# Part 3
# Gauss Curvature flow

Panagiota Daskalopoulos

Columbia University

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# The Gauss Curvature flow - Introduction

- Consider the evolution of a hypersurface $M_t$ in $\mathbb{R}^{n+1}$ by the $\alpha$-Gauss Curvature flow

$$(*_k) \qquad \frac{\partial P}{\partial t} = K^\alpha \, \nu$$

with speed $K^\alpha = (\lambda_1, \cdots, \lambda_n)^\alpha$, $\alpha > 0$.

- This is a well known example of fully-nolinear degenerate diffusion of Monge-Ampére type
- It was introduced by W. Firey in 1974 and has been widely studied especially in the compact case.
- We note important geometric works in the compact case by: K. Tso, B. Chow, R. Hamilton, J. Urbas, B. Andrews, K. Lee, X. Chen, P. Guan, L. Ni, S. Brendle, K. Choi among many others.

# The Gauss Curvature Flow on compact surfaces

- Firey 1974: The GCF ($\alpha = 1$) models the wearing process of tumbling stones subjected to collisions from all directions with uniform frequency.
- Firey: The GCF shrinks strictly convex compact and centrally symmetric surfaces to round points.
- Firey's conjecture: The GCF shrinks any strictly convex compact hypersurface to spherical points.
- Tso 1985: Existence and uniqueness for compact strictly convex and smooth initial data up.
- Andrews 1999: Firey's Conjecture for strictly convex surfaces in dim $n = 2$.
- Brendle, Choi and D., 2017: Firey's Conjecture for the GCF$^{\alpha}$, $\alpha > \frac{1}{n+2}$, flow in any dimension $n \geq 2$.
- Based on previous work by Andrews, Guan and Ni on convergence to self-similar solutions.
- Other works: Andrews, Guan-Ni, Kim-Lee.

# Outline

We will discuss the following topics on GCF:

- GCF on complete non-compact convex hypersurfaces
- Optimal regularity of solutions
- Surfaces with Flat sides
- Firey's Conjecture

# Gauss Curvature flow - the PDE

- If $x_{n+1} = u(x,t)$ defines $M^n$ locally, then the GCF becomes equivalent to the Monge-Ampére type of eq.

$$u_t = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+1}{2}}}.$$

- To understand the nature of the PDE let us look at the case $n = 2$:

$$u_t = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+|Du|^2)^{\frac{3}{2}}}.$$

- The linearized equation at $u$ is

$$h_t = \frac{u_{yy}h_{xx} + u_{xx}h_{yy} - 2u_{xy}h_{xy}}{(1+|Du|^2)^{\frac{3}{2}}} + \text{lower order}$$

- One see that this equation becomes degenerate what points where $u$ is not strictly convex .

# The Regularity of solutions to GCF -Known Results

- Hamilton: Convex surfaces with at most one vanishing principal curvature, will instantly become strictly convex and hence smooth.
- Chopp, Evans and Ishii: If $M^n$ is $C^{3,1}$ at a point $P_0$ and two or more principal curvatures vanish at $P_0$, then $P_0$ will not move for some time $\tau > 0$.
- Andrews: A surface $M^2$ in $\mathbb{R}^3$ evolving by the GCF is always $C^{1,1}$ on $0 < t < T$ and smooth on $t_0 \leq t < T$, for some $t_0 > 0$. This is the optimal regularity in dimension $n = 2$.

  Remark: The regularity of solutions $M^n$ in dimensions $n \geq 3$ poses a much harder question.
- Hamilton: If a surface $M^2$ in $\mathbb{R}^3$ has flat sides, then the flat sides will persist for some time.

# Basic equations under GCF

- $\partial_t g_{ij} = -2K^\alpha h_{ij}, \quad \partial_t g^{ij} = 2K^\alpha h^{ij}$
- $\partial_t \nu = -\nabla K^\alpha$
- $\partial_t h_{ij} = \mathcal{L}\, h_{ij} + \alpha K^\alpha A_{klmn} \nabla_i h_{mn} \nabla_j h_{kl} + \alpha K^\alpha H h_{ij}$
  $-(1+n\alpha) K^\alpha h_{ik} h^k_j$
- $\partial_t K^\alpha = \mathcal{L}\, K^\alpha + \alpha K^{2\alpha} H$
- $\partial_t b^{pq} = \mathcal{L}\, b^{pq} - \alpha K^\alpha b^{ip} b^{jq} B_{klmn} \nabla_i h_{kl} \nabla_j h_{mn} - \alpha K^\alpha H b^{pq}$
  $+(1+n\alpha) K^\alpha g^{pq}$
- $\partial_t v = \mathcal{L}\, v - 2v^{-1} \|\nabla v\|^2_{\mathcal{L}} - \alpha K^\alpha H$

# Basic equations under GCF

- The function $\psi_\beta(p,t) = (M - \beta t - \bar{u}(p,t))_+$ satisfies

$$\partial_t \psi_\beta = \mathcal{L}\,\psi_\beta + (n\alpha - 1)\, v^{-1} K^\alpha - \beta$$

- The function $\bar{\psi}(p,t) = (R^2 - |F(p,t) - \bar{x}_0|^2)_+$ satisfies

$$\partial_t \bar{\psi} \leq \mathcal{L}\,\bar{\psi} + 2\big(n\alpha + 1)(\lambda_{\min}^{-1} + R)K^\alpha$$

# The GCF-flow on complete non-compact graphs

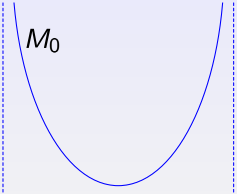
- Jointly with K. Choi, L. Kim and K. Lee we studied the evolution of complete non-compact graphs $M_t$ in $\mathbb{R}^{n+1}$ by the $\alpha$-Gauss Curvature flow

$$(*_k) \qquad \frac{\partial P}{\partial t} = K^\alpha \, \nu$$

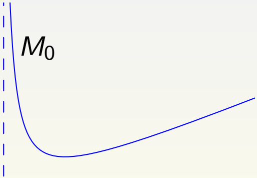
  with speed $K^\alpha = (\lambda_1, \cdots, \lambda_n)^\alpha$, $\alpha > 0$.
- Here $\nu$ is the inner normal.
- We assume that $M_0$ is a complete non-compact strictly convex graph over a domain $\Omega \subset \mathbb{R}^n$.
- The domain $\Omega$ may be bounded or unbounded (e.g. $\Omega = \mathbb{R}^n$).
- H. Wu (1974): a complete non-compact smooth and strictly convex hypersurface $M_0$ in $\mathbb{R}^{n+1}$ is the graph of a function $u_0$ defined on a domain $\Omega \subset \mathbb{R}^n$.

# Examples of the initial hypersurface $M_0$


(a) $\Omega = \mathbb{R}^n$




(b) $\Omega = B_R(0)$




(c) $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$

Figure: Examples of the initial hypersurface $M_0$

# The Main Results

- Theorem 1. Let $M_0 = \{(x, u_0(x)) : x \in \Omega\}$ be a locally uniformly convex graph given by $u_0 : \Omega \to \mathbb{R}$ defined on a convex domain $\Omega \subset \mathbb{R}^n$. Then, for any $\alpha > 0$, there exists a smooth strictly convex solution $u : \Omega \times (0, +\infty) \to \mathbb{R}$ of the $\alpha$-Gauss curvature flow

$$(**_\alpha) \quad u_t = \frac{(\det D^2 u)^\alpha}{(1 + |Du|^2)^{\frac{(n+2)\alpha - 1}{2}}}$$

  such that $\lim_{t \to 0} u(x, t) = u_0(x)$.

- Theorem 2. Let $M_0$ be a smooth complete non-compact and strictly convex hypersurface embedded in $\mathbb{R}^{n+1}$. Then, for any $\alpha > 0$, there exists a smooth complete non-compact and strictly convex solution $M_t$ of the $\alpha$-Gauss curvature flow defined for all time $0 < t < +\infty$ and having initial data $M_0$.

# Proof of Theorem 1 - Main steps for $\alpha = 1$

- For simplicity assume that $\alpha = 1$.
- Assume that $M_0$ is a convex graph in the direction of $\omega := e_{n+1}$. Then, $M_t$ will remain a convex graph.
- Define the heignt function $\bar{u} := \langle F, e_{n+1} \rangle$.
- The proof of Theorem 1 replies on local a'priori geometric bounds which are shown by the maximum principle.
  1. Local gradient estimate on $v := \langle e_{n+1}, \nu \rangle^{-1} = \sqrt{1 + |Du|^2}$.
  2. Local lower bound for the principal curvatures, i.e. on $\lambda_{\min}$.
  3. Local upper speed bound, i.e. on $K$.
- Linearized operator: $\mathcal{L} = K b^{ij} \nabla_i \nabla_j$, where $b^{ij} = (h_{ij})^{-1}$.
- Remark: It is easier to use geometric bounds, rather than pure PDE bounds on the evolution of $u$.

# The Gradient Estimate

- Height function: $\bar{u} := \langle F, e_{n+1} \rangle$. It satisfies $(\bar{u})_t = n^{-1} \mathcal{L} \bar{u}$.
- Cut off function: $\psi_\beta(p,t) = (M - \beta t - \bar{u}(p,t))_+$. It satisfies

$$\partial_t \psi_\beta = \mathcal{L}\, \psi_\beta + (n-1)\, v^{-1} K - \beta.$$

- Gradient: $v = \langle e_{n+1}, \nu \rangle^{-1} = \sqrt{1 + |Du|^2}$ satisfies the equation

$$\partial_t v = \mathcal{L}\, v - 2v^{-1} \|\nabla v\|^2_{\mathcal{L}} - K\, H\, v$$

- Gradient Estimate: Given $\beta > 0$ and $M \geq \beta$:

$$v(p,t)\, \psi_\beta(p,t) \leq M \max\Big\{ \sup_{\bar{u} \leq M} v(p,0),\ \beta^{-1} n^{\frac{1}{n+1}} (n-1) \Big\}$$

# Local lower bound on $\lambda_{\min}$

- Recall that $\psi_\beta(p,t) = (M - \beta t - \bar{u}(p,t))_+$.
- The most crucial estimate is the following lower curvature bound:

$$(\psi_\beta^{-2n}\lambda_{\min})(p,t) \geq M^{-2n} \min\left\{ \inf_{\bar{u}\leq M} \lambda_{\min}(p,0), B_{n,\beta} \right\}$$

  where $B_{n,\beta}$ constant depending on parameters.

- Proof: By a rather involved Pogorelov type computation to bound from above $\psi_\beta^{2n}\lambda_{\min}^{-1}$.

# Local upper bound on $K$

- Let $\psi := (M - \bar{u})_+$, where $\bar{u} := \langle F, e_{n+1} \rangle$ is the height function.
- Recall that $v = \langle e_{n+1}, \nu \rangle^{-1} = \sqrt{1 + |Du|^2}$.
- We show the following local upper bound for the speed $K$.

$$\left( \frac{t}{1+t} \right) (\psi^2 K^{\frac{1}{n}})(p,t) \leq (4n\alpha + 1)^2 (2\theta)^{1+\frac{1}{2n\alpha}} (\theta \Lambda + M^2)$$

where $\theta$ and $\Lambda$ are constants given by

$$\theta = \sup\{ v^2(p,s) : \bar{u}(p,s) < M, \, s \in [0,t] \},$$
$$\Lambda = \sup\{ \lambda_{\min}^{-1}(p,s) : \bar{u}(p,s) < M, \, s \in [0,t] \}.$$

- Proof: Following the CGN trick we set $\varphi(v) := \frac{v^2}{2\theta - v^2}$ and apply the maximum principle on $K^2 \varphi(v)$.
- Remark: The upper bound on $K$ at time $t > 0$ does not depend on an upper bound on $K$ at time $t = 0$.

# The Proof of Long time existence

- We obtain a solution $M_t := \{(x, u(\cdot, t)) : x \in \Omega_t \subset \mathbb{R}^n\}$ as

$$M_t := \lim_{j \to +\infty} \Gamma_t^j$$

  where $\Gamma_t^j$ is a strictly convex closed solution symmetric with respect to the hyperplane $x_{n+1} = j$.

- To pass to the limit we show that our a'priori estimates imply a uniform local $C^{2,\alpha}$ bound for $\Gamma_t^j$.

- Finally we construct barriers to show that $\Omega_t = \Omega$ for all $0 < t < +\infty$.

- This is expected since $K(x, u(x, t)) \to 0$, as $x \to \partial\Omega_t$.

# The Regularity of solutions to GCF

- If $x_{n+1} = u(x,t)$ defines $M^n$ locally, then $u$ evolves by the PDE

$$u_t = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+1}{2}}}.$$

- A strictly convex surface evolving by the GCF remains strictly convex and hence smooth up to its collapsing time $T$.
- The problem of the regularity of solutions in the weakly convex case is a difficult question. It is related to the regularity of solutions of the evolution Monge-Ampére equation

$$u_t = \det D^2 u.$$

- Question: What is the optimal regularity of weakly convex solutions to the Gauss Curvature flow ?

# Optimal regularity for weakly convex surfaces

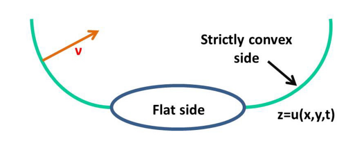
- Theorem (Andrews): Solutions to GCF of dim $n = 2$ in $\mathbb{R}^3$ are always $C^{1,1}$.
- Theorem (D., Savin): Solutions to GCF of dim $n = 3$ in $\mathbb{R}^4$ are always of class $C^{1,\alpha}$.
- Example (D., Savin): In dim $n \geq 4$ there exist self-similar solutions of $u_t = \det D^2 u$ with edges persisting.
- Theorem (D., Savin): If the initial surface $M_0^n$, $n \geq 3$ is of class $C^{1,\beta}$, then the solution $M_t^n$ is of class $C^{1,\alpha}$, $0 < \alpha \leq \beta$.
- Remark: Same results hold for motion by $K^p$, $p > 0$ and for viscosity solutions to

$$\lambda\,(\det D^2 u)^p \leq u_t \leq \Lambda\,(\det D^2 u)^p$$

for $0 < \lambda < \Lambda < \infty$ and $p > 0$.

# Surfaces with Flat Sides

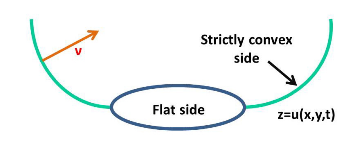
- Assume that the initial surface $M_0$ has a flat side.
- Because of the degenenary of the equation, the flat side will persist at $t > 0$.




$x \geq 0, y \in R^{n-1}$

- The equation becomes degenerate at the fat side.
- The boundary of the flat side $\Gamma_t$ behaves like a free-boundary propagating with finite speed. It will shrink to a point before the surface $M_t$ does.
- Question: What is the optimal regularity of solutions near $\Gamma_t$, $t > 0$ ? Does $\Gamma_t$ become smooth for $t > 0$ ?

- Jointly with Kyeongsu Choi we study the optimal regularity of solutions to the Gauss curvature flow with flat sides.




$x \geq 0, y \in R^{n-1}$

- We establish the optimal $C^{1,\frac{n}{n-1}}$-regularity of the solution.

- The case $n = 2$ was previously studied by D. jointly with R. Hamilton (sort time) and K. Lee (long time).

- The $n$-dim case for $n \geq 3$ is much harder.

# Weakly convex case - Flat sides

- The pressure $p := \left(\frac{n}{n-1}u\right)^{\frac{n-1}{n}}$ satisfies:

$$\partial_t p = \frac{p\,\det\left(p_{ij} + \frac{1}{n-1}p^{-1}p_i p_j\right)}{\left(1 + p^{\frac{2}{n-1}}|Dp|^2\right)^{\frac{n+1}{2}}}.$$

- Non-degenecary condition: We assume that at time $t = 0$ the pressure $p := \left(\frac{n}{n-1}u\right)^{\frac{n-1}{n}}$ satisfies:

$$(*) \qquad |Dp| \geq \lambda > 0 \quad \text{and} \quad p_{\tau\tau} \geq \lambda > 0.$$

- We will establish that $p$ is $C^\infty$ smooth up to the interface, which implies the optimal $C^{1,\frac{1}{n-1}}$-regularity of the solution.

# Weakly convex case - Flat sides

- Theorem. Denote by $T$ the extinction time of the flat side. Let $B_\rho \subset (M_{T_1})_{flat}$, for some $0 < T_1 < T$. Then:
  1. The non-degenecary condition

     $$(*) \quad |Dp| \geq \lambda(\rho) > 0 \quad \text{and} \quad p_{\tau\tau} \geq \lambda(\rho) > 0$$

     holds for $0 < t < T_1$.
  2. the interface $\Gamma_t$ is smooth for $0 < t < T$.
  3. the solution is of optimal class $C^{1,\frac{1}{n-1}}$, on $0 < t < T$.

- Proof. Assume that $B_\rho \subset (M_{T_1})_{flat}$, for some $0 < T_1 < T$. We establish sharp geometric estimates which hold on the strictly convex part of $M_t$, for $0 < t < T_1$. Our estimates depend on $\rho$ and deteriorate as $\rho \to 0$.

# Speed and curvature of the level sets

- Let $u$ be a strictly convex and smooth solution of the (GCF)

$$u_t = \sqrt{1 + |Du|^2}\, K.$$

- Each level set of $u(\cdot, t)$ shrinks with the speed $u_t/|Du|$ along the normal direction to the level set. Hence, the speed of each level set is given by

$$\sigma = \frac{\sqrt{1 + |Du|^2}}{|Du|}\, K$$

- It follows that in our setting to bound the speed of the flat side is equivalent to bound

$$K\, S^{-1}, \qquad \text{for } S := -\langle F, \nu \rangle > 0.$$

# Weakly convex case - Main Geometric estimates

Denote by $M_t^*$ the strictly convex part of our solution $M_t$ and $S := -\langle F, \nu \rangle > 0$ the support function.

- Speed estimate:

$$t^{\frac{n}{n+1}} K \leq C(n, T, \rho, sup|F|)$$

.

- Lower bound on level set speed:

$$\sup_{M_t^*} S\, K^{-1} \leq C \sup_{M_0} S\, K^{-1}.$$

- Short time upper bound on level set speed:

$$K\, S^{-1}(p,t) \leq \left[\mathcal{K}_0^{-1} - (n+1)\,\mathrm{t}\right]^{-1}$$

for $\mathcal{K}_0 := \sup_{M_0^*} K\, S^{-1}$ and on $t < \mathcal{K}_0^{-1}/(n+1)$.

# Weakly convex case - Crucial Geometric estimates

- Crucial estimate:

$$t\, K\lambda_{\min}^{-1}(p,t) + |F|^2(p,t) \le \gamma\,(Q+R)^2$$

where $\gamma = \max\{5, n\}$, $Q = \sup(tK)$, and $R = \sup|F|$.

- Proof: By a rather involved Pogorelov type computation on

$$\bar{Z} := t^2\, K\lambda_{\min}^{-1}(p,t) + t\,|F|^2(p,t) - t\gamma\,(Q+R)^2.$$

- Remark: We applied later a similar computation to prove Firey's conjecture.
- Upper bound on level set speed: If $B^+_{\theta\rho}(0) \prec M_t$, then

$$tKS^{-1} \le C_n\,(\theta^2-1)^{-\gamma}\Big(\frac{R}{\rho}\Big)^{2\gamma+2}\Big[1 + QR^{-1} + \Lambda R^{-2}\Big]$$

where $\gamma = \max\left\{1, \frac{1}{4}(n+1)\right\}$, $Q = \sup tK$, $R = \sup|F|$, $\Lambda = \sup tK\lambda_{\min}^{-1}$.

# Optimal Regularity up to the extinction of the flat side

- The above a priori bounds imply that the non-degeneracy condition is preserved under the GCF.
- The non-degeneracy condition together with linear regularity theory for degenerate equations imply the $C^\infty$ regularity of the free-boundary up to the extinction of the flat side.
- One concludes the optimal $C^{1,\frac{1}{n-1}}$ regularity of the solution.

# Firey's Conjecture

- We will now point out how one of our crucial estimates from the regularity of the free-boundary can be modified to give us the proof of Firey's conjecture !
- Consider a family of compact strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ which evolve by the $\alpha$-Gauss Curvature flow

$$(*_\alpha) \qquad \frac{\partial P}{\partial t} = K^\alpha \, \nu$$

- 1974 -Firey's conjecture: The GCF $(\alpha = 1)$ shrinks a compact surface to a round sphere.
- Theorem (S. Brendle, K. Choi, D. - 2016) Let $\alpha \geq 1/(n+2)$. Then, a solution $M_t$ of $(*_\alpha)$ converges to a round sphere after rescaling, or we have $\alpha = 1/(n+2)$ and the hypersurfaces $M_t$ converges to an ellipsoid after rescaling.

# Frirey's conjecture - Previous results

- B. Chow (1985): the result holds for $\alpha = 1/n$.
- B. Andrews (1999): the result holds for $\alpha = 1$, $n = 2$ and $\alpha = 1/(n+2)$.
- Andrews-Guan-Ni, Andrews, Guan-Ni, Kim-Lee:
  The solution $M_t$ of the $\alpha$-GCF converges after rescaling to a self-similar solution.
- K. Choi and D. (2016): the result holds for $\frac{1}{n} \leq \alpha < 1 + \frac{1}{n}$.

# Classification of self-similar solutions

- Andrews-Guan-Ni, Andrews, Guan-Ni, Kim-Lee: The solution $M_t$ of the $\alpha$-GCF converges after rescaling to a self-similar solution.

- Hence it is sufficient to classify compact self-similar solutions $M = F(M^n)$ which satisfy

$$(**_\alpha) \qquad K^\alpha = \langle F, \nu \rangle.$$

- Theorem (S. Brendle, K. Choi, D - 2016) Let $\alpha \geq 1/(n+2)$. Then, a compact strictly convex solution $M$ of $(**_\alpha)$ is the round sphere, unless $\alpha = 1/(n+2)$ in which case $M$ is an ellipsoid.

- Remark: In the case that $\alpha = 1/(n+2)$ this was shown by Calabi - 1972.

# The Proof

- Case 1: $\alpha \in [\frac{1}{n+2}, \frac{1}{2}]$. Let $b = (h_{ij})^{-1}$ and set

$$Z = K^\alpha \operatorname{tr}(b) - \frac{n\alpha - 1}{2\alpha}|F|^2.$$

- Motivation: $Z$ is constant when $\alpha = \frac{1}{n+2}$ and $M$ is an ellipsoid.
- We show that $Z$ satisfies

$$\alpha K^\alpha b^{ij} \nabla_i \nabla_j Z + (2\alpha - 1) b^{ij} \nabla_i K^\alpha \nabla_j Z \geq 0.$$

- The strong maximum principle implies that $Z$ is constant.
- By examining the case of equality, we show that either $\nabla_i h_{kj} = 0$ or $\alpha = \frac{1}{n+2}$.
- This implies that either $M$ is a round sphere, or $\alpha = \frac{1}{n+2}$ and $M$ is an ellipsoid.

# The Proof

- Case 2: $\alpha \in (1/2, +\infty)$. We consider the quantity

$$W = n\,K^{\alpha}\lambda_{\min}^{-1} - \frac{n\alpha - 1}{2\alpha}\,|F|^2.$$

- By applying the maximum principle, we show that any maximum point for $W$ is umbilic.
- Recall that $Z = K^{\alpha}\,\mathrm{tr}(b) - \frac{n\alpha-1}{2\alpha}|F|^2$, $b := (h_{ij})^{-1}$.
- Hence a maximum point of $W$ is also a maximum point of $Z$.
- Applying the strong maximum principle to $Z$, we are able to show that $Z$ and $W$ are both constant.
- This implies that $M$ is a round sphere.
- The proof is complete !!!