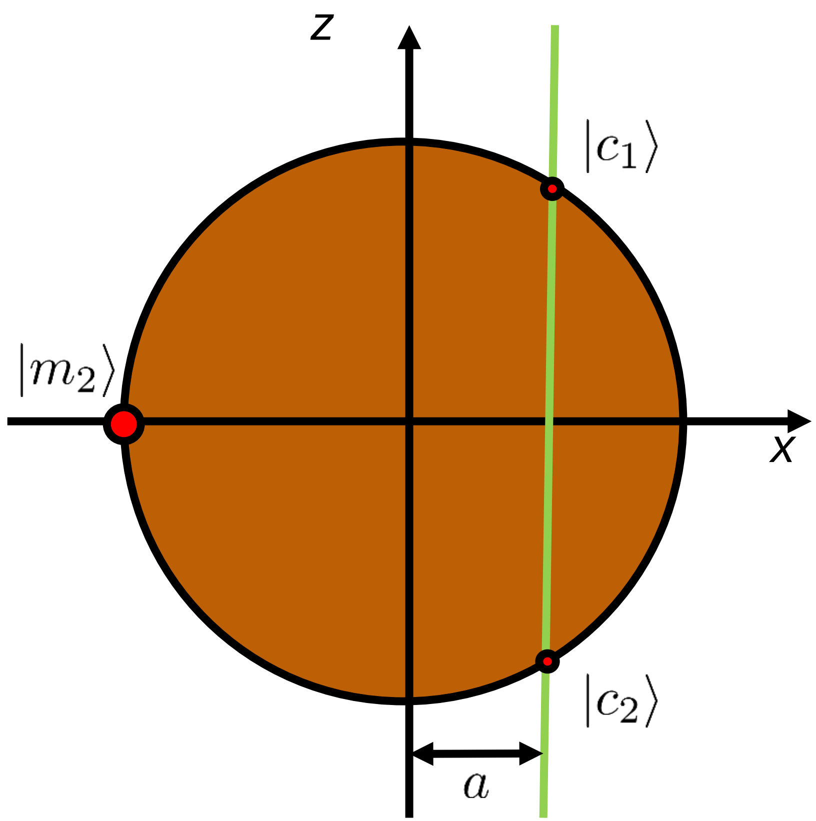
$$\begin{aligned} \text{maximize} \qquad & p = \sum_n p_n \\ \text{subject to} \qquad & \sum_n p_n F_n^\dagger F_n \leq \mathbb{1} \\ & p_n \geq 0 \qquad \text{for all } n \end{aligned}$$

Qubit transformations



States with maximal superposition

Golden unit – exists for qubits



States with maximal superposition

- Not existent in general.
- Exist in the limit of coherence theory.

- How to proof? Counter example for qutrits.
  1. Classical rank can never increase.
  2. Maximize the $\boldsymbol{l_1}$-measure of superposition.

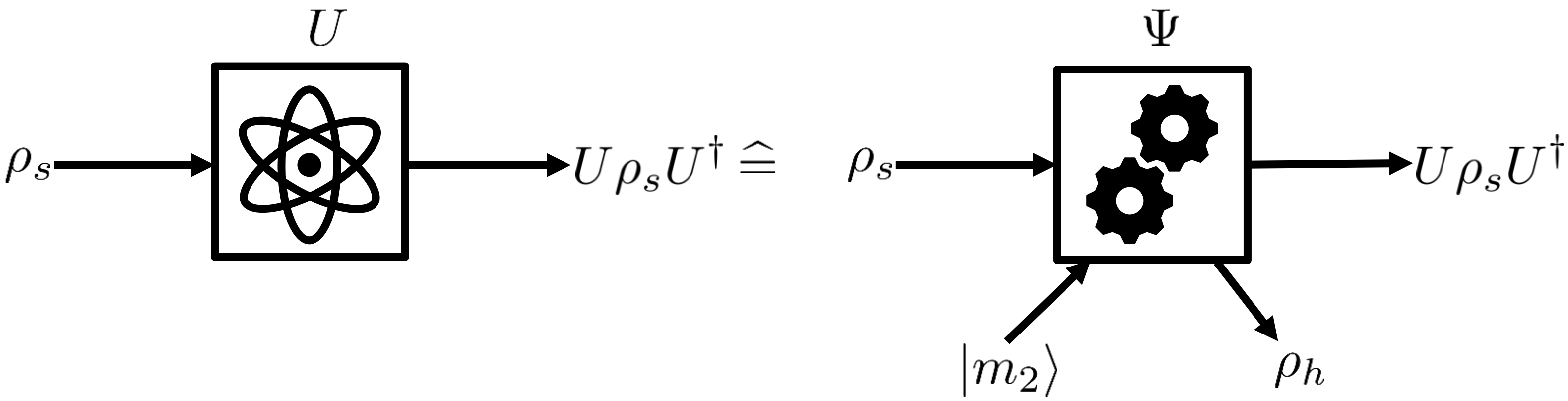
$$|m_d\rangle = \sum_{j=1}^{d} m_j \, |c_j\rangle$$

  3. Consider transformation to max-rank states and formulate semidefinite program.
  4. Bound solution by duality .

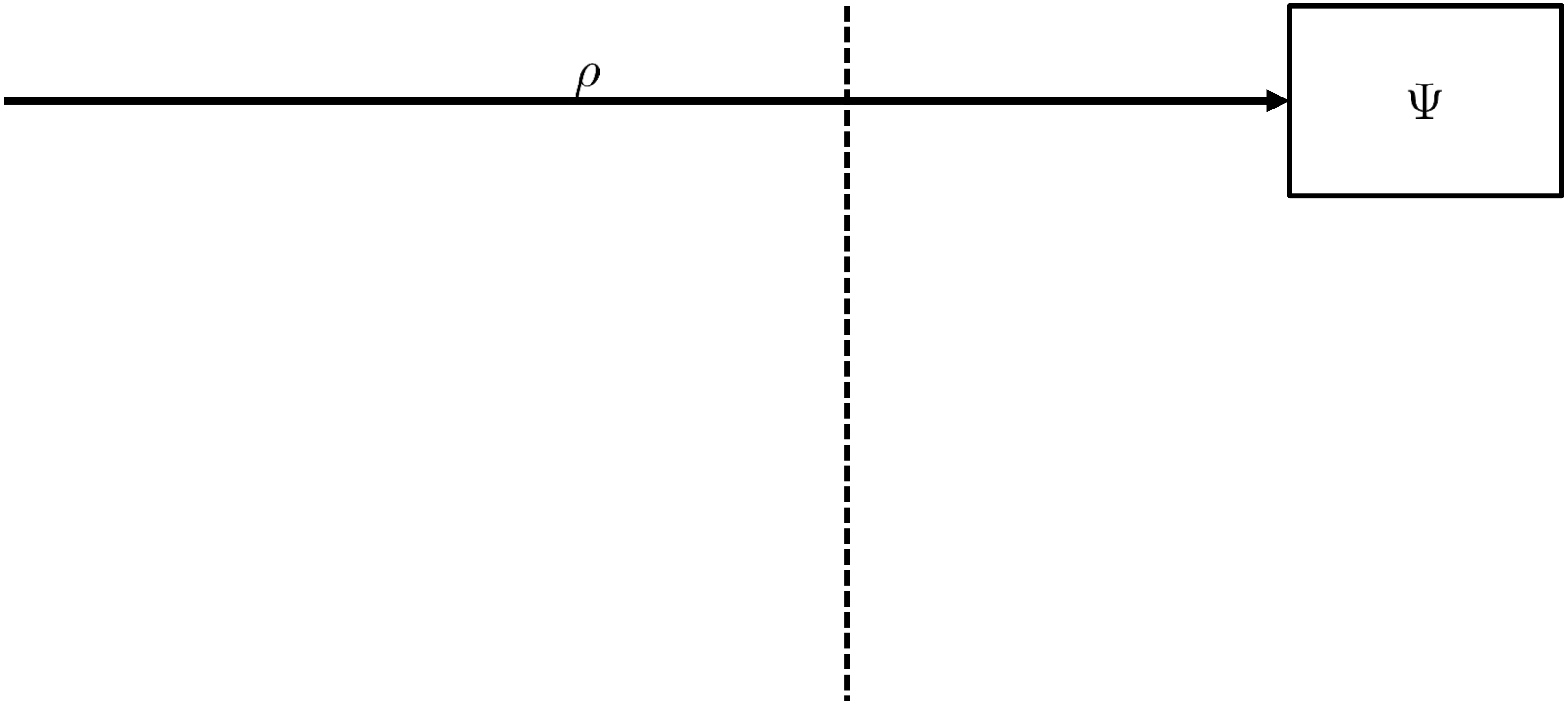
Unitary qubit operations

Theorem: For every qubit-unitary U there exists a fixed $\Psi \in \mathcal{FO}$ independent of ρ_s acting on two qubits such that

$$\Psi\left(\rho_s \otimes |m_2\rangle\langle m_2|\right) = \left(U\rho_s U^\dagger\right) \otimes \rho_h.$$



Superposition as a resource in decision tasks



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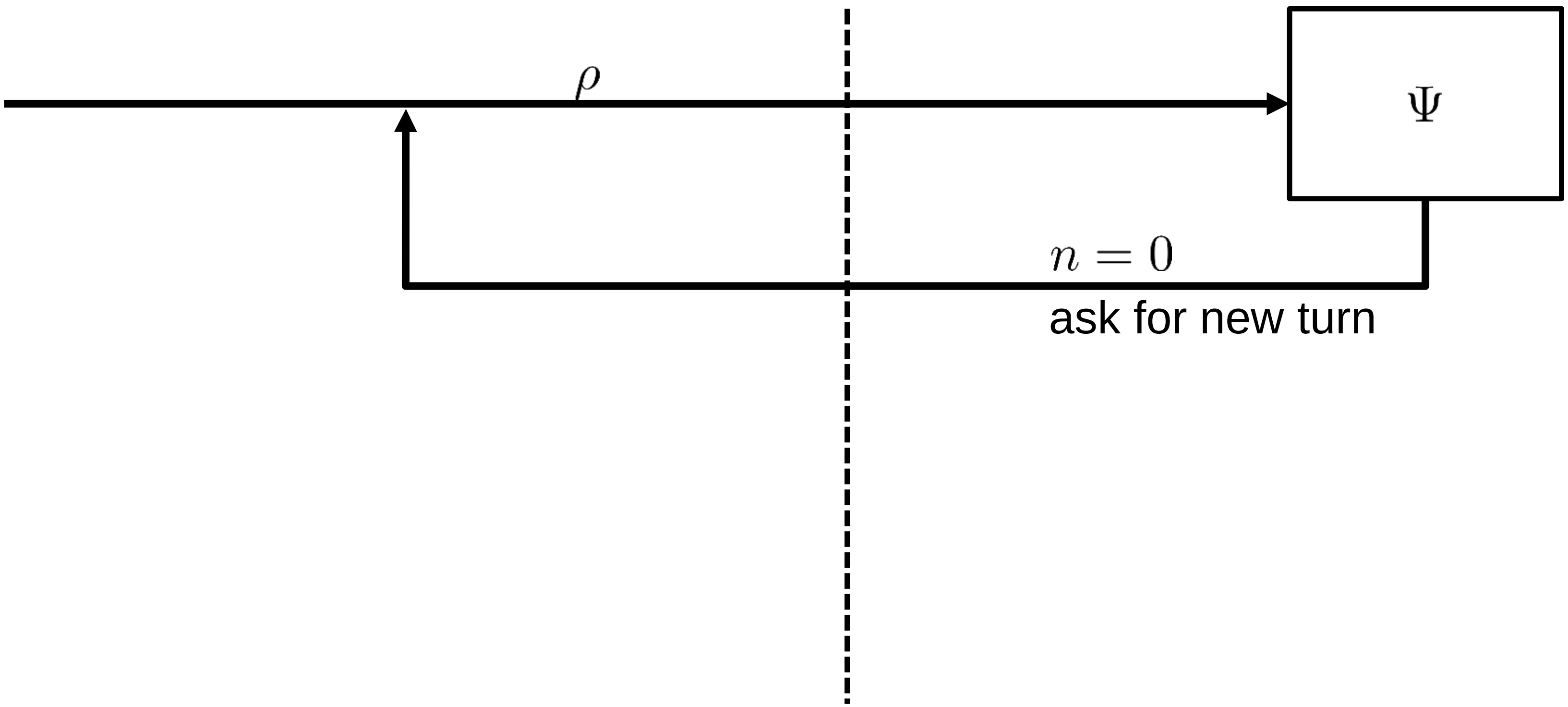

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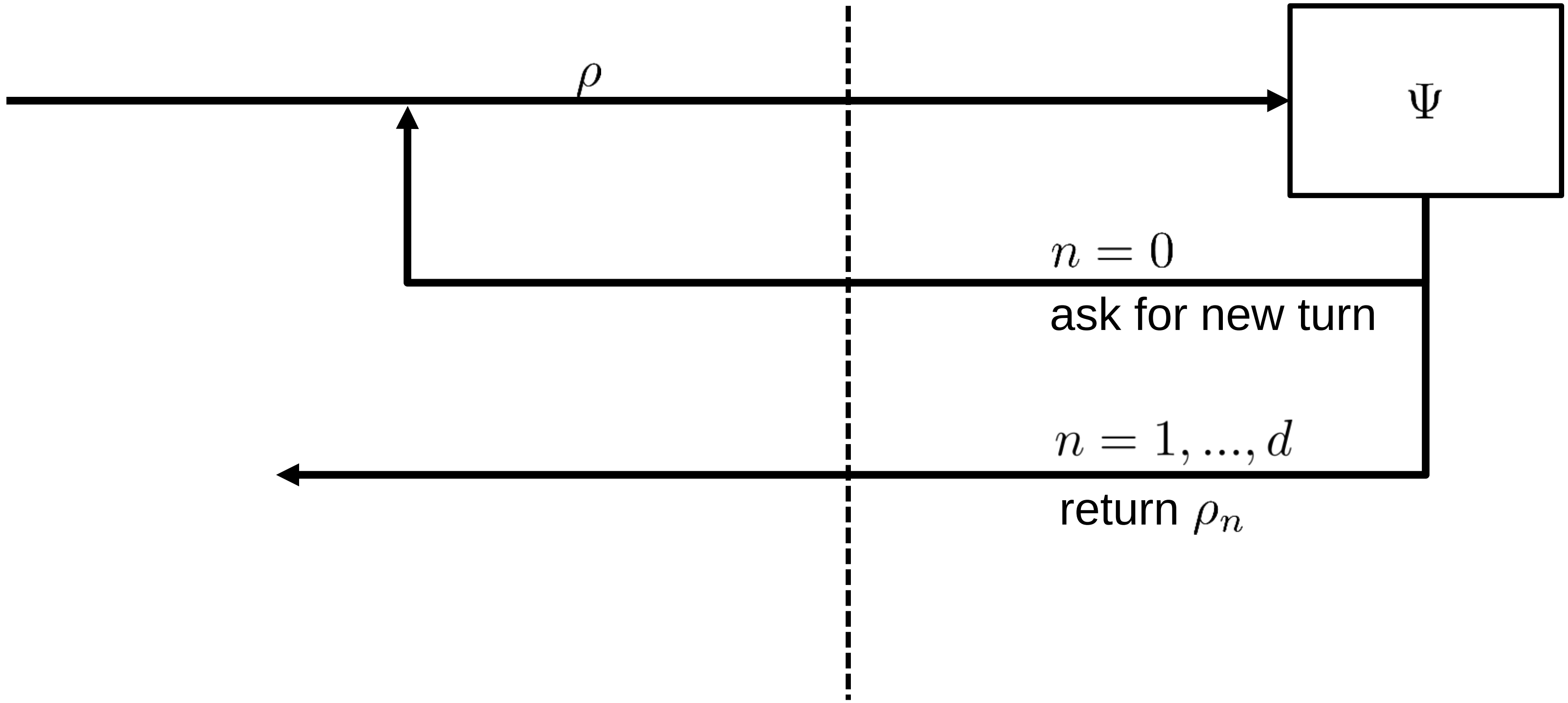

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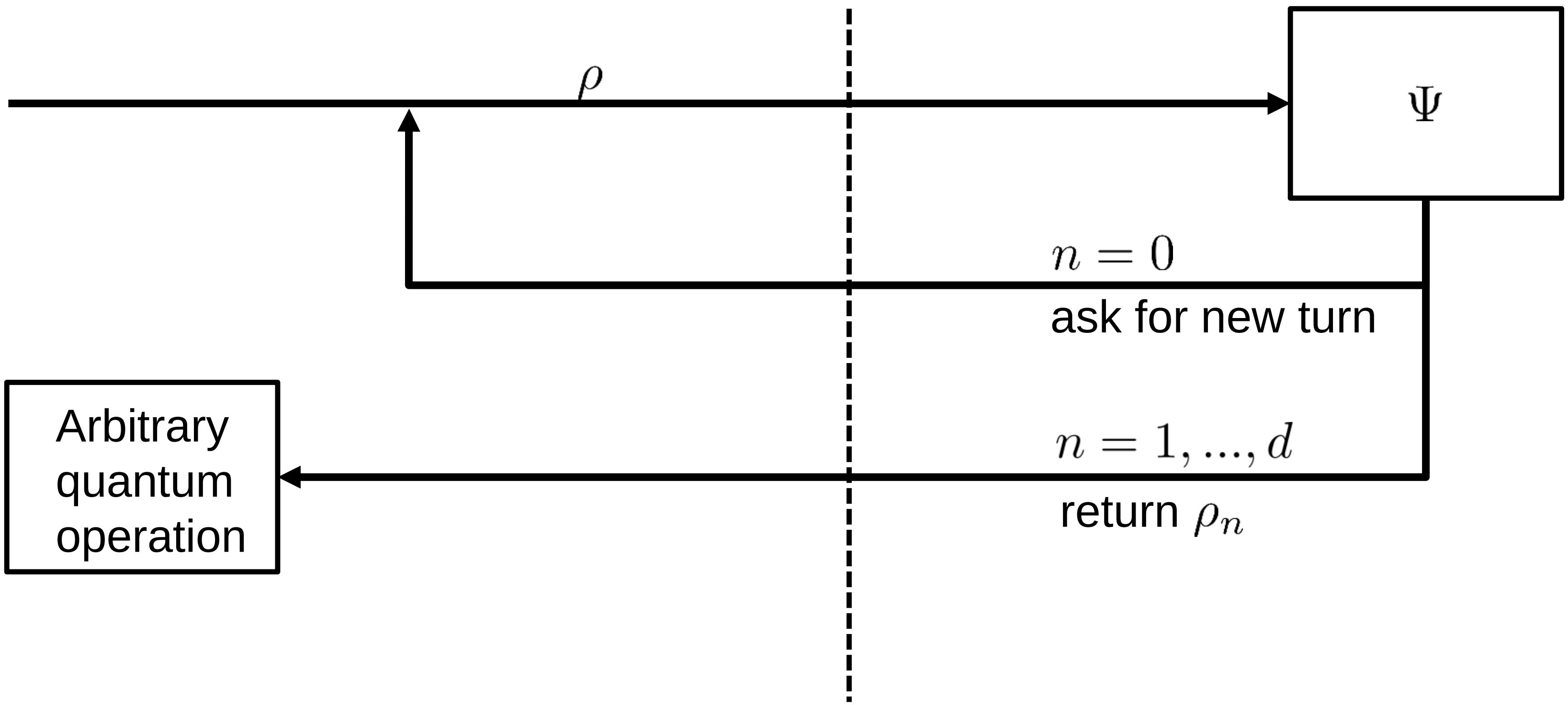

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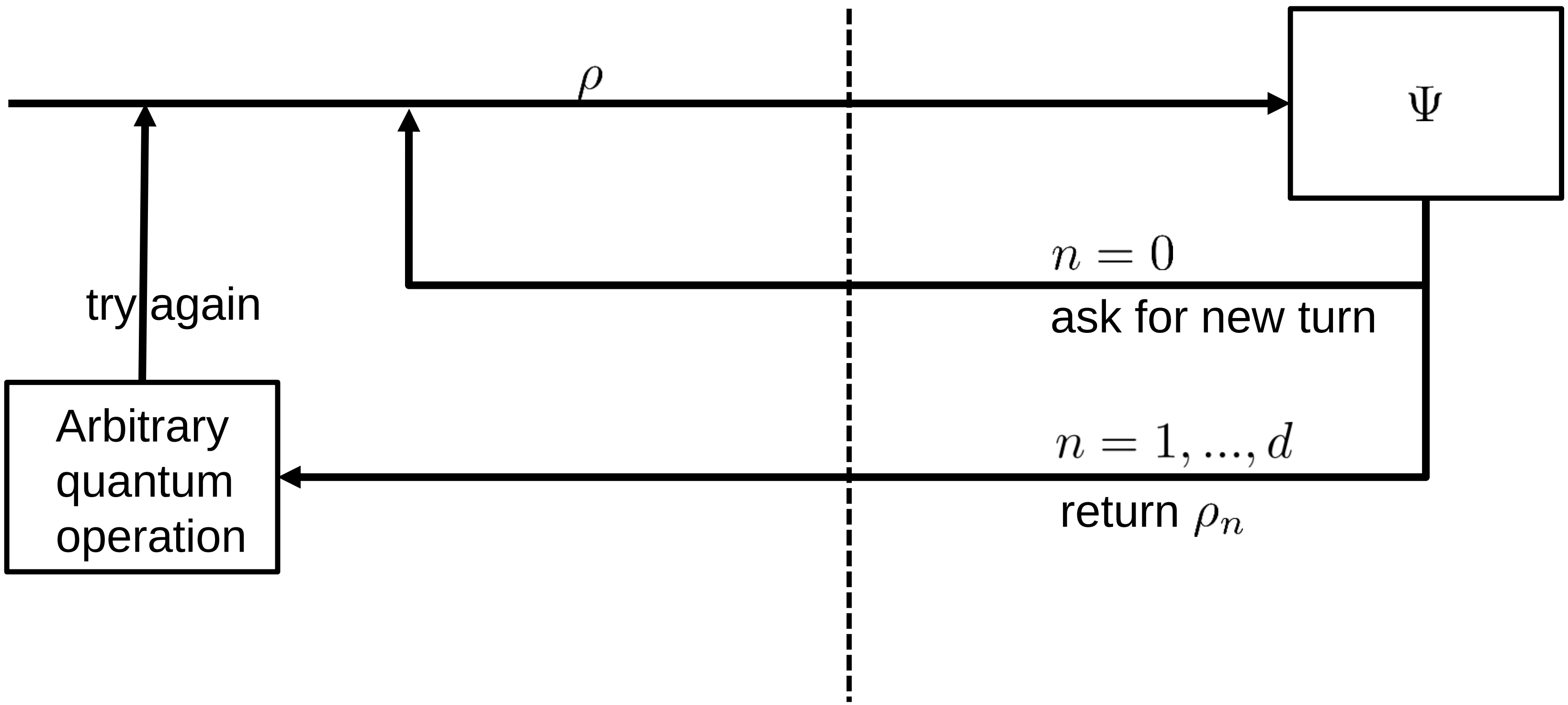

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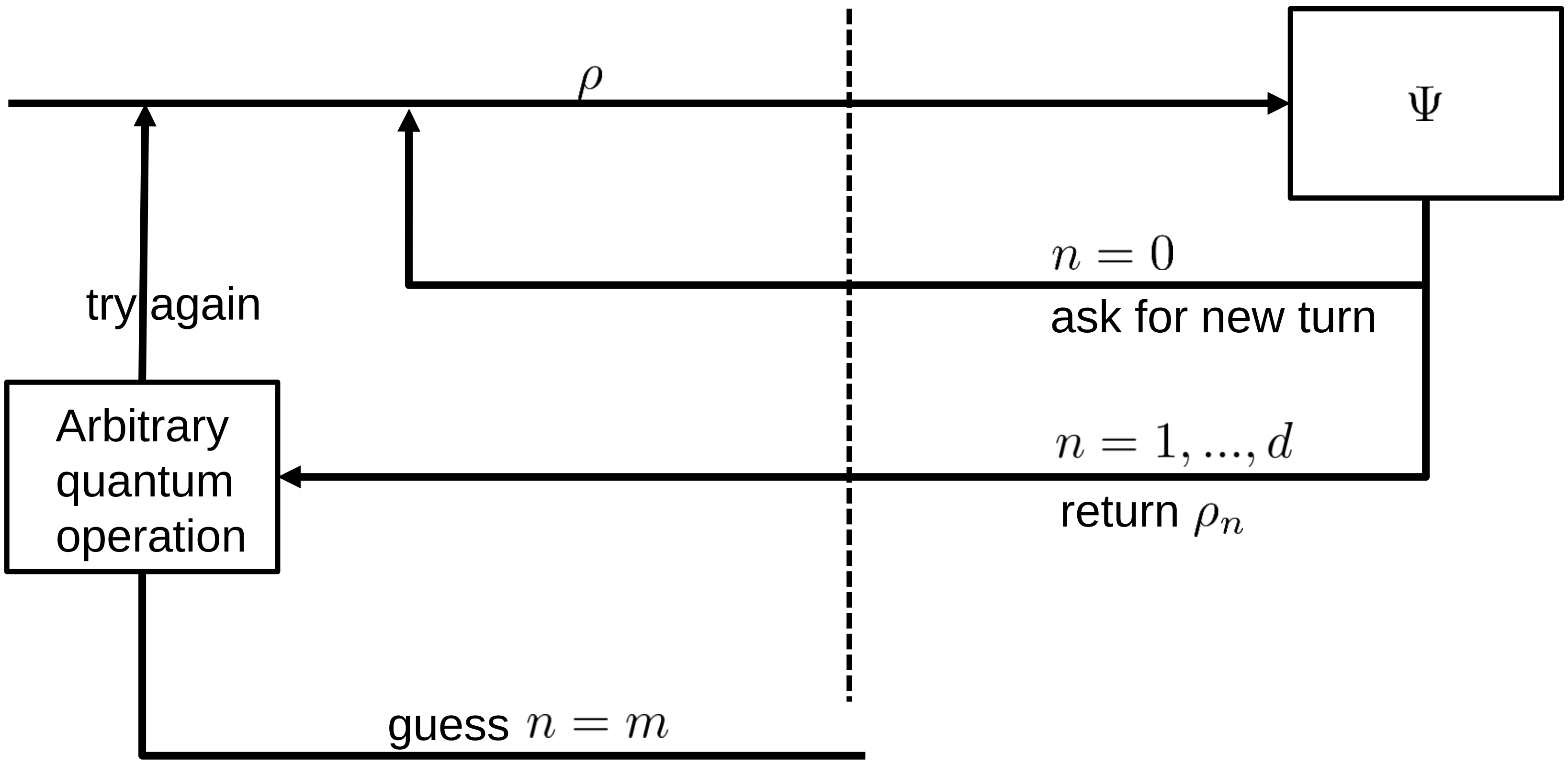

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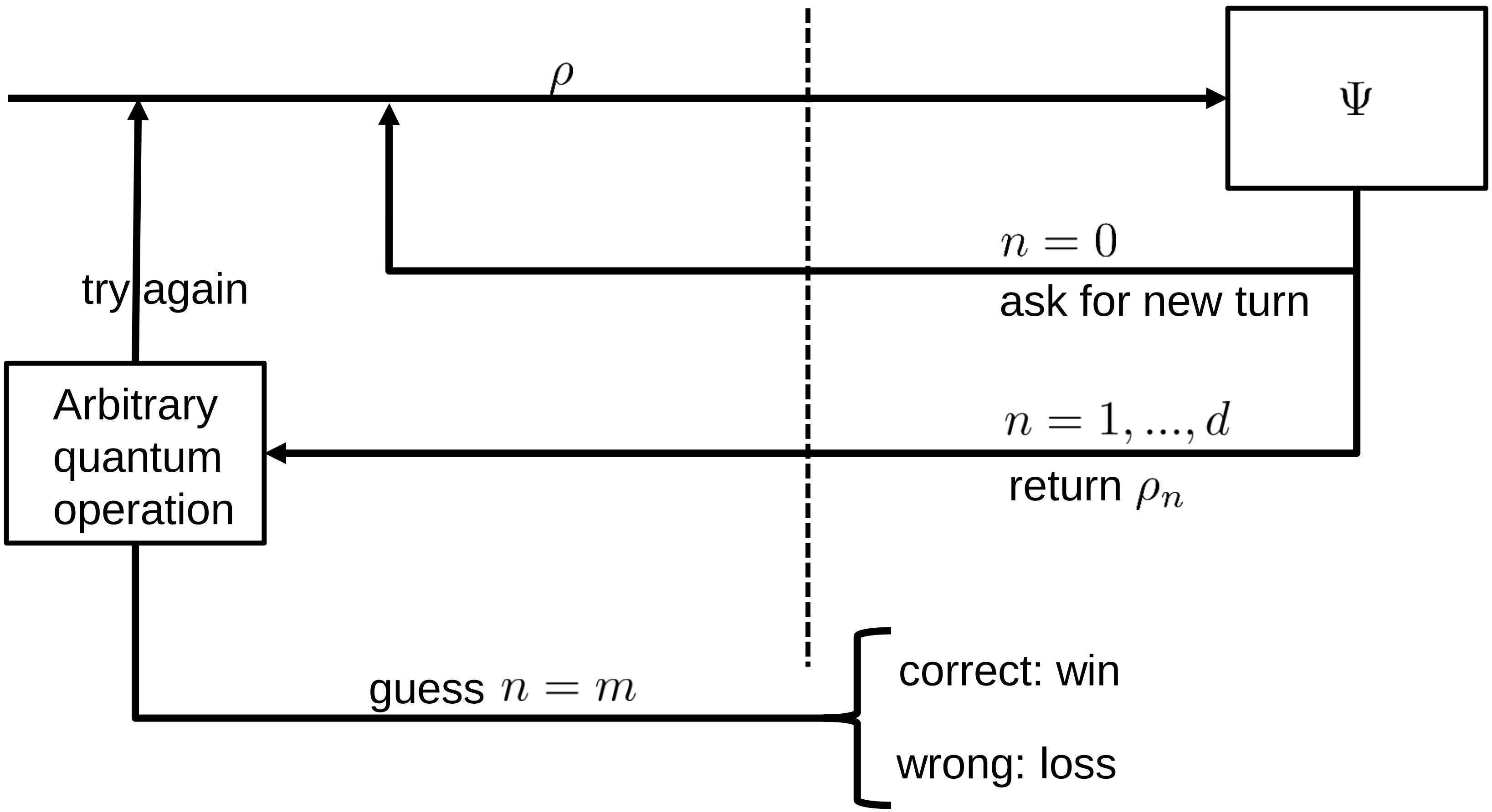
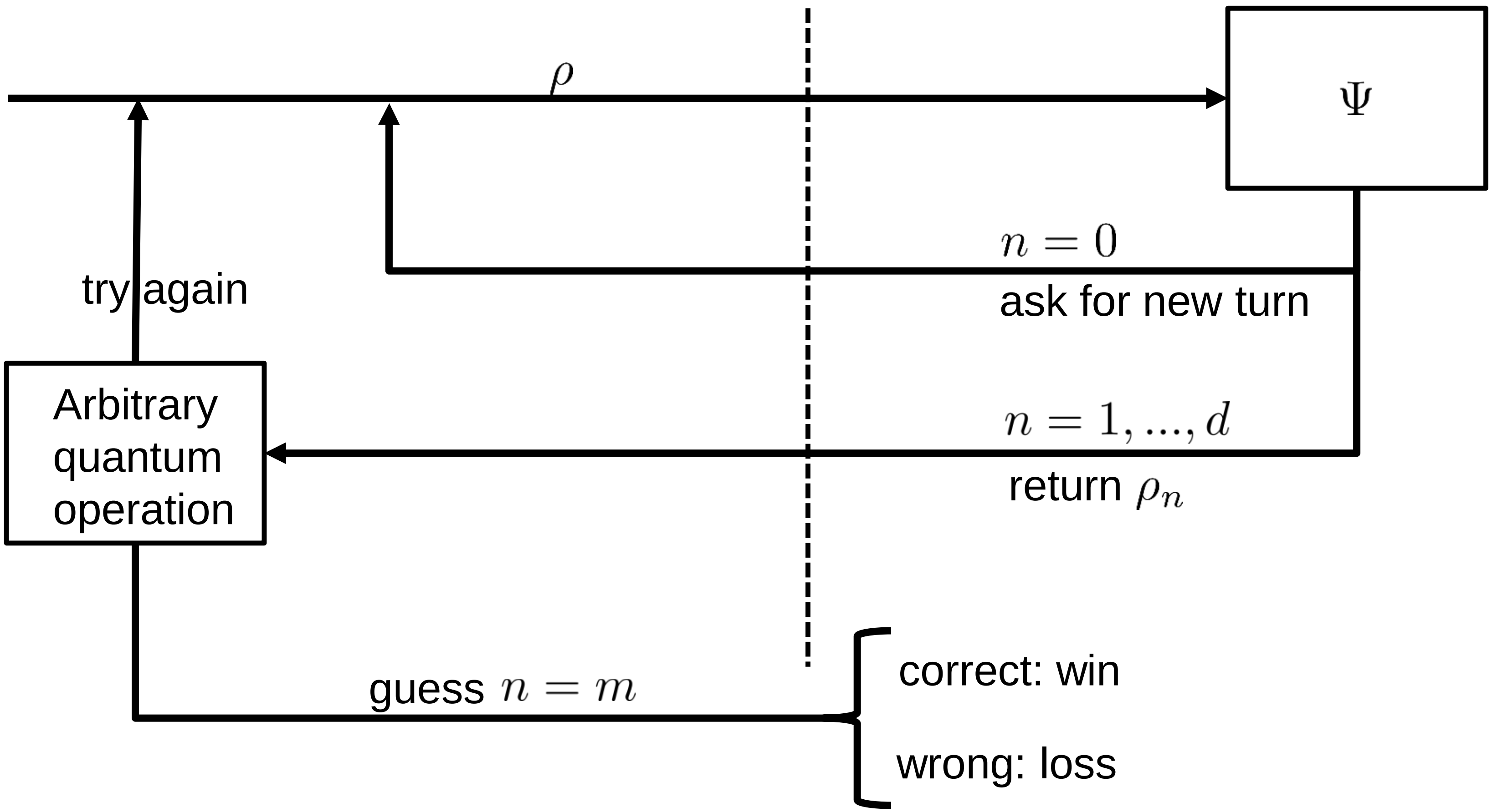


Superposition as a resource in decision tasks

$$\Psi = \{K_n\}_{n=0}^{d}; \quad \rho_n = K_n \rho K_n^{\dagger}/p_n : K_n = \sqrt{p/d}\sum_{j=1}^{d} e^{\frac{2\pi i j n}{d}} |c_j\rangle\langle c_j^{\perp}| \ \forall n > 0.$$



Conclusions

Summary

- Relevance
  - Definition of non-classicality using entanglement
  - Generalization of coherence theory
  - Step toward optical non-classicality
- Mathematical structure
  - Free maps and free completion of maps
- Superposition measures
- Superposition manipulation
  - Semidefinite program
  - States with maximal superposition
- Advantages by superposition
  - Unitary qubit operations
  - Decision task

Outlook

- Manipulation
  - Mixed state, catalytic and approximate transformations
  - Transformations in the asymptotic limit
  - Counterpart to Nielsen's theorem
- Combine with further restrictions
  - Distributed scenarios
  - Energy conservation
- Generalizations
  - Drop linear independence
  - Infinite dimensional systems
  - Continuous settings