Optimal quantum driving of a thermal machine

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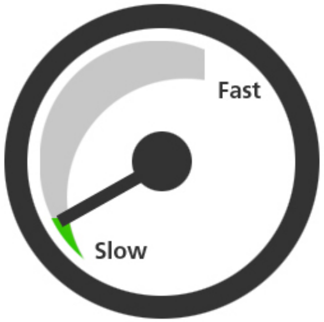
Outline



1. Slow driving of quantum thermal machines
(close to thermodynamic equilibrium)

- General theory of slowly driven master equations
- Efficiency at maximum power for heat engines

2. Optimal driving of quantum thermal machines
(strongly out of equilibrium)

- Optimality of finite-time Carnot cycles
- Full solution for a two-level system heat engine

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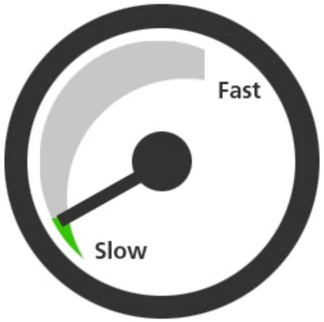

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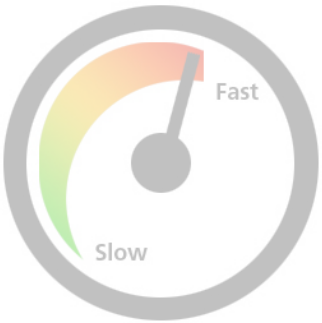

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(strongly out of equilibrium)

- Optimality of finite-time Carnot cycles
- Full solution for a two-level system heat engine

Master equations

Classical Markov process

$$\mathbf{p} = (p_1, p_2, ..., p_n)^\top$$

$$\dot{\mathbf{p}}(t) = L(t)\mathbf{p}(t)$$

Liouvillian matrix

$$L_{i\neq j}(t) \geq 0 \qquad \sum_i L_{ij}(t) = 0$$

Quantum Markov process

$$\rho = \sum_{i,j=1}^{n} \rho_{ij} |i\rangle\langle j|$$

$$\dot{\rho}(t) = \mathcal{L}(t)\rho(t)$$

Liouvillian superoperator

$$\mathcal{L}\rho = -i[H,\rho] + \sum_{j=1}^{n^2-1} 2L_j\rho L_j^\dagger - L_j^\dagger L_j \rho - \rho L_j^\dagger L_j$$

Equilibrium states

$$\dot{\rho}_0 = \mathcal{L}\rho_0 = 0$$ → ρ_0 is a **fixed point** of the map = **equilibrium state**

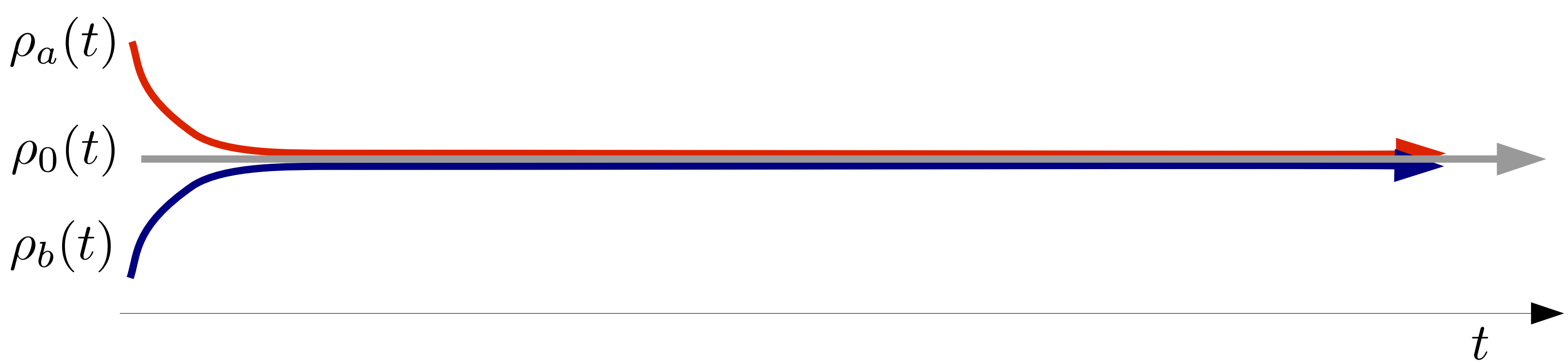
ρ_0 corresponds to an eigenvector of $\mathcal{L}$ with eigenvalue zero

$\mathcal{L}^\dagger(\mathbb{I}) = 0$
(trace preserving condition) → There is at least one equilibrium state ρ_0

If ρ_0 is **unique** the master equation is usually called "**mixing**" or "**relaxing**"

(assuming convergence from every initial state)

Mixing process



Slowly driven master equations

Time dependent master equation: $\dot{\rho}(t) = \mathcal{L}(t)\rho(t)$

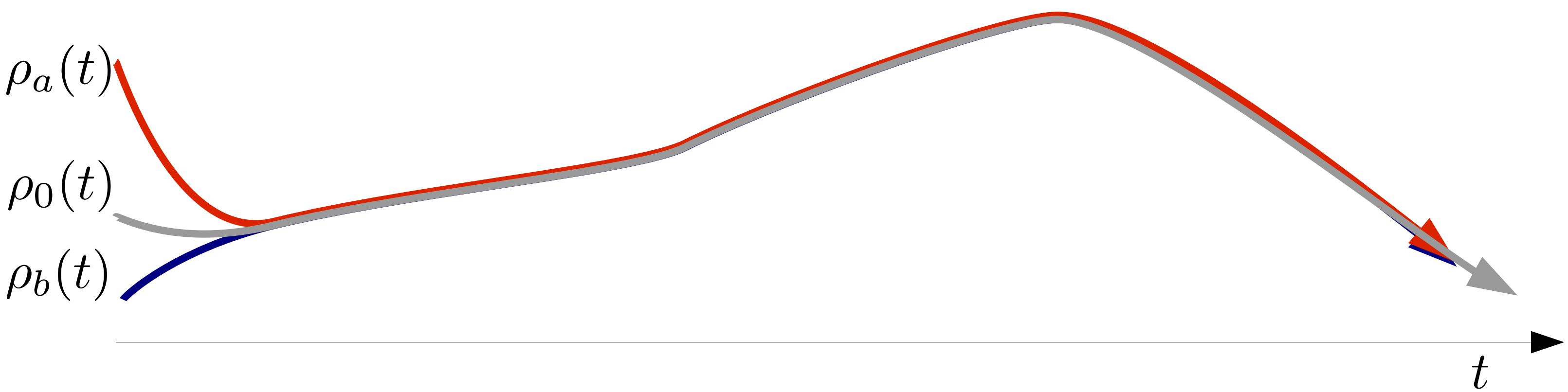
If $\mathcal{L}(t)$ is relaxing for every t : $\mathcal{L}(t)\rho_0(t) = 0$

unique instantaneous equilibrium state

Slow driving regime

[external driving time-scale] $\tau \gg \tau_S$ [characteristic time-scale of the system]

Quasi-static limit $\tau \to \infty$



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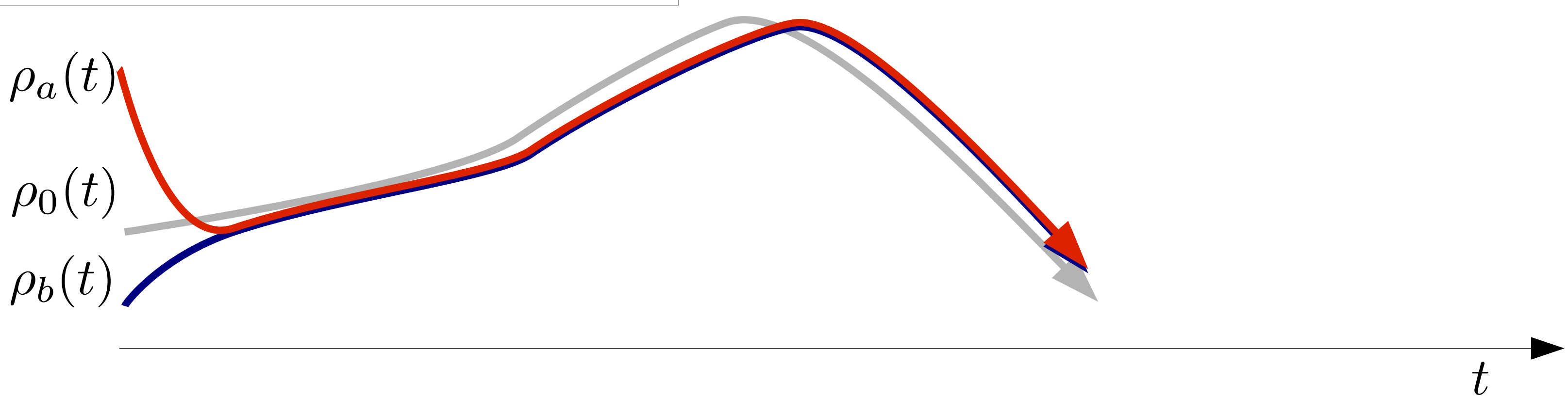
Slow driving regime

[external driving time-scale] $\tau \gg \tau_S$ [characteristic time-scale of the system]

Finite driving time $\tau < \infty$



Perturbation theory of slowly driven quantum systems

Time scaling

$$t \in [0,\tau] \rightarrow t' \in [0,1] \qquad \begin{cases} \tilde{\rho}(t') = \rho(\tau t'), \\ \tilde{\mathcal{L}}(t') = \mathcal{L}(\tau t') \end{cases}$$

$$\dot{\rho}(t) = \mathcal{L}(t)\rho(t) \longrightarrow \dot{\tilde{\rho}}(t') = \tau \tilde{\mathcal{L}}(t')\tilde{\rho}(t'),$$

time-length of the process (τ)

"shape" of the process ($\tilde{\mathcal{L}}$)

Perturbation series ansatz: $\tilde{\rho}(t') = \tilde{\rho}_0(t') + \frac{\tilde{\rho}_1(t')}{\tau} + \frac{\tilde{\rho}_2(t')}{\tau^2} + \ldots$

might not converge!

$$\left. \begin{aligned} 0 &= \tilde{\mathcal{L}}(t')\tilde{\rho}_0(t') \\ \dot{\tilde{\rho}}_j(t') &= \tilde{\mathcal{L}}(t')\tilde{\rho}_{j+1}(t') \end{aligned} \right\} \longrightarrow$$

Solution:

$$\tilde{\rho}_j(t') = \left[[\tilde{\mathcal{L}}(t')\mathcal{P}]^{-1} \frac{d}{dt'} \right]^j \rho_0(t') \qquad j = 1, 2, \ldots$$

Projector on the traceless subspace

$$\mathcal{P}X = X - \mathrm{tr}(X)\mathbb{I}$$

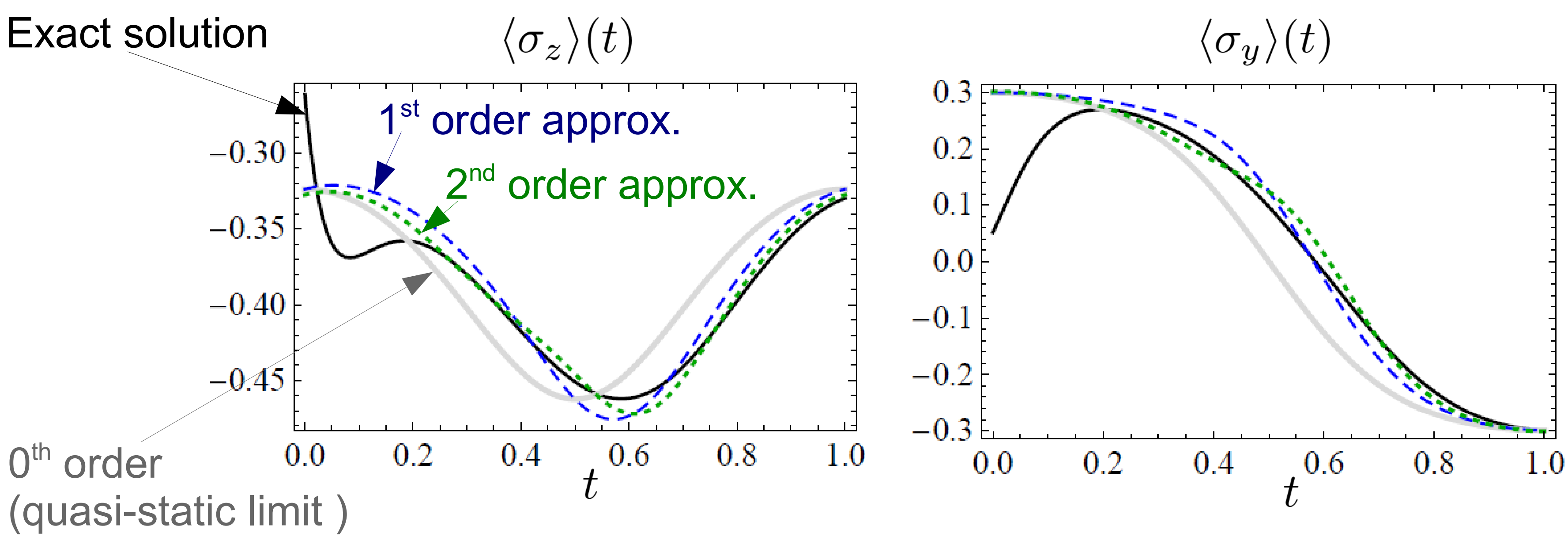
$$\rho(t) = \frac{1}{1 - [\tilde{\mathcal{L}}(t')\mathcal{P}]^{-1}\frac{d}{dt}}\rho_0(t)$$

Example: slowly driven two-level system

$N(t) = [\exp(\beta\hbar\omega(t)) - 1]^{-1}$

modulation (sinusoidal in this case)

$$\dot{\rho}(t) = -\frac{i}{2\hbar}[\rho(t), \Delta(t)\sigma_x] + \gamma(N(t)+1)\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\right)$$
$$+\gamma N(t)\left(\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\right)$$



Finite-time thermodynamics

Thermal master equations:

$$\begin{cases} \mathcal{L}(t)\rho_0(t) = 0 \\ \rho_0(t) = \dfrac{\exp[-\beta H(t)]}{Z(t)}, \quad Z(t) = \mathrm{tr}\{\exp[-\beta H(t)] \end{cases}$$

$$\tilde{\rho}(t') = \tilde{\rho}_0(t') + \frac{\tilde{\rho}_1(t')}{\tau} + \frac{\tilde{\rho}_2(t')}{\tau^2} + \ldots$$

$$\tilde{\rho}_j(t') = \left[[\tilde{\mathcal{L}}(t')\mathcal{P}]^{-1} \frac{d}{dt'} \right]^j \frac{\exp[-\beta \tilde{H}(t')]}{\tilde{Z}(t')}$$

Quasi-static evolution

Finite-time corrections

$$S(t) = -\mathrm{tr}[\rho(t)\log(\rho(t))] = S_0(t) + S_1(t)/\tau + S_2(t)/\tau^2 + \ldots$$

$$U(t) = \mathrm{tr}[H(t)\rho(t)] = U_0(t) + U_1(t)/\tau + U_2(t)/\tau^2 + \ldots$$

$$Q = \int_0^\tau \mathrm{Tr}[\dot{\rho}(t)H(t)]dt = \int_0^1 \mathrm{Tr}[\dot{\tilde{\rho}}(t')\tilde{H}(t')]dt' = Q_0(t) + Q_1/\tau + Q_2/\tau^2 + \ldots$$

$$W = \int_0^\tau \mathrm{Tr}[\rho(t)\dot{H}(t)]dt = \int_0^1 \mathrm{Tr}[\tilde{\rho}(t')\dot{\tilde{H}}(t')]dt' = W_0(t) + W_1/\tau + W_2/\tau^2 + \ldots$$

Reversible thermodynamics

Irreversible corrections

First order irreversible corrections

$$U_1(t) = \mathrm{tr}\left[\tilde{H}(t')\rho_1(t')\right]_{t'=t/\tau},$$

$$= \mathrm{tr}\left[\tilde{H}(t')[\tilde{\mathcal{L}}(t')\mathcal{P}]^{-1}\frac{d}{dt'}\tilde{\rho}_0(t')\right]_{t'=t/\tau},$$

$$S_1(t) = -\mathrm{Tr}[\tilde{\rho}_1(t')\log(\tilde{\rho}_0(t'))]_{t'=t/\tau} = \beta U_1(t),$$

$$Q_1 = \int_0^1 \mathrm{tr}\left[\tilde{H}(t')\dot{\tilde{\rho}}_1(t')\right]dt'$$

$$= \int_0^1 \mathrm{tr}\left[\tilde{H}(t')\tilde{\mathcal{L}}(t')\tilde{\rho}_2(t')\right]dt'$$

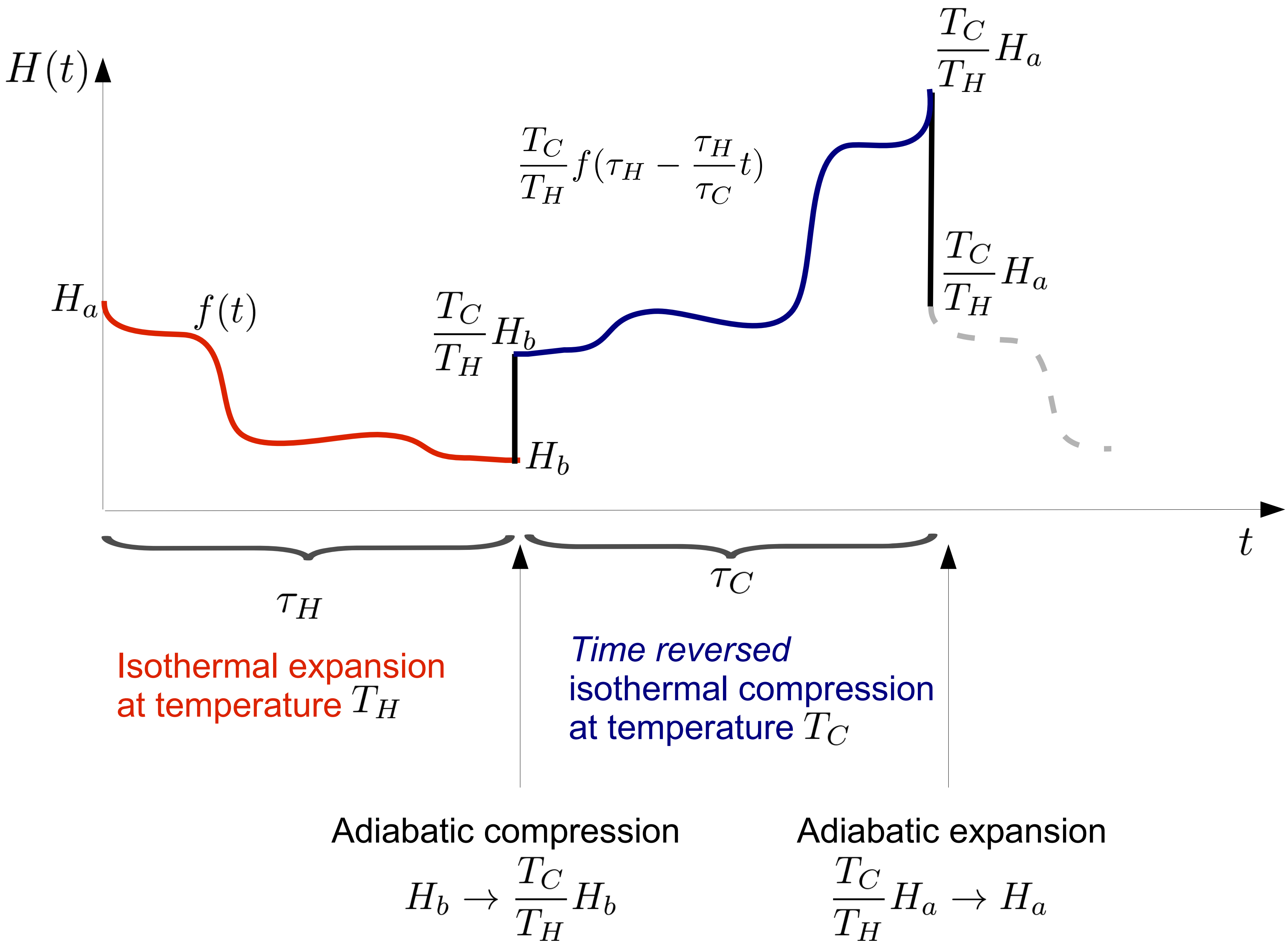
$$= \int_0^1 \mathrm{tr}\left[\tilde{H}(t')\frac{d}{dt'}[\tilde{\mathcal{L}}(t')\mathcal{P}]^{-1}\frac{d}{dt'}\tilde{\rho}_0(t')\right]dt',$$

1st law $W_1 = \Delta U_1 - Q_1.$

2nd law $\begin{cases} Q_1 < 0 \\ W_1 > 0 \end{cases}$

Important property: Q_1 is invariant for a time reversed protocol

Finite-time Carnot cycle



Efficiency at maximum power

Limit of many cycles → Initial conditions are lost and also the quantum state becomes periodic, $\Delta U = 0$

1st order perturbation theory

$$W = \cancel{\Delta U} - Q^H - Q^C$$

Power

$$P = \frac{-W}{\tau_H + \tau_C} \simeq \frac{Q_0^H + Q_1^H/\tau_H + Q_0^C + Q_1^C/\tau_C}{\tau_H + \tau_C}$$

Efficiency

$$\eta \simeq 1 + \frac{Q_0^C + Q_1^C/\tau_C}{Q_0^H + Q_1^H/\tau_H} \xrightarrow{\tau_H, \tau_C \longrightarrow \infty}$$

Carnot efficiency

$$\eta_C = 1 - \frac{T_C}{T_H}$$

Max Power $P^* = \max_{\tau_H, \tau_C} P$

Efficiency at max Power $\eta^* = \left[\frac{2}{\eta_C} - \frac{1}{1 + \sqrt{Q_1^C/Q_1^H}}\right]^{-1}$

We know how to compute finite-time heat corrections

Schmiedl, Seifert. EPL 81.2 20003 (2007)

Esposito *et al.*, PRL 105, 150603 (2010)

Efficiency at maximum power

$$\eta^* = \left[\frac{2}{\eta_C} - \frac{1}{1+\sqrt{Q_1^C/Q_1^H}}\right]^{-1}$$

If $\beta(t)H(t)$ is continuous and differentiable

$$Q_1^{(H,C)} = \int_0^1 \mathrm{tr}\left[\tilde{H}^{(H,C)}(t')\frac{d}{dt'}[\tilde{\mathcal{L}}^{(H,C)}(t')\mathcal{P}]^{-1}\frac{d}{dt'}\tilde{\rho}_0^{(H,C)}(t')\right]dt'$$

(depends on the particular protocol)

Pseudo-time reversal symmetry of the cycle

$$\tilde{H}^{(C)}(t') = \frac{T_C}{T_H}\tilde{H}^{(H)}(1-t')$$

Scaling properties of thermal Liouvillians
(derives from macroscopic derivation)

$$\mathcal{L}(\lambda H, \{L_j\}, \lambda^{-1}\beta) = \mathcal{L}(H, \{\lambda^\alpha L_j\}, \beta)$$

Spectral density exponent

$$J(\omega) = \eta\omega^\alpha$$

$$\frac{Q_1^C}{Q_1^H} = \left(\frac{T_C}{T_H}\right)^{1-\alpha}$$

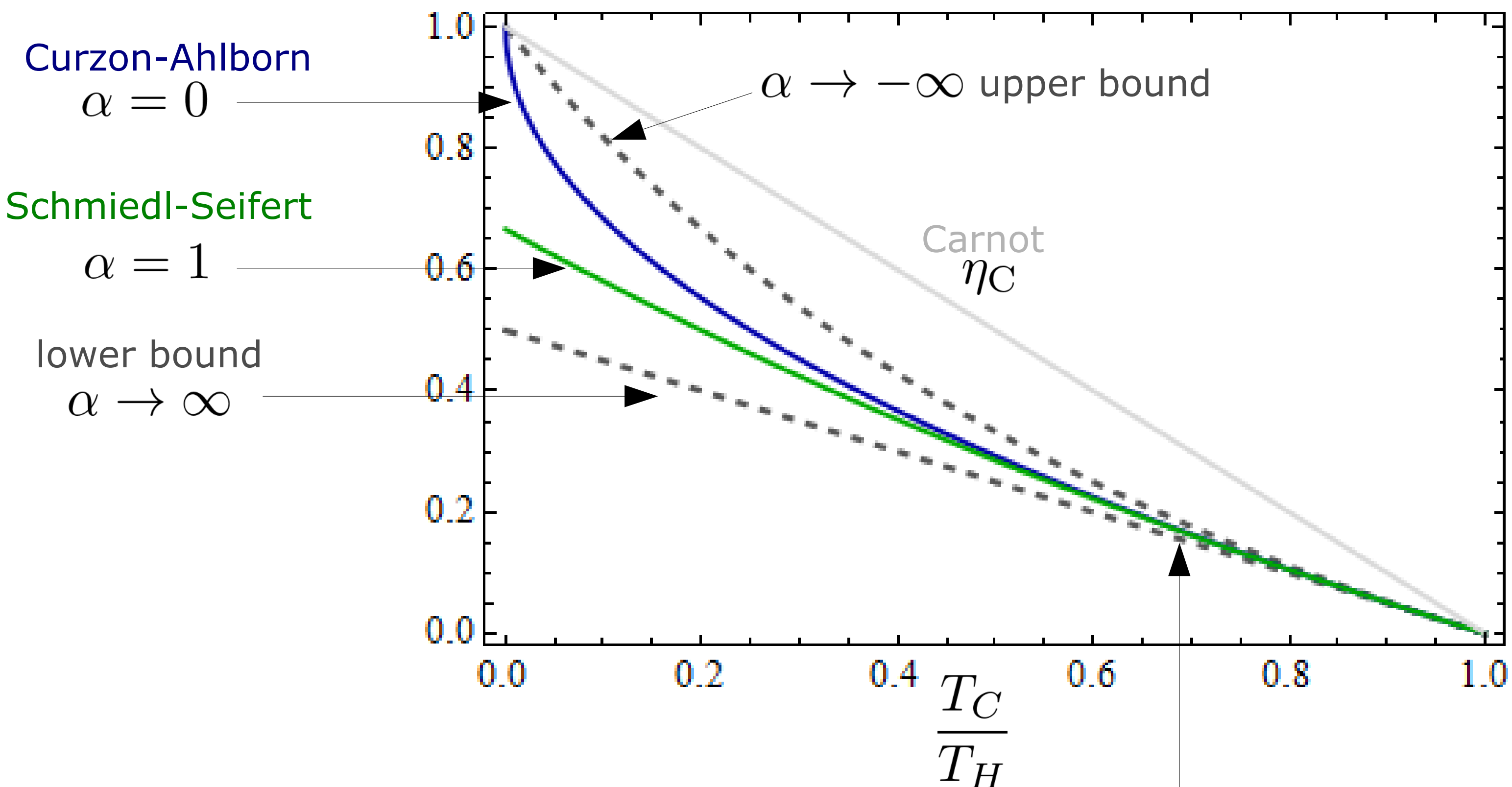
Universal scaling for all protocols

Efficiency at maximum power

Thermal bath spectral density		**Efficiency at maximum power**	
	$J(\omega) = \eta\omega^{\alpha}$	$$\eta^* = \left[\frac{2}{1 - T_C/T_H} - \frac{1}{1 + (T_C/T_H)^{(1-\alpha)/2}}\right]^{-1}$$	
Flat bath	$J(\omega) = \eta$	$\eta^*\|_{\alpha=0} = 1 - \sqrt{\frac{T_C}{T_H}}$	Curzon, Ahlborn, AJP 43, 22 (1975) Chambadal, *L.c..n.*, 4 1-58 (1957)
Ohmic bath	$J(\omega) = \eta\omega$	$\eta^*\|_{\alpha=1} = 2\eta_C/(4 - \eta_C)$	Schmiedl, Seifert. EPL 81.2 20003 (2007)
Infinitely super-Ohmic bath	$J(\omega) = \eta\omega^{\alpha\to\infty}$	$\eta^*\|_{\alpha\to\infty} = \frac{\eta_C}{2}$	Esposito *et al.,* PRL 105, 150603 (2010) Schmiedl, Seifert. EPL 81.2 20003 (2007) Benenti, *et al. ArXiv:1608.05595 (2016)*
Infinitely sub-Ohmic bath	$J(\omega) = \eta\omega^{\alpha\to-\infty}$	$\eta^*\|_{\alpha\to-\infty} = \frac{2\eta_C}{2 - \eta_C}$	

Efficiency at maximum power

$$\eta^* = \left[\frac{2}{1 - T_C/T_H} - \frac{1}{1 + (T_C/T_H)^{(1-\alpha)/2}}\right]^{-1}$$



CAUTION
PROCEED WITH CAUTION

Only within 1st order perturbation theory

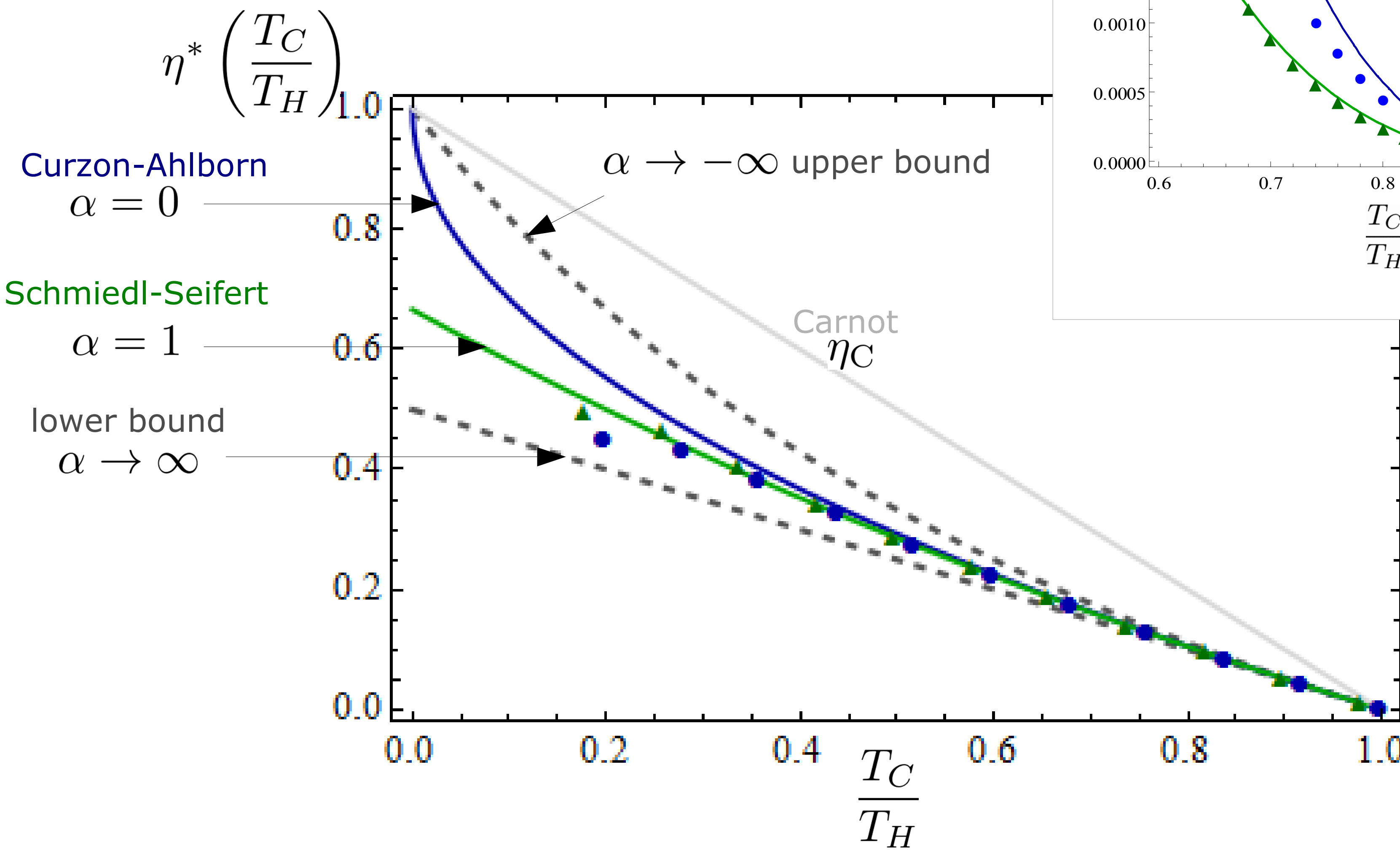
Only for sufficiently smooth cycles

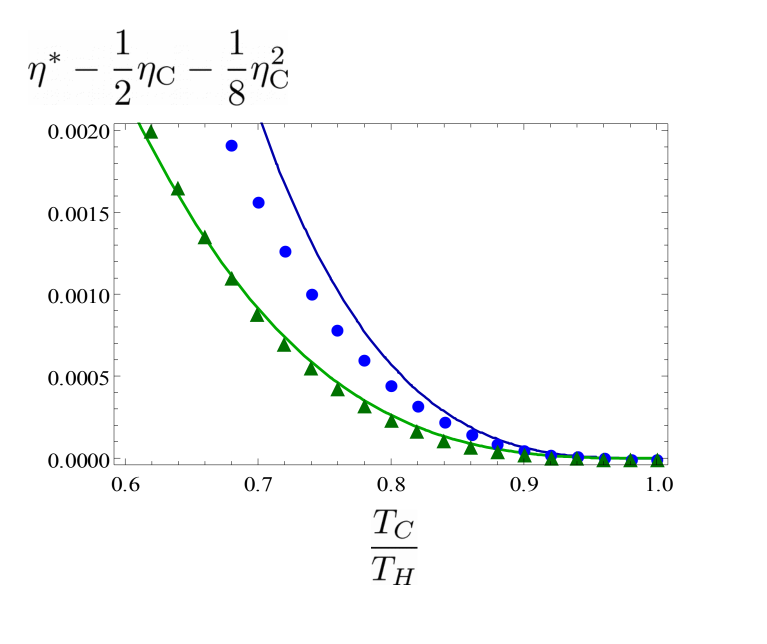
$\beta(t)H(t) \in C^1$

$$\eta^* = \frac{1}{2}\eta_C + \frac{1}{8}\eta_C^2 + \frac{2-\alpha}{32}\eta_C^3 + O(\eta_C^4)$$

Efficiency at maximum power

Exact simulation based on a single qubit in flat or Ohmic thermal baths: ▲ $\alpha = 0$ ▲ $\alpha = 1$





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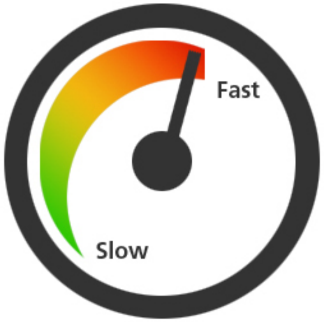

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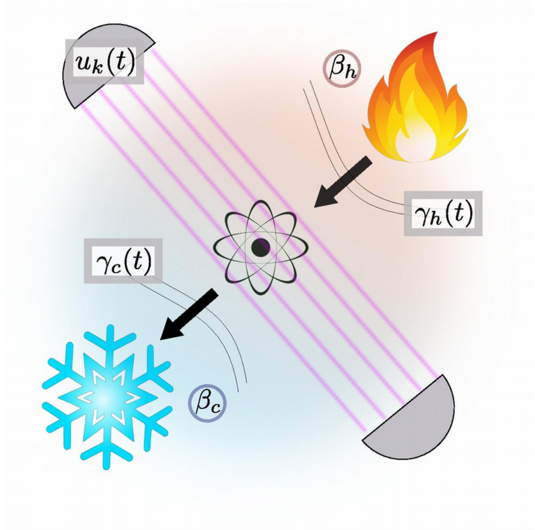
(strongly out of equilibrium)

- Optimality of finite-time Carnot cycles
- Full solution for a two-level system heat engine

General questions

What is the optimal driving of a thermal machine ?

Given a d-level quantum system and two heat baths, what is the maximum power that we can extract?



Methods

~~Slow-driving perturbation theory~~ (because we are far from equilibrium)

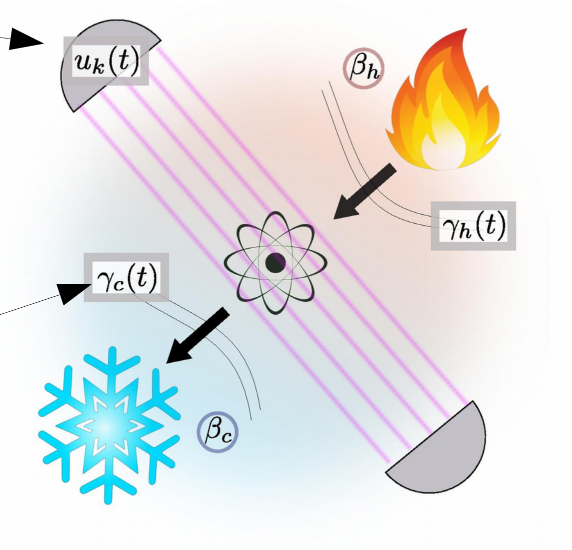
Optimal control theory approach (*Pontryagin's minimum principle*)

Optimal control of a thermal machine

Hamiltonian driving

$$\frac{d\hat{\rho}(t)}{dt} = \mathcal{L}_{\mathbf{u}(t)}[\hat{\rho}(t)] := -i[\hat{H}_{\mathbf{u}(t)}, \hat{\rho}(t)] + \mathcal{D}_{\mathbf{u}(t)}[\hat{\rho}(t)]$$

Dissipative control



Heat released by the system:

$$Q := -\int_0^\tau \left\langle \hat{H}_{\mathbf{u}(t)} \, \mathcal{L}_{\mathbf{u}(t)}[\hat{\rho}(t)] \right\rangle dt$$

Work done by the system:

$$W := -\int_0^\tau \left\langle \hat{\rho}(t) \, \frac{d\hat{H}_{\mathbf{u}(t)}}{dt} \right\rangle dt$$

Optimal control problem

minimize Q

with respect to all control strategies $\mathbf{u}(t)$

for fixed: $\tau, \rho(0), \rho(\tau)$

Pontryagin's approach

(similar to Hamiltonian formalism applied to control theory)

Extended functional

Lagrange multipliers

$$\mathcal{J} := Q + \int_0^\tau \left\{ \lambda(t)(\underbrace{\langle\hat{\rho}(t)\rangle - 1}_{\text{normalization}}) + \left\langle \hat{\pi}(t) \left(\underbrace{\mathcal{L}_{\mathbf{u}(t)}[\hat{\rho}(t)] - \frac{d\hat{\rho}(t)}{dt}}_{\text{master equation}} \right) \right\rangle \right\} dt$$

Pseudo Hamiltonian

$$\mathcal{H}(t) := \left\langle (\hat{\pi}(t) - \hat{H}_{\mathbf{u}(t)})\mathcal{L}_{\mathbf{u}(t)}[\hat{\rho}(t)] \right\rangle + \lambda(t)(\langle\hat{\rho}(t)\rangle - 1)$$

$$\mathcal{J} = \int_0^\tau \left\{ \mathcal{H}(t) - \left\langle \hat{\pi}(t) \frac{d\hat{\rho}(t)}{dt} \right\rangle \right\} dt$$

Analogue of Hamilton equations:

$$\frac{d\hat{\rho}(t)}{dt} = \frac{\partial \mathcal{H}(t)}{\partial \hat{\pi}(t)}\,, \qquad \frac{d\hat{\pi}(t)}{dt} = -\frac{\partial \mathcal{H}(t)}{\partial \hat{\rho}(t)}\,,$$

Analogue of energy conservation: $\mathcal{H}(t) = \mathcal{K}$ (constant conserved quantity)

Pontryagin's minimum principle

Necessary conditions for optimal control strategies minimizing the extended functional

$$\mathcal{J} := Q + \int_0^\tau \left\{ \lambda(t)(\langle\hat{\rho}(t)\rangle - 1) + \left\langle \hat{\pi}(t) \left(\mathcal{L}_{\mathbf{u}(t)}[\hat{\rho}(t)] - \frac{d\hat{\rho}(t)}{dt} \right) \right\rangle \right\} dt$$

are such that:

1. there exists a non-zero costate $\hat{\pi}(t)$ evolving according to:

$$\frac{d\hat{\pi}(t)}{dt} = -\frac{\partial \mathcal{H}(t)}{\partial \hat{\rho}(t)} = -\left\{ \mathcal{L}^\dagger_{\mathbf{u}(t)}[\hat{\pi}(t) - \hat{H}_{\mathbf{u}(t)}] + \lambda(t) \right\}$$

2. the pseudo Hamiltonian $\mathcal{H}(t) := \left\langle (\hat{\pi}(t) - \hat{H}_{\mathbf{u}(t)})\mathcal{L}_{\mathbf{u}(t)}[\hat{\rho}(t)] \right\rangle + \lambda(t)(\langle\hat{\rho}(t)\rangle - 1)$

is minimized by the control function $\mathbf{u}(t)$ for all $t \in [0, \tau]$

3. the pseudo Hamiltonian is constant $\mathcal{H}(t) = \mathcal{K}$

Thermodynamic link between $\mathcal{K}$ and maximum power

Does $\mathcal{K}$ have a physical meaning?

Assume that we want to maximize the power of a cyclic engine $P = W/\tau = -Q/\tau$

Its variation w.r.t. τ is:

$$\delta \mathcal{J} = \delta Q = \mathcal{H}(\tau)\delta\tau = \mathcal{K}\delta\tau$$

$$\delta P = -\frac{\delta Q}{\tau} + \frac{Q\delta\tau}{\tau^2} = -\frac{\delta Q}{\tau} - \frac{P\delta\tau}{\tau}$$

$$\delta P = -(\mathcal{K} + P)\frac{\delta\tau}{\tau} \Longrightarrow$$

optimal solutions must satisfy: $P_{\max} = -\mathcal{K}$

Optimization procedure:

Determine $\mathcal{K}^* := \min_{\mathcal{K} \in A} \mathcal{K}$

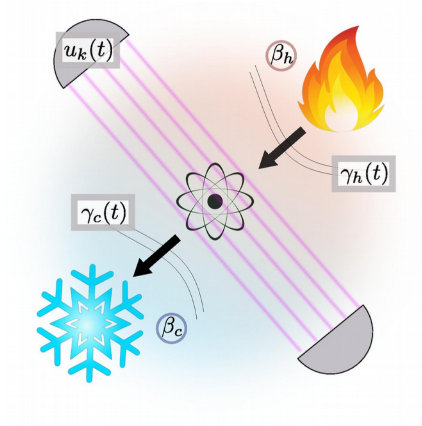
Check if $P|_{\mathcal{K}=\mathcal{K}^*} \stackrel{?}{=} -\mathcal{K}^*$

YES → $P_{\max} := \max_{\substack{\text{all cycles,} \\ \text{all } \tau > 0}} \frac{W}{\tau} = -\mathcal{K}^*$

NO → Try a larger $\mathcal{K}$

The optimal driving of a generic quantum heat engine reduces to the optimization of a **single degree of freedom** $\mathcal{K}$ within its accessible region A .

Optimal cycle for a d-level quantum heat engine



$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_{\mathbf{u}(t)}, \hat{\rho}(t)] + \gamma_c(t)\mathcal{D}^{(c)}_{\mathbf{u}(t)}[\hat{\rho}(t)] + \gamma_h(t)\mathcal{D}^{(h)}_{\mathbf{u}(t)}[\hat{\rho}(t)]$$

Upper bound on the total dissipation rate:

$$\gamma_c(t) + \gamma_h(t) \leq \Gamma$$

The optimal control for $\gamma_c(t)$ and $\gamma_h(t)$ turns out to be of "*bang-bang*" type:

2 alternatives:

- $\gamma_c(t) = \Gamma,\ \gamma_h(t) = 0$ (strong coupling **only** with the cold bath)
- $\gamma_h(t) = \Gamma,\ \gamma_c(t) = 0$ (strong coupling **only** with the hot bath)

Optimal control for the Hamiltonian $\hat{H}_{\mathbf{u}(t)}$ turns out to be given by differentiable solutions (**isothermal processes**) separated by discontinuous jumps (**adiabatic quenches**).

Maximum power quantum heat engines are achieved by a **finite-time Carnot cycle**

Power maximization: take the minimum $\mathcal{K}$ such that $P = -\mathcal{K}$

Example: full solution for a 2-level system

control on the energy level

$$\hat{H}(t) = u(t)|1\rangle\langle 1]$$

Gibbs thermalizing dissipators

$$\mathcal{D}_{u(t)}^{(c,h)}[\hat{\rho}(t)] = \frac{e^{-\beta_{(c,h)}u(t)}|1\rangle\langle 1| + |0\rangle\langle 0|}{e^{-\beta_{(c,h)}} + 1} - \hat{\rho}(t)$$

Quantum state (diagonal): $\hat{\rho}(t) := p(t)|1\rangle\langle 1| + [1 - p(t)]|0\rangle\langle 0|$

Pontryagin's costate: $\hat{\pi}(t) = q(t)(|0\rangle\langle 0| - |1\rangle\langle 1|)$

Pseudo Hamiltonian: $\mathcal{H} = \Gamma(-p + \frac{1}{1 + e^{\beta_{c,h}}})(2q + u) = \mathcal{K}$ (constant of motion)

Pseudo Hamilton equations: $\frac{dp(t)}{dt} = \Gamma\Big[\frac{1}{1 + e^{\beta_{c,h}u(t)}} - p(t)\Big]$ (master equation)

$$\frac{dq(t)}{dt} = \frac{\Gamma}{2}[2q(t) + u(t)]$$ (costate equation)

Optimal solutions for a 2-level system

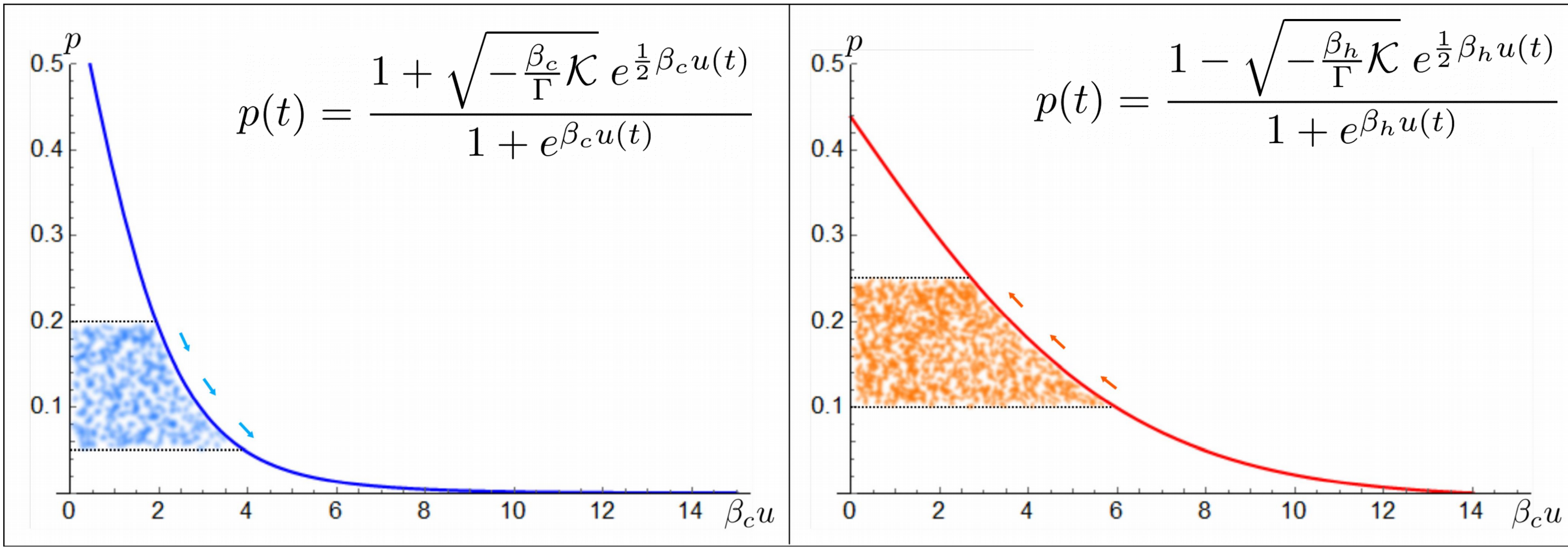
$$\hat{H}(t) = u(t)|1\rangle\langle 1|$$

$$\hat{\rho}(t) = p(t)|1\rangle\langle 1| + [1 - p(t)]|0\rangle\langle 0|$$

Optimal trajectories in the (u, p) plane

Cold isotherm

Hot isotherm



$$\mathcal{K} = -0.05\,\Gamma/\beta_c \qquad \beta_h = 0.3\,\beta_c$$

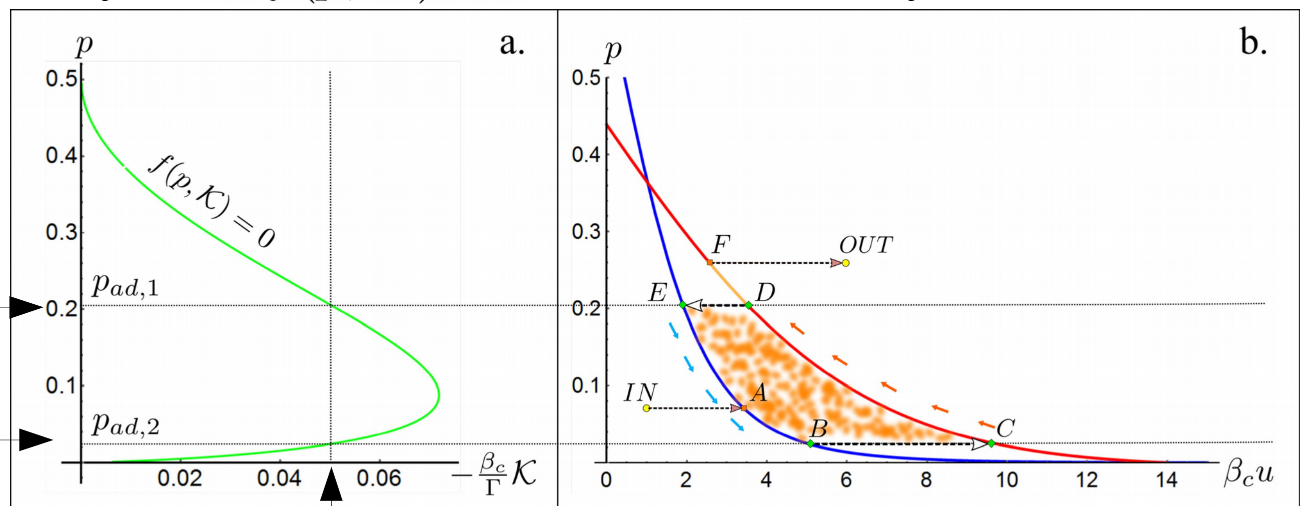
Optimal solutions for a 2-level system

$$\hat{H}(t) = u(t)|1\rangle\langle 1|$$
$$\hat{\rho}(t) = p(t)|1\rangle\langle 1| + [1 - p(t)]|0\rangle\langle 0|$$

$\hat{\pi}(t)$ is also a continuous cycle $\Leftrightarrow f(p, K) = 0$

Carnot cycle at fixed $\mathcal{K}$

Populations for adiabats



$\mathcal{K} = -0.05\,\Gamma/\beta_c$

$\beta_h = 0.3\,\beta_c$

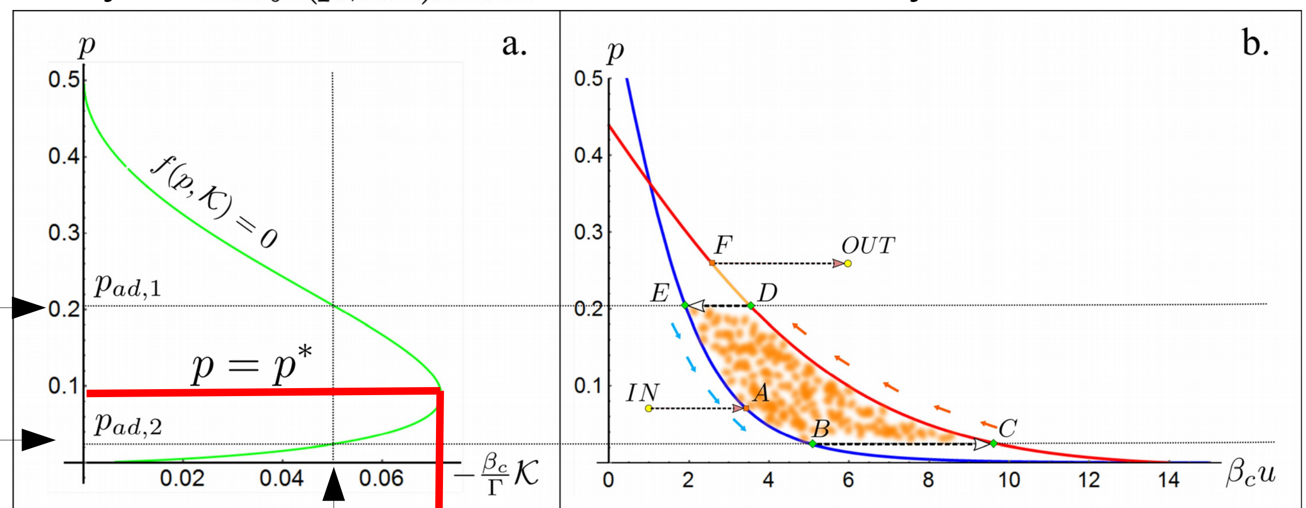
$\mathcal{K}$ completely determines the Carnot cycle.

Optimal solutions for a 2-level system

$$\hat{H}(t) = u(t)|1\rangle\langle 1|$$
$$\hat{\rho}(t) = p(t)|1\rangle\langle 1| + [1 - p(t)]|0\rangle\langle 0|$$

$\hat{\pi}(t)$ is also a continuous cycle $\Leftrightarrow f(p, K) = 0$

Carnot cycle at fixed $\mathcal{K}$



Populations for adiabats

$\mathcal{K} = -0.05\,\Gamma/\beta_c$

$\beta_h = 0.3\,\beta_c$

$\mathcal{K} = \mathcal{K}^*$

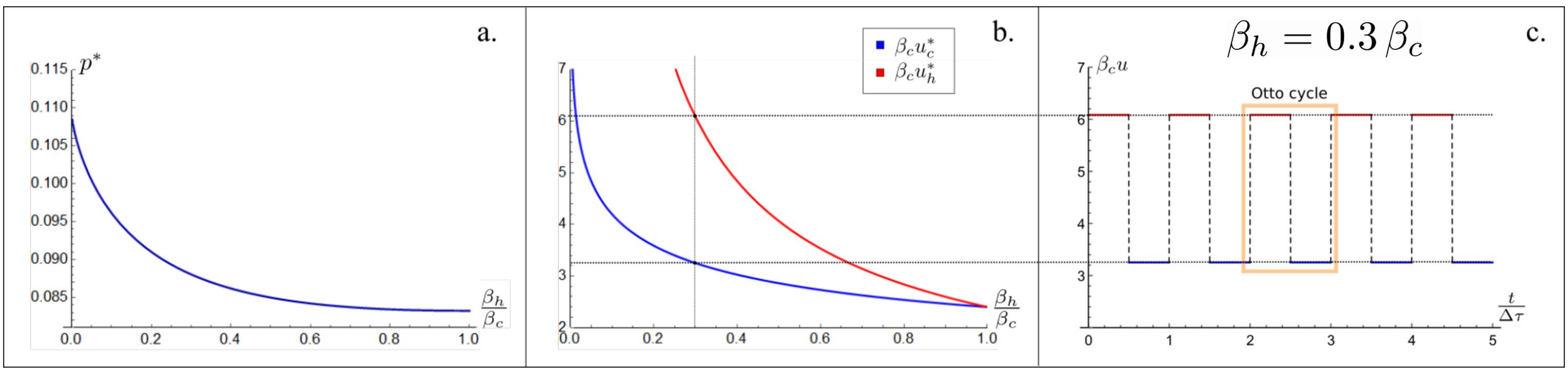
$\mathcal{K}$ completely determines the Carnot cycle.

The maximum power is achieved for $\mathcal{K} = \mathcal{K}^*$ corresponding to an infinitesimal cycle performed around the optimal non-equilibrium state $p = p^*$

Maximum power cycle for a 2-level system

$$\hat{H}(t) = u(t)|1\rangle\langle 1|$$
$$\hat{\rho}(t) = p(t)|1\rangle\langle 1| + [1 - p(t)]|0\rangle\langle 0|$$

Optimal state | Optimal energy levels | Optimal control

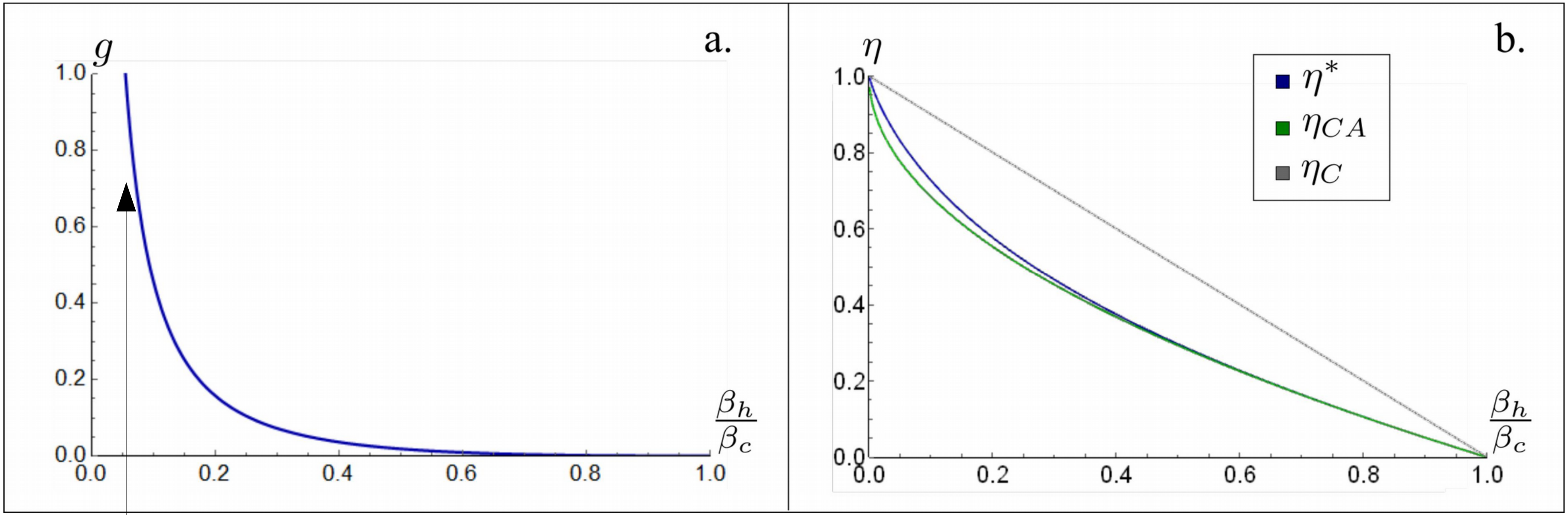


Maximum power cycle for a 2-level system

$$\hat{H}(t) = u(t)|1\rangle\langle 1|$$
$$\hat{\rho}(t) = p(t)|1\rangle\langle 1| + [1-p(t)]|0\rangle\langle 0|$$

Maximum power $P_{\max} = -\mathcal{K}^* = \frac{\Gamma}{\beta_c} g\left(\frac{\beta_h}{\beta_c}\right)$

Efficiency at maximum power



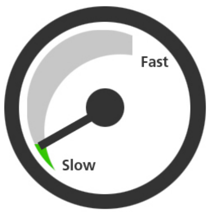
$\frac{\beta_h}{\beta_c} \to 0$ (high power limit)

$$g(z) \simeq \frac{0.06961}{z} \quad \Rightarrow P_{\max} \simeq 0.06961 \frac{\Gamma}{\beta_h}$$

$$\eta^* = 1 - \frac{u_c^*}{u_h^*}$$

Remark: same efficiency as for a quasi-static Otto cycle

Conclusions



1. Slow driving of quantum thermal machines [1]

- Perturbation theory of slowly driven master equations
- Universal formula for the efficiency at maximum power

2. Optimal driving of quantum thermal machines [2]

- Optimal control theory approach (*Pontryagin's minimum principle*)
- Optimal processes are finite-time Carnot cycles
- Maximum power = conserved quantity of the control problem: $-\mathcal{K}$
- Full solution for a two-level system heat engine

[1] Cavina, AM, Giovannetti, Phys. Rev. Lett. (2017).

[2] Cavina, AM, Carlini, Giovannetti, arXiv: (2017).