Motion by curvature of networks in the plane/5

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Restarting the flow after a singularity

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Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

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Hence, we are able to restart the flow after a singularity, possibly losing the uniqueness.

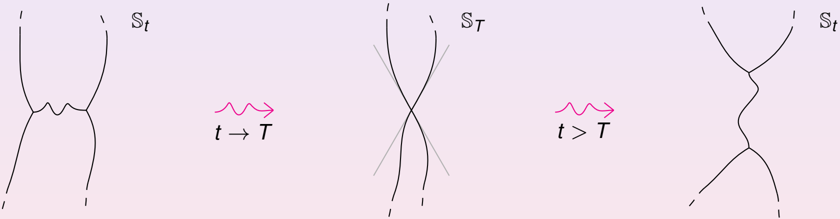
We anyway state the following conjecture:

Conjecture

The limit network $\mathbb{S}_T$ has always bounded curvature.

Restarting the flow after a singularity – The special case of bounded curvature/Standard transition

In the case of collapse of an isolated curve with bounded curvature, the INS Theorem produces the curvature flow that we observed in the simulation.



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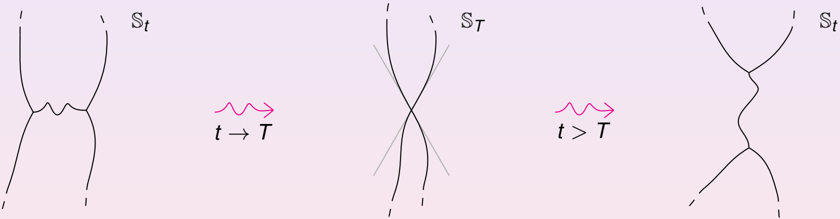

There is actually hope for uniqueness in this case, that we call *standard transition*.

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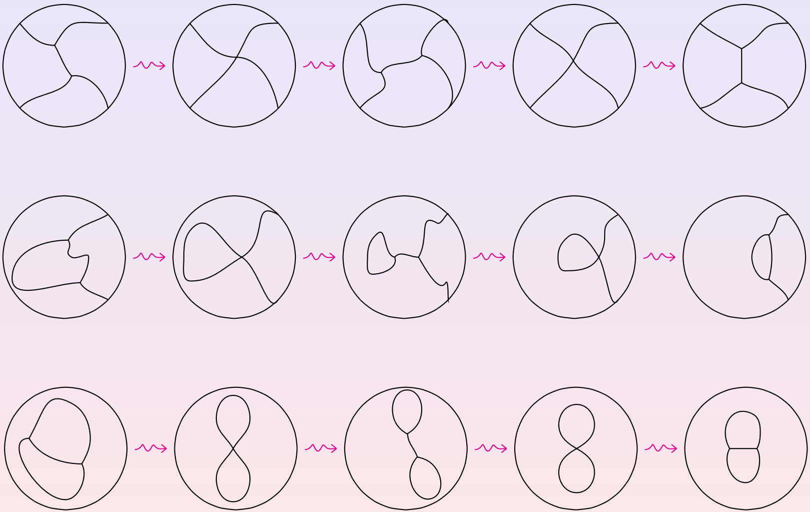
Conjecture

The limit of $\mathbb{S}_t$ as $t \to +\infty$, is unique (the full sequence of networks converges).

Possible “oscillation of shape” phenomenon



Possible infinite “oscillations” via standard transitions from a shape to another and viceversa.

Open problems and research directions

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Theorem (CM, M. Novaga, A. Pluda – 2015)

- *If during the flow the triple–junctions stay uniformly far each other, then* **M1** *is true.*
- *If the initial network has at most two triple junctions, then* **M1** *is true.*

Analysis of singularity formation for some flows of networks with "few" triple junctions can then be made rigorous (under uniqueness of the limit hypothesis in some cases).

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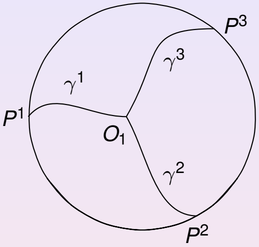
The Triod – A. Magni, CM, M. Novaga, V. Tortorelli



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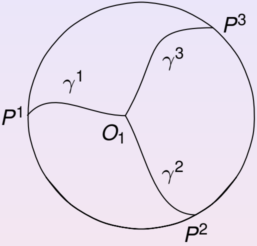
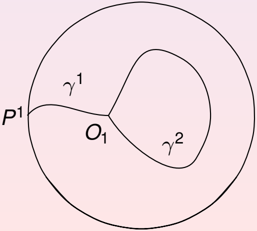
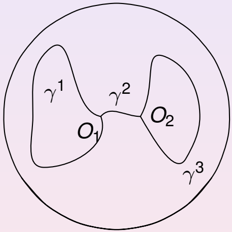


The Spoon – A. Pluda



2 triple junctions – CM, M. Novaga, A. Pluda

The Eyeglasses



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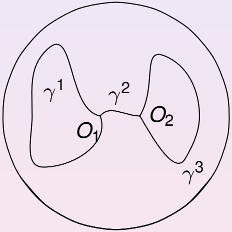
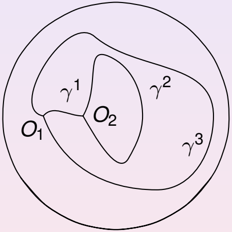
The Eyeglasses and... the Broken Eyeglasses





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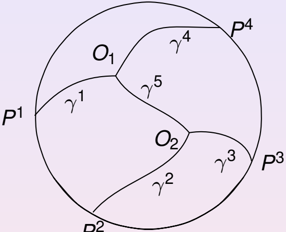
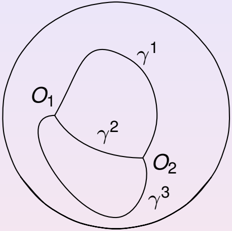
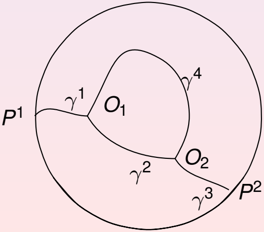
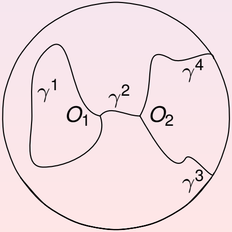
The “Steiner”, Theta, Lens and Island









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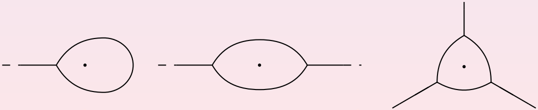

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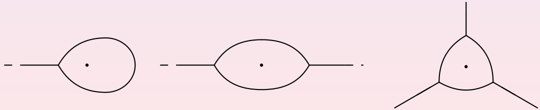

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- *Generically, only regions with at most three edges can collapse*

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- Short time existence/uniqueness for special initial interfaces by Depner – Garcke – Kohsaka
- Short time existence/estimates for special initial interfaces by Schulze – White

Thanks for your attention