Vacuum polarization effects probed at high-intensity laser facilities

↓

(strong) electromagnetic fields

Winter College on Extreme Non-linear Optics, Attosecond Science and High-field Physics

Felix Karbstein

14. February 2018

f.karbstein@gsi.de

Introduction

classical understanding : vacuum is empty
no particles / fields
no boundary conditions
zero temperature

e.m. fields : theory of electrodynamics

$$\mathcal{L}_{ED} = \mathcal{L}_{MW} + \mathcal{L}_j$$

$$\mathcal{L}_{MW} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_j = -\underset{\substack{\uparrow \\ \text{source}}}{j_\mu} A^\mu$$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
field strength tensor

$$\mathcal{L} = \mathcal{L}(A_\nu, \partial_\mu A_\nu) \qquad \text{vgl. } L(x, \dot{x})$$

$$\xrightarrow{\text{EoMs}} \quad \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\rightsquigarrow \mathcal{L}(A_\nu, F_{\mu\nu})$$

$$\longrightarrow \quad \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \underbrace{\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}}_{= \delta_\alpha{}^\mu \delta_\beta{}^\nu - \delta_\beta{}^\mu \delta_\alpha{}^\nu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - 2\partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = 0$$

i.e. for $\mathcal{L} = \mathcal{L}_{ED}$

$$j^\nu = 2\partial_\mu \frac{1}{2} F^{\mu\nu} = \partial_\mu F^{\mu\nu}$$

Maxwell eqs.

& in vacuo : $j^\nu = 0 \quad \rightsquigarrow \boxed{\partial_\mu F^{\mu\nu} = 0}$

→ superposition principle holds:

if A_i^ν with associated $F_i^{\mu\nu}$ are solutions, then clearly also $\sum_i A_i^\nu$.

this is different in the "true" vacuum

$\simeq$ the $\boxed{\text{quantum}}$ vacuum

→ vacuum of QFT here: QED

QVac is not empty but permeated by fluctuations of the fields of the considered QFT in form of virtual processes. In QED: electrons positrons & photons

These virtual processes influence the dynamics of (strong) macroscopic electromagnetic fields.

Theory of quantum electrodynamics (QED)

$$\mathcal{L}_{QED} = \bar{\Psi}\,(i\partial^\mu \gamma_\mu + \underset{\substack{\uparrow \\ e^- \text{ charge}}}{e} A^\mu \gamma_\mu - \underset{\substack{\uparrow \\ \text{mass}}}{m_e})\,\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$(\hbar = c = 1 = \varepsilon_0)$

$$= \bar{\Psi}\,(i\partial^\mu \gamma_\mu - m_e)\,\Psi + e A^\mu \bar{\Psi} \gamma_\mu \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Ψ: 4-component complex Dirac spinor (anti-commuting Grassmann-valued field)

γ matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = -2 g_{\mu\nu}$

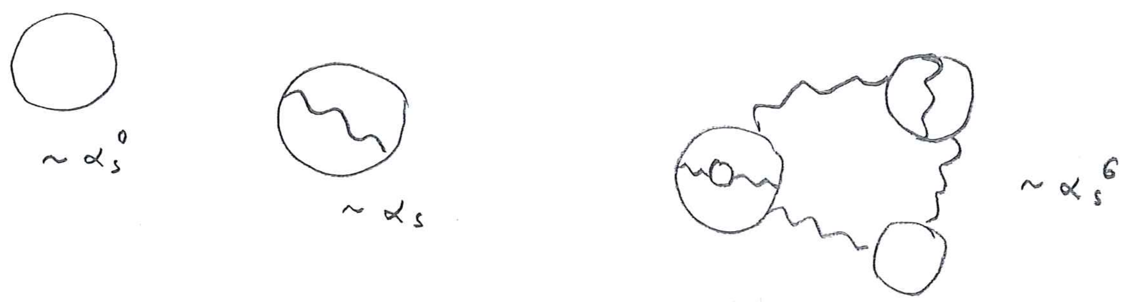
with metric $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

basic principle: all processes that can be drawn can happen (if they are not forbidden)

→ more suppressed with increasing loop order $\sim \alpha_s^{\ell-1}$

→ vacuum of QED: $\alpha_s \overset{(SI)}{=} \frac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \frac{1}{137}$

Feynman diagrams with internal lines only



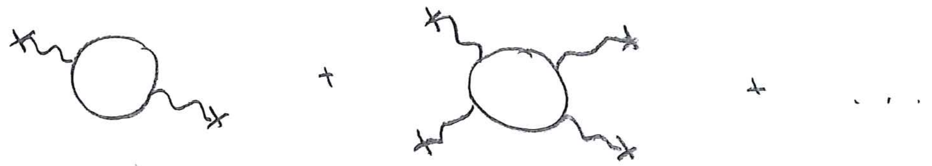
↪ no coupling to observer, not probed.

QED + external e.m. fields

$$\mathcal{L}_{extQED} = \mathcal{L}_{QED} + \bar{\Psi}\gamma_\mu \Psi e A^\mu_{ext} - \frac{1}{4} F_{ext,\mu\nu} F^{\mu\nu}_{ext}$$

→ Vacuum fluctuations induce effective self-couplings of e.m. fields.

At dominant / 1-loop order: $\sim \alpha_s^0$



linear & <u>nonlinear</u> couplings of e.m. fields.

$\simeq$ QED vacuum + external e.m. fields

Question: How does the effective theory <u>of e.m. fields</u> in the (quantum) vacuum look like?

⇢ What are the effects to be observed?

General properties of $\mathcal{L}_{eff}$ are

(i) Lorentz scalar

(ii) gauge invariance

(iii) mass dimension: $[\mathcal{L}_{eff}] = 4$

(iv) CP invariance (as no CP violation known in QED) (assuming locality of $\mathcal{L}_{eff}$)

⇝ what scalar combinations can be formed from $F^{\mu\nu}$ alone?

just two, namely

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(\vec{B}^2 - \frac{\vec{E}^2}{c^2}\right) \quad \& \quad \mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = -\frac{\vec{E}}{c} \cdot \vec{B}$$

$\hat{=}$ Lorentz scalar

with $\tilde{F}^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$

$(\epsilon^{0123} = 1)$ totally antisym. tensor

$\hat{=}$ pseudo-scalar, CP odd

where we used the defining relations

$$\vec{E}(x) = -\vec{\nabla} A^0(x) - \partial_t \vec{A}(x)$$
$$\vec{B}(x) = \vec{\nabla} \times \vec{A}(x) .$$

i.e. building blocks for $\mathcal{L}_{eff}$ are $\mathcal{F}$ & $\mathcal{G}^2$.

As in general $\partial_\alpha F_{\mu\nu} \neq 0$, also ∂_μ is a building block.

In turn, $\mathcal{L}_{eff}$ is expected to be of the form

$$\mathcal{L}_{eff} = -\mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2 + \mathcal{O}(\mathcal{F}^3, \mathcal{F}\mathcal{G}^2)$$
$$+ d_1 F_{\mu\nu} \partial^2 F^{\mu\nu} + \mathcal{O}(\partial^2 F^4, \partial^4 F^2)$$

where terms which are of higher order such as $\partial_\mu F^{\mu\nu}$ have already been dropped.

⇝ Gradient expansion in field strength with coeffs $c_{1,2}, d_1$ of mass dimension

$$[c_{1,2}] = -4 \quad , \quad [d_1] = -2$$

⇝ as long as no other dimensionful scale is available, the coefficients $c_{1,2}, d_1, \ldots$ have to vanish, and

$\mathcal{L}_{eff} \to -\mathcal{F}$ $\simeq$ classical ED in vacuo.

⇝ however, allowing for quantum fluctuations the e.m. field can couple to charged particle/antiparticle fluctuations ⇝ dimensional parameter: mass scale m

in QED, electron mass m_e

$m_e \simeq 511\,\mathrm{keV}$

$\rightsquigarrow$ length scale $\lambda\!\!\!^{-}_c = \frac{\hbar}{m_e c} \simeq 3{,}8 \cdot 10^{-13}\,\mathrm{m}$

time scale $\tau_c = \frac{\hbar}{m_e c^2} \simeq 1{,}3 \cdot 10^{-21}\,\mathrm{s}$

$\uparrow$

Compton

$\rightsquigarrow$ magnetic field $\frac{m_e^2 c^2}{\hbar} \simeq 1{,}3 \cdot 10^{9}$ Tesla

electric field $\frac{m_e^2 c^3}{\hbar} \simeq 4{,}0 \cdot 10^{17}\,\frac{\mathrm{V}}{\mathrm{m}}$

Hence, we expect $c_{1,2} \sim \frac{1}{m_e^4}$, $d_1 \sim \frac{1}{m_e^2}$

If a e.m. field is "<u>slowly varying</u>" on length / time scales L, T larger larger than $\lambda\!\!\!^{-}_c, \tau_c$, neglecting the derivative terms in the expansion of $\mathcal{L}_{\mathrm{eff}}$ is a very good approximation

as $d_1 \partial^2 \sim \frac{\partial^2}{m_e^2} \sim \left(\frac{\frac{1}{L}}{\frac{1}{\lambda\!\!\!^{-}_c}}\right)^2 \sim \left(\frac{\lambda\!\!\!^{-}_c}{L}\right)^2 \sim \left(\frac{\tau_c}{T}\right)^2 \ll 1$

$\rightsquigarrow$ Macroscopic e.m. fields available in the laboratory, in particular laser fields, are of slowly varying type, such that $\mathcal{L}_{\mathrm{eff}} \simeq \mathcal{L}_{\mathrm{eff}}(\mathcal{F}, \mathcal{G}^2)$ provides an sufficiently accurate description

$\rightsquigarrow$ In other words, it suffices to determine $\mathcal{L}_{\mathrm{eff}}$ in constant e.m. fields, $\partial_\alpha F^{\mu\nu} = 0$, only.

If, in addition, $\{c_1 \mathcal{F}, c_2 \mathcal{G}\} \ll 1 \iff |\vec{B}| \ll 1{,}3 \cdot 10^{9}$ Tesla
$|\vec{E}| \ll 4{,}0 \cdot 10^{17}\,\frac{\mathrm{V}}{\mathrm{m}}$

also the nonlinear terms can be neglected, and $\mathcal{L}_{\mathrm{eff}} \simeq -\mathcal{F}$.

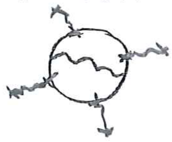
High-intensity laser fields reach

$$|\vec{B}| \sim \mathcal{O}(10^5 - 10^6)\,\mathrm{T}, \quad |\vec{E}| \sim \mathcal{O}(10^{13} - 10^{14})\,\frac{\mathrm{V}}{\mathrm{m}}$$

The coefficients $c_{1,2}$ can be determined explicitly from QED

Euler, Kockel: Naturwiss. 23, 246 (1935)
Euler: Ann. Phys. 5, 26 (1936)

& 1 loop, all orders: Heisenberg, Euler: Z. Phys. 98, 714 (1936)
2 loop: Ritus: Sov. Phys. JETP 42, 774 (1975)



$\rightsquigarrow$ This results in

$$c_1 = \frac{8}{45}\frac{\alpha^2}{m_e^4}\frac{\hbar}{c^2} \times \left(1 + \frac{40}{9\pi}\alpha + \mathcal{O}(\alpha^2)\right) \quad \& \quad c_2 = \frac{14}{45}\frac{\alpha^2}{m_e^4}\frac{\hbar}{c^2} \times \left(1 + \frac{1315}{252\pi}\alpha + \mathcal{O}(\alpha^2)\right).$$

such that $\mathcal{L}_{eff} \simeq \underbrace{-\mathcal{F}}_{\mathcal{L}_{MW}} + \underbrace{c_1\mathcal{F}^2 + c_2\mathcal{G}^2}_{\equiv \Delta\mathcal{L}\ \left(\xrightarrow{\hbar\to 0} 0\right)}$ is fully determined.

$\rightsquigarrow$ EoMs: $\partial_\mu \dfrac{\partial \mathcal{L}_{eff}}{\partial F_{\mu\nu}} = 0$ 'quantum Maxwell eqs.'

$$\rightsquigarrow \partial_\mu \Big(F^{\mu\nu} \underbrace{- 2c_1 \mathcal{F} F^{\mu\nu} - 2c_2 \mathcal{G} \tilde{F}^{\mu\nu}}_{\xrightarrow{\hbar\to 0} 0} \Big) = 0$$

Signatures of QED vacuum nonlinearities in high-intensity laser experiments

To this end, we decompose $F^{\mu\nu} \to F^{\mu\nu} + f^{\mu\nu}$ into an applied e.m field and a signal photon field and write

$F^{\mu\nu}$: applied, $\simeq$ large; $f^{\mu\nu}$: signal, $\simeq$ small

$$\Delta\mathcal{L}(F+f) = \Delta\mathcal{L}(F) + f^{\mu\nu}\frac{\partial \Delta\mathcal{L}(F)}{\partial F^{\mu\nu}} + \mathcal{O}(f^2)$$

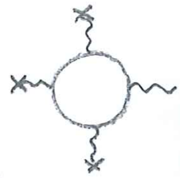
We are interested in signal photons a^μ

$$\rightsquigarrow \Delta\mathcal{L}(F+f) = \Delta\mathcal{L}(F) + \overset{\text{assym. of } F}{2}\,\partial^\mu a^\nu \frac{\partial \Delta\mathcal{L}(F)}{\partial F^{\mu\nu}} + \mathcal{O}(a^2)$$

$$= \Delta\mathcal{L}(F) \underset{\text{Partial Integration}}{-} 2 a^\nu \partial^\mu \frac{\partial \Delta\mathcal{L}(F)}{\partial F^{\mu\nu}} + \mathcal{O}(a^2)$$

$$= \Delta\mathcal{L}(F) - j_\nu a^\nu + \mathcal{O}(a^2)$$

where we defined $j_\nu := 2\partial^\mu \dfrac{\partial \Delta\mathcal{L}(F)}{\partial F^{\mu\nu}}$



$\simeq$ signal photon current

We are interested in signal photons to be detected far outside the interaction region.

The eff. theory for signal photon propagation is

$$\mathcal{L}_\gamma \simeq -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - j_\nu a^\nu \qquad \text{outside the interaction region.}$$

$\xrightarrow{\text{EoM}}$ $\partial_\mu f^{\mu\nu} = j^\nu$ use Lorenz gauge $\partial_\mu a^\mu = 0$

$\Box a^\nu = j^\nu$

PS, sec. 6.1
1412.0951 [hep-ph]
(11) with 2nd term $\vec{k} \to -\vec{k}$

$$\to a^\nu(x) = \int d^4x' \, \underset{\text{retarded}}{G^R}(x,x') \, j^\nu(x')$$

$$G^R(x,x') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \left(\frac{1}{i} e^{ik(x-x')} + \text{c.c.} \right) \Big|_{k^0=|\vec{k}|}$$

at asymptotic times

$$\to a^\nu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} \left(\frac{e^{ikx}}{i} \underbrace{\int d^4x' \, e^{-ikx'} j^\nu(x')}_{=: j^\nu(k)} + \text{c.c.} \right) \Big|_{k^0=|\vec{k}|}$$

$$= \text{Re} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^0} \frac{e^{ikx}}{i} j^\nu(k)$$

$\to$ the associated electric/magnetic fields are (cf. rels. on p. 4)

$$\vec{e}(x) = -\vec{\nabla} a^0(x) - \partial_t \vec{a}(x)$$

$$= \text{Re} \int \frac{d^3k}{(2\pi)^3} \underbrace{\left(\vec{j}(k) - \hat{\vec{k}} j^0(k) \right)}_{=: \vec{e}(k)} e^{ikx} \Big|_{k^0=|\vec{k}|}$$

$$\vec{b}(x) = \vec{\nabla} \times \vec{a}(x)$$

$$= \text{Re} \int \frac{d^3k}{(2\pi)^3} \underbrace{\hat{\vec{k}} \times \vec{j}(k)}_{=: \vec{b}(k)} e^{ikx} \Big|_{k^0=|\vec{k}|}$$

Note that $\vec{k}.\vec{e}(k)\,|_{k^0=|\vec{k}|} = \left(\vec{k}.\vec{j}(k) - |\vec{k}|\,j^0(k)\right)|_{k^0=|\vec{k}|}$

$= k_\mu j^\mu(k)\,|_{k^0=|\vec{k}|} = 0$ Ward Id.

$\leftrightarrow$ the electric field is transverse

Moreover, $\vec{k}\times\vec{e}(k) = \vec{k}\times\vec{j}(k)$, such that $\vec{b}(k) = \hat{\vec{k}}\times\vec{e}(k)$.

In turn, $|\vec{e}(k)| = |\vec{b}(k)| = |\vec{j}(k)|$.

The energy put in the signal field is

$$W = \frac{1}{2}\int d^3x\,\left(|\vec{e}(x)|^2 + |\vec{b}(x)|^2\right).$$

$$\cdot \int d^3x\,|\vec{e}(x)|^2 = \frac{1}{4}\int d^3x \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\left(\vec{e}(k)\,e^{ikx} + \vec{e}^{\,*}(k)\,e^{-ikx}\right)$$
$$\times\left(\vec{e}(k')\,e^{ik'x} + \vec{e}^{\,*}(k')\,e^{-ik'x}\right)\Big|_{k^0=|\vec{k}|,\,k'^0=|\vec{k}'|}$$

$$= \frac{1}{4}\int\frac{d^3k}{(2\pi)^3}\left(\vec{e}(k).\vec{e}(-k)\,e^{2ik^0t} + \vec{e}^{\,*}(k)\vec{e}^{\,*}(-k)\,e^{-2ik^0t}\right.$$
$$\left. + 2\,\vec{e}(k)\,\vec{e}^{\,*}(k)\right)\Big|_{k^0=|\vec{k}|}$$

with $\vec{b}(k).\vec{b}(-k) = (\hat{\vec{k}}\times\vec{e}(k)).(-\hat{\vec{k}}\times\vec{e}(-k)) = -\vec{e}(k).\vec{e}(-k)$

$\vec{b}^{\,*}(k).\vec{b}(-k) = -\vec{e}^{\,*}(k).\vec{e}^{\,*}(-k)$.

$\rightsquigarrow$ we obtain $\quad W = \frac{1}{4}\int\frac{d^3k}{(2\pi)^3}\left(|\vec{e}(k)|^2 + |\vec{b}(k)|^2\right)\Big|_{|\vec{k}|=k^0}$

$$= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,|\vec{e}(k)|^2\Big|_{|\vec{k}|=k^0}$$

Note that $|\vec{e}(k)|^2 = \sum_{p=1,2}|\vec{\varepsilon}_p(k).\vec{e}(k)|^2 = \sum_{p=1,2}|\vec{\varepsilon}_p(k).\vec{j}(k)|^2$

$\uparrow$ 2 transverse polarization vectors

$$\rightsquigarrow W = \int\frac{d^3k}{(2\pi)^3}\sum_{p=1,2}\frac{1}{2}|\vec{\varepsilon}_p(k).\vec{j}(k)|^2\Big|_{k^0=|\vec{k}|}$$

$$\leftrightsquigarrow d^3W_p = \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}|\vec{\varepsilon}_p(k).\vec{j}(k)|^2\Big|_{k^0=|\vec{k}|}$$

differential field energy put in mode p.

diff. signal photon number

$$\rightsquigarrow d^3N_p := \frac{d^3W_p}{|k^0|} = \frac{d^3k}{(2\pi)^3}\,\frac{1}{2k^0}\,|\vec{\varepsilon}_p(k).\vec{j}(k)|^2\Big|_{k^0=|\vec{k}|}$$

We have $j_\nu(k) = \int d^4x \, e^{-ikx} \, 2 \partial^\mu \dfrac{\partial \Delta \mathcal{L}(F)}{\partial F^{\mu\nu}}$

$$= 2ik^\mu \int d^4x \, e^{-ikx} \frac{\partial \Delta \mathcal{L}(F)}{\partial F^{\mu\nu}}$$

$$\simeq 2ik^\mu \int d^4x \, e^{-ikx} \left(c_1 \mathcal{F} F_{\mu\nu} + c_2 \mathcal{G} \tilde{F}_{\mu\nu} \right)$$

Note that

$$k^\mu F_{\mu\nu} \Big|_{k^0 = |\vec{k}|} = k^0 \left(\hat{\vec{k}} . \vec{E} \, , - \hat{\vec{k}} \times \vec{B} - \vec{E} \right)$$

$$k^\mu \tilde{F}_{\mu\nu} \Big|_{k^0 = |\vec{k}|} = k^0 \left(\hat{\vec{k}} . \vec{B} \, , \, \hat{\vec{k}} \times \vec{E} - \vec{B} \right)$$

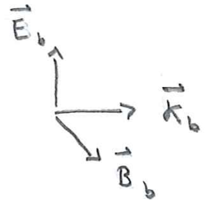
such that finally

$$\vec{\varepsilon}_p(k) . \vec{j}(k) \Big|_{k^0 = |\vec{k}|} = 2ik^0 \int d^4x \, e^{-ikx} \Big(- c_1 \mathcal{F} \, \vec{\varepsilon}_p(k) . (\hat{\vec{k}} \times \vec{B} + \vec{E}) + c_2 \mathcal{G} \, \vec{\varepsilon}_p(k) . (\hat{\vec{k}} \times \vec{E} - \vec{B}) \Big) \Big|_{k^0 = |\vec{k}|}$$

$\rightarrow$ Everything boils down to a 4d Fourier transform of the 'field configuration'.

<u>Applications / examples</u> assuming linearly polarized laser beams

A given beam 'b' fulfills



$$\left.\begin{aligned} \hat{\vec{B}}_b . \hat{\vec{E}}_b = \hat{\vec{E}}_b . \hat{\vec{k}}_b = \hat{\vec{B}}_b . \hat{\vec{k}}_b = 0 \\ \hat{\vec{E}}_b \times \hat{\vec{B}}_b = \hat{\vec{k}}_b \end{aligned}\right\} \text{position independent}$$

single field profile $\mathcal{E}_b(x)$

$$\vec{E}_b(x) = \mathcal{E}_b(x) \, \hat{\vec{E}}_b$$

$$\vec{B}_b(x) = \mathcal{E}_b(x) \, \hat{\vec{B}}_b$$

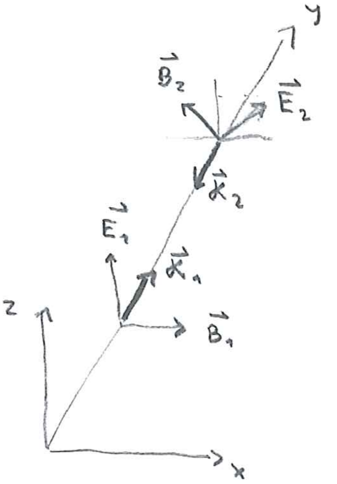
(1) One laser beam $\rightarrow \mathcal{F} = \mathcal{G} = 0$

$\rightarrow$ no signal photons !

$\hookrightarrow$ need at least two (paraxial) beams to obtain signal. (or 1 beam + other field source).

(2) Consider specific two beam scenario

for simplicity: given kinematics / polarizations and toy-model laser pulses



$$\hat{\vec{E}}_1 = \vec{e}_z \qquad \hat{\vec{E}}_2 = \frac{\vec{e}_x + \vec{e}_z}{\sqrt{2}}$$

$$\hat{\vec{B}}_1 = \vec{e}_x \qquad \hat{\vec{B}}_2 = \frac{\vec{e}_z - \vec{e}_x}{\sqrt{2}}$$

$$\hat{\vec{k}}_1 = \vec{e}_y \qquad \hat{\vec{k}}_2 = -\vec{e}_y$$

we are interested in polarization flipped signal photons from beam 1 propagating along $\hat{\vec{k}} \simeq \vec{e}_y$, $\vec{e}_p(k) = \vec{e}_x$.

"vacuum birefringence"

Discussion along the lines of F.K. & C. Sundqvist, PRD 94, 013004 (2016)

$\rightsquigarrow$ The fields are

$$\vec{E} = \left(\mathcal{E}_1 + \frac{\mathcal{E}_2}{\sqrt{2}}\right)\vec{e}_z + \frac{\mathcal{E}_2}{\sqrt{2}}\vec{e}_x$$

$$\vec{B} = \left(\mathcal{E}_1 - \frac{\mathcal{E}_2}{\sqrt{2}}\right)\vec{e}_x + \frac{\mathcal{E}_2}{\sqrt{2}}\vec{e}_z$$

such that
$$\mathcal{F} = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) = \left[\left(\mathcal{E}_1 - \frac{\mathcal{E}_2}{\sqrt{2}}\right)^2 - \left(\mathcal{E}_1 + \frac{\mathcal{E}_2}{\sqrt{2}}\right)^2\right]\frac{1}{2}$$

$$= -4\,\mathcal{E}_1 \frac{\mathcal{E}_2}{\sqrt{2}}\frac{1}{2} = -\sqrt{2}\,\mathcal{E}_1\mathcal{E}_2$$

$$\mathcal{G} = -\vec{E}\cdot\vec{B} = -\frac{\mathcal{E}_2}{\sqrt{2}}\left(\mathcal{E}_1 - \frac{\mathcal{E}_2}{\sqrt{2}}\right) - \frac{\mathcal{E}_2}{\sqrt{2}}\left(\mathcal{E}_1 + \frac{\mathcal{E}_2}{\sqrt{2}}\right)$$

$$= -\sqrt{2}\,\mathcal{E}_1\mathcal{E}_2$$

$$\hat{\vec{k}} \times \vec{B} = \vec{e}_y \times \left[\left(\mathcal{E}_1 - \frac{\mathcal{E}_2}{\sqrt{2}}\right)\vec{e}_x + \frac{\mathcal{E}_2}{\sqrt{2}}\vec{e}_z\right] = -\left(\mathcal{E}_1 - \frac{\mathcal{E}_2}{\sqrt{2}}\right)\vec{e}_z + \frac{\mathcal{E}_2}{\sqrt{2}}\vec{e}_x$$

$$\hat{\vec{k}} \times \vec{E} = \vec{e}_y \times \left[\left(\mathcal{E}_1 + \frac{\mathcal{E}_2}{\sqrt{2}}\right)\vec{e}_z + \frac{\mathcal{E}_2}{\sqrt{2}}\vec{e}_x\right] = \left(\mathcal{E}_1 + \frac{\mathcal{E}_2}{\sqrt{2}}\right)\vec{e}_x - \frac{\mathcal{E}_2}{\sqrt{2}}\vec{e}_z$$

$$\rightsquigarrow \vec{\varepsilon}_p(k)\cdot(\hat{\vec{k}} \times \vec{B} + \vec{E}) = \frac{\mathcal{E}_2}{\sqrt{2}}\,2 = \sqrt{2}\,\mathcal{E}_2$$

$$\rightsquigarrow \vec{\varepsilon}_p(k).(\hat{\vec{k}} \times \vec{E} - \vec{B}) = \varepsilon_1 + \frac{\varepsilon_2}{\sqrt{2}} - \varepsilon_1 + \frac{\varepsilon_2}{\sqrt{2}} = \sqrt{2}\,\varepsilon_2$$

and hence

$$\vec{\varepsilon}_p(k).\vec{j}(k)\big|_{k^0=|\vec{k}|} = 2ik^0 \int d^4x\, e^{-ikx} \Big(+c_1\sqrt{2}\,\varepsilon_1\varepsilon_2\,\sqrt{2}\,\varepsilon_2 + c_2(-\sqrt{2}\,\varepsilon_1\varepsilon_2)\sqrt{2}\,\varepsilon_2 \Big)$$

$$= 4ik^0 \int d^4x\, e^{-ikx} (c_1 - c_2)\, \varepsilon_1(x)\, \varepsilon_2^2(x)$$

$$= 4ik^0 (c_1 - c_2) \int d^4x\, e^{-ikx}\, \varepsilon_1(x)\, \varepsilon_2^2(x)$$

$$\rightsquigarrow d^3N_\perp = \frac{d^3k}{(2\pi)^3}\, 8k^0 (c_1 - c_2)^2 \left| \int d^4x\, e^{-ikx}\, \varepsilon_1(x)\,\varepsilon_2^2(x) \right|^2 \Bigg|_{k^0=|\vec{k}|}$$

To evaluate this quantity explicitly, we need a model for the profile function (which can be integrated easily).

We use $\varepsilon_1 = \varepsilon_+$

$\varepsilon_2 = \varepsilon_-$

$$\text{with } \underbrace{\varepsilon_\pm = \varepsilon_{0\pm}}_{\text{amplitude}}\ \underbrace{e^{-\frac{(y\mp t)^2}{(\tau/2)^2}}}_{\text{(same) pulse duration}}\ \underbrace{e^{-\frac{x^2+z^2}{w^2}}}_{\text{(same) finite transv. extent}}\ \underbrace{\cos(\omega_\pm (y \mp t))}_{\text{photon energy}}$$

$\leftrightsquigarrow$ no widening of beam as fct. of distance from focus $\rightsquigarrow$ overestimation of effects.

$$\rightsquigarrow \varepsilon_2^2 = \varepsilon_{0-}^2\, e^{-2\frac{(y+t)^2}{(\tau/2)^2}}\, e^{-2\frac{x^2+z^2}{w^2}}\, \frac{1}{2}\Big(\underbrace{1}_{\text{elastic}} + \underbrace{\cos(2\omega_-(y+t))}_{\text{inelastic}} \Big)$$

contributions

Here, we limit ourselves to the elastic contribution (no photons from beam 2 absorbed / released)

$$\rightsquigarrow \varepsilon_1\varepsilon_2^2 \simeq \varepsilon_{0+}\varepsilon_{0-}^2\, e^{-2\frac{(y+t)^2}{(\tau/2)^2} - \frac{(y-t)^2}{(\tau/2)^2}}\, e^{-3\frac{x^2+z^2}{w^2}}\, \frac{1}{4}\sum_{\ell=\pm} e^{i\ell\omega_+(y-t)}$$

$$= \varepsilon_{0+}\varepsilon_{0-}^2\, e^{-\frac{4}{\tau^2}(3y^2+3t^2+2yt) - 3\frac{x^2+z^2}{w^2}}\, \frac{1}{4}\sum_{\ell=\pm} e^{i\ell\omega_+(y-t)}$$

can be straightforwardly integrated

$$\to \int d^4x \, e^{-ikx} \, \mathcal{E}_1 \mathcal{E}_2^2 \Big|_{k^0 = |\vec{k}|} \underset{\uparrow}{=} \mathcal{E}_{0+} \mathcal{E}_{0-}^2 \, \frac{1}{4} \sum_{\ell = \pm} \tau^2 w^2 \, \frac{\sqrt{2}\,\pi^2}{48}$$

Computer algebra

$$\times \, e^{-\frac{1}{12} w^2 (k_x^2 + k_z^2)} \, e^{-\frac{\tau^2}{16}\left(\omega_+^2 - \ell\,\omega_+ (k^0 + k_y) + \frac{3}{8}((k^0)^2 + k_y^2) + \frac{1}{4} k^0 k_y\right)} \Big|_{k^0 = |\vec{k}|}$$

Use coordinates where $k_y = k \cos \vartheta$ $\qquad k = |\vec{k}| = k^0$

$k_x^2 + k_z^2 = k^2 \sin^2 \vartheta$

Angular symmetry around y axis.

$$\leftrightarrow \quad d^3k \doteq d\varphi \, d\cos\vartheta \, k^2 \, dk$$

Note that the $\ell = -1$ contribution is suppressed in forward direction.

$\to$ will be neglected

$$\to \left| \int d^4x \, e^{-ikx} \, \mathcal{E}_1 \mathcal{E}_2^2 \right|^2 \Big|_{k^0 = |\vec{k}|} \simeq \mathcal{E}_{0+}^2 \mathcal{E}_{0-}^4 \, \frac{2\pi^4}{192^2} \, \tau^4 w^4 \, e^{-\frac{w^2}{6} k^2 \sin^2\vartheta}$$

$$\times \, e^{-\frac{\tau^2}{8}\left[\omega_+^2 - \omega_+ k (1 + \cos\vartheta) + \frac{3}{8} k^2 (1 + \cos^2\vartheta) + \frac{1}{4} k^2 \cos\vartheta\right]}$$

As in the vicinity of the forward direction we have $\vartheta \ll 1$, we can employ $\cos\vartheta \simeq 1 - \frac{\vartheta^2}{2}$, $\sin\vartheta \simeq \vartheta$ to obtain

$$\left| \int d^4x \, e^{-ikx} \, \mathcal{E}_1 \mathcal{E}_2^2 \right|^2 \Big|_{k^0 = |\vec{k}|} \simeq \mathcal{E}_{0+}^2 \mathcal{E}_{0-}^4 \, \frac{2\pi^4}{192^2} \, \tau^4 w^4 \, e^{-\frac{w^2}{6} k^2 \vartheta^2}$$

$$\times \, e^{-\frac{\tau^2}{8}\left[\omega_+^2 - 2\omega_+ k \left(1 - \frac{\vartheta^2}{4}\right) + \frac{3}{4} k^2 \left(1 - \frac{\vartheta^2}{2}\right) + \frac{1}{4} k^2 \left(1 - \frac{\vartheta^2}{2}\right)\right]}$$

$$= \mathcal{E}_{0+}^2 \mathcal{E}_{0-}^4 \, \frac{2\pi^4}{192^2} \, \tau^4 w^4 \, e^{-\frac{\tau^2}{8}(\omega_+ - k)^2} \, e^{-\frac{w^2}{6} k^2 \vartheta^2} \, e^{-\frac{\tau^2}{16} k (\omega_+ - k) \vartheta^2}$$

The laser pulse energy is given by $W_b \overset{\text{p. 8}}{=} \int d^3x \, \mathcal{E}_b^2(x)$.

An integration yields (for $\omega_b \tau \gg 1$)

$$W_b = \left(\frac{\pi}{2}\right)^{3/2} \frac{1}{4} \tau w^2 \mathcal{E}_{0b}^2$$

$$\leftrightarrow \quad \mathcal{E}_{0b}^2 = 4 \left(\frac{2}{\pi}\right)^{3/2} \frac{W_b}{\tau w^2} \quad \leftarrow \text{known for lasers (better than } \mathcal{E}_{0b})$$

Therewith

$$d^3 N_\perp \simeq \frac{d^3k}{(2\pi)^3} \frac{4}{9} \sqrt{\frac{2}{\pi}} \, k (c_1 - c_2)^2 (\tau w)^4 \frac{W_+}{w^2\tau} \left(\frac{W_-}{w^2\tau} \right)^2$$
$$\times e^{-\frac{\tau^2}{8}(\omega_+ - k)^2} \, e^{-\frac{w^2}{6} k^2 \vartheta^2} \, e^{-\frac{\tau^2}{16} k (\omega_+ - k) \vartheta^2}$$

As for $\vartheta \ll 1$ the main contribution to the k-integral will stem from $k \simeq \omega_+$, we can set $k = \omega_+$ in the exponentials $\sim \vartheta^2$, such that

$$d^3 N_\perp \simeq \frac{d^3k}{(2\pi)^3} \frac{4}{9} \sqrt{\frac{2}{\pi}} \, k (c_1 - c_2)^2 (\tau w)^4 \frac{W_+}{w^2\tau} \left(\frac{W_-}{w^2\tau} \right)^2$$
$$\times e^{-\frac{\tau^2}{8}(\omega_+ - k)^2} \, e^{-\frac{w^2}{6} \omega_+^2 \vartheta^2}$$

$$\rightsquigarrow \frac{dN_\perp}{d\cos\vartheta} \simeq \frac{4}{9\pi^2} (c_1 - c_2)^2 (\tau w)^4 \frac{W_+}{w^2\tau} \left(\frac{W_-}{w^2\tau} \right)^2 \frac{\omega_+^3}{\tau} \left(1 + \frac{12}{(\omega_+\tau)^2} \right)$$
$$\times e^{-\frac{w^2}{6} \omega_+^2 \vartheta^2} \qquad (*)$$

where we approximated $\int_0^\infty dk \simeq \int_{-\infty}^\infty dk$.

For $\vartheta \ll 1$, we have $|d\cos\vartheta| \simeq |\vartheta \, d\vartheta|$.
In the limit of $w\omega_+ \gg 1$, we can moreover extend the ϑ integration domain to $0 \ldots \infty$, resulting in

$$\frac{N_\perp}{N} \simeq \frac{4}{3\pi^2} (c_1 - c_2)^2 (\omega_+ \tau)^2 \left(\frac{W_-}{w^2\tau} \right)^2 \left(1 + \frac{12}{(\omega_+\tau)^2} \right)$$

where we used $W_+ = N \omega_+$
$\uparrow$
number of laser photons

$\rightsquigarrow$ For $W_- = 30\,\text{J}$, $\tau = 30\,\text{fs}$ $\rightsquigarrow$ 1 PW laser

$w = 1\,\mu\text{m}$, $\omega_+ = 12914\,\text{eV}$

Note that $\begin{cases} 1\,\text{J} \simeq 6{,}24 \cdot 10^{18}\,\text{eV} \\ 1\,\mu\text{m} \simeq 5{,}07 \frac{1}{\text{eV}} \\ 1\,\text{fs} \simeq 1{,}52 \frac{1}{\text{eV}} \end{cases}$

we obtain

B. Marx, et al., Opt. Comm. 284, 915 (2011)

$$\frac{N_\perp}{N} \simeq 1{,}3 \cdot 10^{-11}$$

$\rightsquigarrow$ best polarimeter $5{,}7 \cdot 10^{-10} \simeq P$ @ 12914 eV

Assuming the $b=+$ beam to have the far field divergence of a Gaussian beam, we have

$$\frac{dN}{d\cos\vartheta} = (\omega_+ w)^2\, N\, e^{-2\vartheta^2\left(\frac{\omega_+ w}{2}\right)^2}$$

$$= (\omega_+ w)^2\, N\, e^{-\frac{w^2}{2}\omega_+^2\vartheta^2} \quad \leftarrow \text{decays faster than } (*)$$

$$\left(\text{Factor } 1+\frac{12}{(\omega_+\tau)^2}\simeq 1\right)$$

$$\rightsquigarrow \left(\frac{dN_\perp}{d\cos\vartheta}\right) \Big/ \left(\frac{dN}{d\cos\vartheta}\right) = \frac{4}{9\pi^2}(c_1-c_2)^2(\omega_+\tau)^2 \times \left(\frac{W_-}{w^2\tau}\right)^2 e^{\frac{w^2}{3}\omega_+^2\vartheta^2}$$

$$P \overset{!}{=} \frac{4}{9\pi^2}(c_1-c_2)^2(\omega_+\tau)^2\left(\frac{W_-}{w^2\tau}\right)^2 e^{\frac{w^2}{3}\omega_+^2\vartheta_p^2}$$

$$\rightsquigarrow \vartheta_p = \sqrt{\frac{3}{(w\omega_+)^2}\ln\left\{\frac{9\pi^2}{4}P\,\frac{1}{(c_1-c_2)^2}\,\frac{1}{(\omega_+\tau)^2}\left(\frac{w^2\tau}{W_-}\right)^2\right\}}$$

Noting that $\displaystyle\int_{\vartheta_p}^{\infty} d\vartheta\,\vartheta\, e^{-\frac{w^2}{6}\omega_+^2\vartheta^2} = \frac{1}{2}e^{-\frac{w^2}{6}\omega_+^2\vartheta_p^2}\,\frac{6}{(w\omega_+)^2}$

we obtain

$$N_{\perp,>\vartheta_p} \simeq \frac{8}{9\pi^3}|c_1-c_2|^3\frac{1}{\sqrt{P}}\left(\frac{W_-}{w^2\tau}\right)^3\frac{W_+}{w^2\tau}(\tau w)^2(\omega_+\tau)^2$$

$$= \frac{64}{30375}\,\frac{1}{\pi^3}\,\frac{\alpha^6}{m_e^{12}}\,\frac{1}{\sqrt{P}}\,\frac{W_-^3}{w^6}\,N\omega_+^3 \simeq 1{,}13$$

$\uparrow$

$N\simeq 10^{12}$

(3) 3-beam scenario

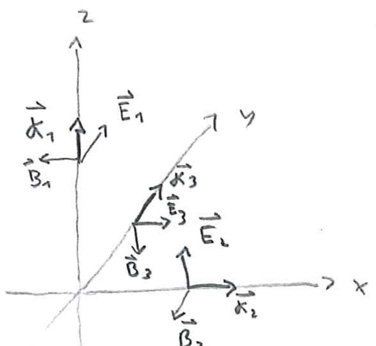
for simplicity: given kinematics / polarizations

toy model laser pulses as above $\tau_b=\tau$

$w_b = w$

& in addition polarization insensitive measurement

$\rightsquigarrow$ all induced signal photons



cf. Lundström, et. al., PRL 96, 083602 (2006)

Gies, et.al., arXiv:1712.06450 [hep-ph]

$\hat{\vec{\kappa}}_1 = \vec{e}_z$, $\hat{\vec{\kappa}}_2 = \vec{e}_x$, $\hat{\vec{\kappa}}_3 = \vec{e}_y$

$\hat{\vec{E}}_1 = \vec{e}_y$, $\hat{\vec{E}}_2 = \vec{e}_z$, $\hat{\vec{E}}_3 = \vec{e}_x$

$\hat{\vec{B}}_1 = -\vec{e}_x$, $\hat{\vec{B}}_2 = -\vec{e}_y$, $\hat{\vec{B}}_3 = -\vec{e}_z$