

## Motion by curvature of networks in the plane

CARLO MANTEGAZZA

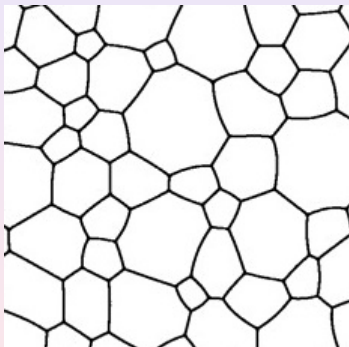
ICTP – Trieste  
11/15 June 2017

## Motion by curvature of networks in the plane – *Joint project with*

- ▶ *Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005*
- ▶ *Annibale Magni & Matteo Novaga, 2010 – 2014*
- ▶ *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
- ▶ *Pietro Baldi & Emanuele Haus, 2015 –*

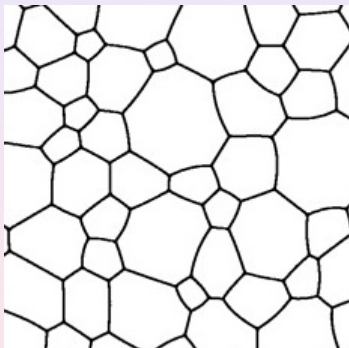
## Motion by curvature of networks in the plane – *Joint project with*

- ▶ *Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005*
- ▶ *Annibale Magni & Matteo Novaga, 2010 – 2014*
- ▶ *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
- ▶ *Pietro Baldi & Emanuele Haus, 2015 –*



## Motion by curvature of networks in the plane – *Joint project with*

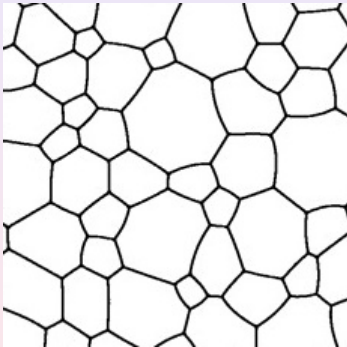
- ▶ *Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005*
- ▶ *Annibale Magni & Matteo Novaga, 2010 – 2014*
- ▶ *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
- ▶ *Pietro Baldi & Emanuele Haus, 2015 –*



This is a (toy) model for the time evolution of the interfaces of a multiphase system where the energy is given only by the area of such interfaces.

## Motion by curvature of networks in the plane – *Joint project with*

- ▶ *Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005*
- ▶ *Annibale Magni & Matteo Novaga, 2010 – 2014*
- ▶ *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
- ▶ *Pietro Baldi & Emanuele Haus, 2015 –*



This is a (toy) model for the time evolution of the interfaces of a multiphase system where the energy is given only by the area of such interfaces.

Even if it is still possible to use several ideas from the “parametric” classical approach to mean curvature flow (differential geometry/maximum principle), some extra variational methods are needed due to the presence of multi-points, that makes the network a singular set (possibly, the simplest).

# Some simple observations from simulations

## Some simple observations from simulations

Larger regions “eat” smaller regions. More precisely, the area of a region bounded by more than 6 edges grows, less than 6 edges it decreases (and possibly the region collapses).

## Some simple observations from simulations

Larger regions “eat” smaller regions. More precisely, the area of a region bounded by more than 6 edges grows, less than 6 edges it decreases (and possibly the region collapses).

With the exception of the times when a structural change happens (vanishing of a curve or of a region), there are only *triple* junctions and the three concurring curves form angles of 120 degrees. We call such networks *regular*.

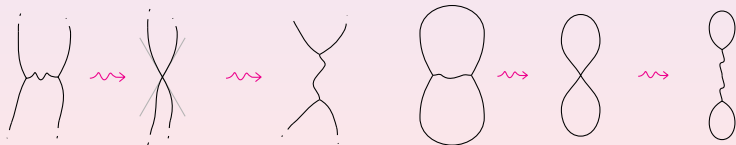


## Some simple observations from simulations

Larger regions “eat” smaller regions. More precisely, the area of a region bounded by more than 6 edges grows, less than 6 edges it decreases (and possibly the region collapses).

With the exception of the times when a structural change happens (vanishing of a curve or of a region), there are only *triple* junctions and the three concurring curves form angles of 120 degrees. We call such networks *regular*.

If there is no vanishing of a region, there is a collapse of only two triple junctions along a vanishing curve connecting them, producing a 4–point in the network. Immediately after such a collapse, the network becomes again *regular*: a new pair of triple junctions emerges from the 4–point (standard transition).

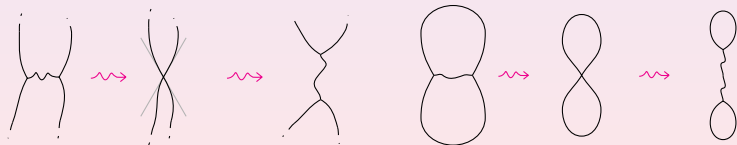


## Some simple observations from simulations

Larger regions “eat” smaller regions. More precisely, the area of a region bounded by more than 6 edges grows, less than 6 edges it decreases (and possibly the region collapses).

With the exception of the times when a structural change happens (vanishing of a curve or of a region), there are only *triple* junctions and the three concurring curves form angles of 120 degrees. We call such networks *regular*.

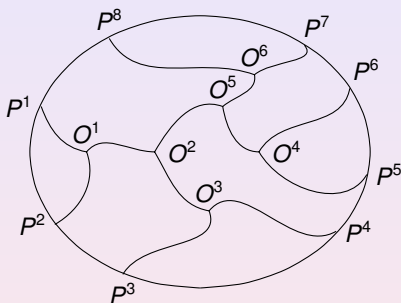
If there is no vanishing of a region, there is a collapse of only two triple junctions along a vanishing curve connecting them, producing a 4–point in the network. Immediately after such a collapse, the network becomes again *regular*: a new pair of triple junctions emerges from the 4–point (standard transition).



Actually, despite the (apparently) simple problem/behavior/statements, to show in a mathematically satisfactory way these observations, a lot of “technology” from analysis and geometry is needed.

## Regular networks

Let  $\Omega$  be an open, regular and convex subset of  $\mathbb{R}^2$ .



### Definition

A regular network  $\mathbb{S} = \bigcup_{i=1}^n \sigma^i([0, 1])$  in  $\Omega$  is a connected set described by a finite family of curves  $\sigma^i : [0, 1] \rightarrow \overline{\Omega}$  (sufficiently regular) such that:

## Regular networks

1. the curves cannot intersect each other or self-intersect in their “interior”, but they can meet only at their end-points;

## Regular networks

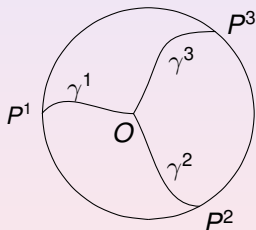
1. the curves cannot intersect each other or self-intersect in their “interior”, but they can meet only at their end-points;
2. if a curve of the network “touches” the boundary of  $\Omega$  at a fixed point  $P \in \partial\Omega$ , no other end-point of a curve can coincide with that point;

## Regular networks

1. the curves cannot intersect each other or self-intersect in their “interior”, but they can meet only at their end-points;
2. if a curve of the network “touches” the boundary of  $\Omega$  at a fixed point  $P \in \partial\Omega$ , no other end-point of a curve can coincide with that point;
3. all the junctions points  $O^1, O^2, \dots, O^m \in \Omega$  have order three, considering  $\mathbb{S}$  as a planar graph, and at each of them the three concurring curves  $\{\sigma^{p_i}\}_{i=1,2,3}$  meet in such a way that the external unit tangent vectors  $\tau^{p_i}$  satisfy  $\sum \tau^{p_i} = 0$  (the curves form three angles of 120 degrees at  $O^p$ ).

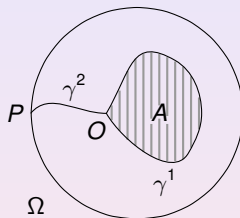
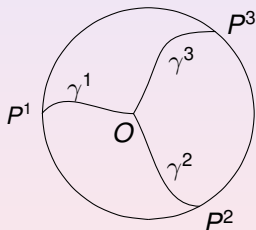
## Examples: The triod and the spoon

A triod  $\mathbb{T}$  is a network composed only by three regular, embedded curves  $\gamma^j : [0, 1] \rightarrow \bar{\Omega}$ .



## Examples: The triod and the spoon

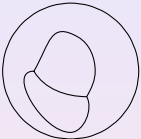
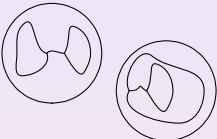
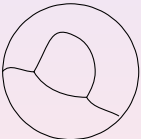
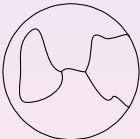
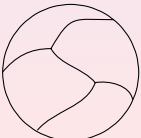
A triod  $\mathbb{T}$  is a network composed only by three regular, embedded curves  $\gamma^j : [0, 1] \rightarrow \overline{\Omega}$ .



A spoon  $\Gamma = \gamma^1([0, 1]) \cup \gamma^2([0, 1])$  is the union of two regular, embedded curves  $\gamma^1, \gamma^2 : [0, 1] \rightarrow \overline{\Omega}$ .



## Examples: Networks with two triple junctions

	0 closed curves	1 closed curve	2 closed curves
0 end-points on $\partial\Omega$	 <p>Theta</p>		 <p>Eyeglasses</p>
2 end-points on $\partial\Omega$	 <p>Lens</p>	 <p>Island</p>	
4 end-points on $\partial\Omega$	 <p>Tree</p>		

## Motion by curvature

### Definition

We say that a regular network moves by curvature if each of its time-dependent curves  $\gamma^i : [0, 1] \times [0, T) \rightarrow \mathbb{R}^2$  satisfies

$$\gamma_t^i(x, t)^\perp = \underline{k}^i(x, t)$$

## Motion by curvature

### Definition

We say that a regular network moves by curvature if each of its time-dependent curves  $\gamma^i : [0, 1] \times [0, T) \rightarrow \mathbb{R}^2$  satisfies

$$\gamma_t^i(x, t)^\perp = \underline{k}^i(x, t) = \frac{\langle \gamma_{xx}^i(x, t) | \nu^i(x, t) \rangle}{|\gamma_x^i(x, t)|^2} \nu^i(x, t) = \left( \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \right)^\perp.$$

## Motion by curvature

### Definition

We say that a regular network moves by curvature if each of its time-dependent curves  $\gamma^i : [0, 1] \times [0, T) \rightarrow \mathbb{R}^2$  satisfies

$$\gamma_t^i(x, t)^\perp = \underline{k}^i(x, t) = \frac{\langle \gamma_{xx}^i(x, t) | \nu^i(x, t) \rangle}{|\gamma_x^i(x, t)|^2} \nu^i(x, t) = \left( \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \right)^\perp.$$

To be more precise, a family of regular networks  $\mathbb{S}_t$  is a motion by curvature in a time interval  $[0, T)$  if the functions  $\gamma^i : [0, 1] \times [0, T) \rightarrow \bar{\Omega}$  are at least  $C^2$  in space and  $C^1$  in time and satisfy the following system:

$$\begin{cases} \gamma_x^i(x, t) \neq 0 \\ \sum \tau^i(O, t) = 0 & \text{at every 3-point} \\ \gamma_t^i = k^i \nu^i + \lambda^i \tau^i & \text{for some continuous functions } \lambda^i \end{cases} \quad (*)$$

with fixed end-points on  $\partial\Omega$ .

## Motion by curvature

With the right choice of the tangential component of the velocity the problem becomes a *non-degenerate* system of quasilinear parabolic equations (with several geometric properties).

## Motion by curvature

With the right choice of the tangential component of the velocity the problem becomes a *non-degenerate* system of quasilinear parabolic equations (with several geometric properties).

$$\gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2}$$

## Motion by curvature

With the right choice of the tangential component of the velocity the problem becomes a *non-degenerate* system of quasilinear parabolic equations (with several geometric properties).

$$\gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \implies \gamma_t^i(x, t)^\perp = \left( \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \right)^\perp$$

## Motion by curvature

With the right choice of the tangential component of the velocity the problem becomes a *non-degenerate* system of quasilinear parabolic equations (with several geometric properties).

$$\gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \implies \gamma_t^i(x, t)^\perp = \left( \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \right)^\perp = \underline{k}^i(x, t)$$



## Motion by curvature

With the right choice of the tangential component of the velocity the problem becomes a *non-degenerate* system of quasilinear parabolic equations (with several geometric properties).

$$\gamma_t^i(x, t) = \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \implies \gamma_t^i(x, t)^\perp = \left( \frac{\gamma_{xx}^i(x, t)}{|\gamma_x^i(x, t)|^2} \right)^\perp = \underline{k}^i(x, t)$$

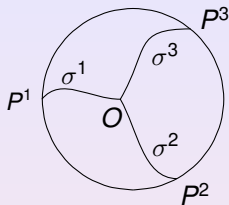
### Definition

A curvature flow  $\gamma^i$  for the initial, regular  $C^2$  network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which satisfies

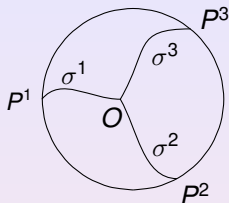
$$\gamma_t^i = \frac{\gamma_{xx}^i}{|\gamma_x^i|^2}$$

for every  $t > 0$  will be called a special curvature flow of  $\mathbb{S}_0$ .

## Short time existence (Triod)



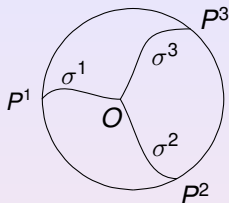
## Short time existence (Triod)



## Theorem (Bronsard–Reitich, 1992 &amp; CM–Novaga–Tortorelli, 2004)

*For any initial, regular  $C^{2+\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1)$ , which is 2-compatible, there exists a unique special flow in the class  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T))$ , in a maximal time interval  $[0, T)$ .*

## Short time existence (Triod)



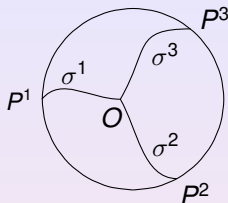
## Theorem (Bronsard–Reitich, 1992 &amp; CM–Novaga–Tortorelli, 2004)

For any initial, regular  $C^{2+\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1)$ , which is 2-compatible, there exists a unique special flow in the class  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T))$ , in a maximal time interval  $[0, T)$ .

A triod is 2-compatible if

$$\frac{\sigma_{xx}^i(0)}{|\sigma_x^i(0)|^2} = \frac{\sigma_{xx}^j(0)}{|\sigma_x^j(0)|^2} \text{ for every } i, j \in \{1, 2, 3\}$$

## Short time existence (Triod)



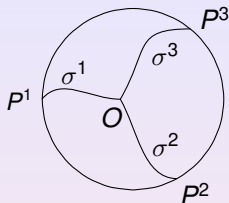
## Theorem (Bronsard–Reitich, 1992 &amp; CM–Novaga–Tortorelli, 2004)

For any initial, regular  $C^{2+\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1)$ , which is 2-compatible, there exists a unique special flow in the class  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ , in a maximal time interval  $[0, T)$ .

A triod is 2-compatible if

$$\frac{\sigma_{xx}^i(0)}{|\sigma_x^i(0)|^2} = \frac{\sigma_{xx}^j(0)}{|\sigma_x^j(0)|^2} \text{ for every } i, j \in \{1, 2, 3\} \implies \sum_{i=1}^3 k^i(O) = 0 \text{ at the 3-point}$$

## Short time existence (Triod)



## Theorem (Bronsard–Reitich, 1992 &amp; CM–Novaga–Tortorelli, 2004)

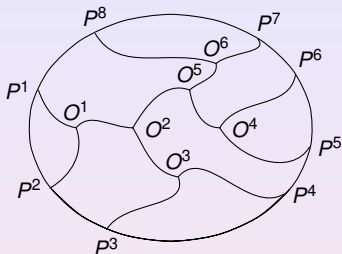
For any initial, regular  $C^{2+\alpha}$  triod  $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0, 1])$ , with  $\alpha \in (0, 1)$ , which is 2-compatible, there exists a unique special flow in the class  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T))$ , in a maximal time interval  $[0, T)$ .

A triod is 2-compatible if

$$\frac{\sigma_{xx}^i(0)}{|\sigma_x^i(0)|^2} = \frac{\sigma_{xx}^j(0)}{|\sigma_x^j(0)|^2} \text{ for every } i, j \in \{1, 2, 3\} \implies \sum_{i=1}^3 k^i(O) = 0 \text{ at the 3-point}$$

Viceversa, if the sum of the curvatures is zero at the 3-point there is a reparametrization making the triod 2-compatible (geometric 2-compatibility).

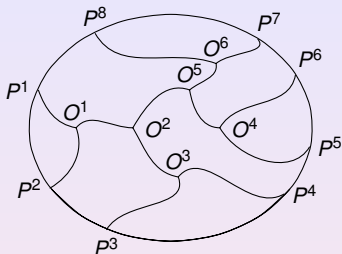
## Short time existence



### Theorem

For any initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is regular,  $C^{2+\alpha}$  with  $\alpha \in (0, 1)$ , 2-compatible, there exists a unique special flow in the class  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ , in a maximal time interval  $[0, T)$ .

## Short time existence



### Theorem

For any initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is regular,  $C^{2+\alpha}$  with  $\alpha \in (0, 1)$ , 2-compatible, there exists a unique special flow in the class  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ , in a maximal time interval  $[0, T)$ .

### Theorem

For any initial smooth, regular network  $\mathbb{S}_0$  there exists a unique smooth special flow in a maximal time interval  $[0, T)$ .



## Short time existence

Theorem [Bronsard–Reitich]:

$\mathbb{T}_0 \in C^{2+\alpha}$ , 2-compatible  
special flow  $\gamma_t = \gamma_{xx}/|\gamma_x|^2$

↓

there exists a unique solution  
 $\mathbb{T}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ .

well-posedness  
of the system →

Theorem:

$\mathbb{S}_0 \in C^{2+\alpha}$ , 2-compatible  
special flow  $\gamma_t = \gamma_{xx}/|\gamma_x|^2$

↓

there exists a unique solution  
 $\mathbb{S}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ .

↓

Theorem:

$\mathbb{S}_0 \in C^{2+\alpha}$ , 2-compatible  
'general' flow  $\gamma_t = k\nu + \lambda\tau$

↓

there exists a solution  
 $\mathbb{S}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ .

## Short time existence

Theorem [Bronsard–Reitich]:

$\mathbb{T}_0 \in C^{2+\alpha}$ , 2-compatible  
special flow  $\gamma_t = \gamma_{xx}/|\gamma_x|^2$



there exists a unique solution  
 $\mathbb{T}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ .

well-posedness  
of the system →

Theorem:

$\mathbb{S}_0 \in C^{2+\alpha}$ , 2-compatible  
special flow  $\gamma_t = \gamma_{xx}/|\gamma_x|^2$



there exists a unique solution  
 $\mathbb{S}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ .



Theorem:

$\mathbb{S}_0 \in C^{2+\alpha}$ , 2-compatible  
'general' flow  $\gamma_t = k\nu + \lambda\tau$



there exists a solution  
 $\mathbb{S}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ .

### Theorem

*For any initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is regular,  $C^{2+\alpha}$  with  $\alpha \in (0, 1)$ , 2-compatible, there exists a  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$  curvature flow of  $\mathbb{S}_0$  in a maximal positive time interval  $[0, T)$ .*

## Short time existence and uniqueness

A network is *geometrically 2-compatible* if

$$\sum_{i=1}^3 k^i(O) = 0 \quad \text{at every 3-point } O$$

## Short time existence and uniqueness

A network is *geometrically 2-compatible* if

$$\sum_{i=1}^3 k^i(O) = 0 \quad \text{at every 3-point } O$$

### Proposition

*For any initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is regular,  $C^{2+\alpha}$  with  $\alpha \in (0, 1)$ , geometrically 2-compatible, there exists a geometrically unique curvature flow in the class of maps  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T))$ , in a maximal positive time interval  $[0, T)$ .*

Geometric uniqueness = Uniqueness up to reparametrizations.

## Short time existence and uniqueness

A network is *geometrically 2-compatible* if

$$\sum_{i=1}^3 k^i(O) = 0 \quad \text{at every 3-point } O$$

### Proposition

For any initial network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  which is regular,  $C^{2+\alpha}$  with  $\alpha \in (0, 1)$ , geometrically 2-compatible, there exists a geometrically unique curvature flow in the class of maps  $C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T])$ , in a maximal positive time interval  $[0, T)$ .

Geometric uniqueness = Uniqueness up to reparametrizations.

Theorem:

$\mathbb{S}_0 \in C^{2+\alpha}$ , 2-compatible  
special flow  $\gamma_t = \gamma_{xx}/|\gamma_x|^2$



there exists a unique solution  
 $\mathbb{S}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T))$ .



Proposition:

$\mathbb{S}_0 \in C^{2+\alpha}$ , geom. 2-compatible  
'general' flow  $\gamma_t = k\nu + \lambda\tau$



there exists a geom. unique solution  
 $\mathbb{S}_t \in C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T))$ .

## Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

## Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

### Theorem

*For any initial  $C^2$  regular network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  there exists a curvature flow  $\gamma^i$ , in a maximal time interval  $[0, T)$ .*

*Such flow  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is a smooth flow for every time  $t > 0$ .*

## Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

### Theorem

*For any initial  $C^2$  regular network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  there exists a curvature flow  $\gamma^i$ , in a maximal time interval  $[0, T)$ .*

*Such flow  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is a smooth flow for every time  $t > 0$ .*

- ▶ The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2-compatibility), but only to have angles of 120 degrees between the concurring curves at every 3-point, that is, to be regular. Clearly, the curvature function is no more necessarily continuous at  $t = 0$  at the triple junctions.



## Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

### Theorem

*For any initial  $C^2$  regular network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  there exists a curvature flow  $\gamma^i$ , in a maximal time interval  $[0, T)$ .*

*Such flow  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is a smooth flow for every time  $t > 0$ .*

- ▶ The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2-compatibility), but only to have angles of 120 degrees between the concurring curves at every 3-point, that is, to be regular. Clearly, the curvature function is no more necessarily continuous at  $t = 0$  at the triple junctions.
- ▶ The geometric uniqueness of the solution found in this theorem is an open problem.

## Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

### Theorem

*For any initial  $C^2$  regular network  $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0, 1])$  there exists a curvature flow  $\gamma^i$ , in a maximal time interval  $[0, T)$ .*

*Such flow  $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0, 1], t)$  is a smooth flow for every time  $t > 0$ .*

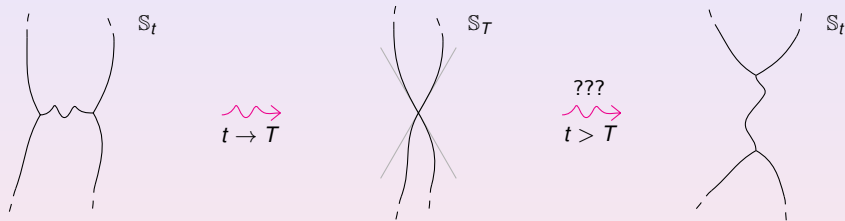
- ▶ The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2-compatibility), but only to have angles of 120 degrees between the concurring curves at every 3-point, that is, to be regular. Clearly, the curvature function is no more necessarily continuous at  $t = 0$  at the triple junctions.
- ▶ The geometric uniqueness of the solution found in this theorem is an open problem.
- ▶ General problem with uniqueness: it depends on the class of curves where we look for solutions. No proof of uniqueness in the “natural” class of  $C^2$  in space,  $C^1$  in time curves. Lack of maximum principle.

## Short time existence for non-regular networks

All these results and the method work exclusively for *regular* networks, moreover, in order to be able to “restart” the flow after some “collapse” (change of topological structure of the network) we really need an existence theorem also for *non-regular* networks.

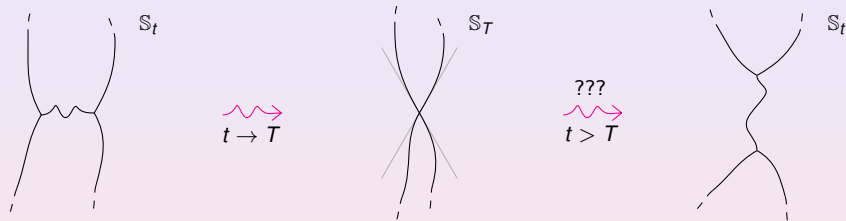
## Short time existence for non-regular networks

All these results and the method work exclusively for *regular* networks, moreover, in order to be able to “restart” the flow after some “collapse” (change of topological structure of the network) we really need an existence theorem also for *non-regular* networks.



## Short time existence for non-regular networks

All these results and the method work exclusively for *regular* networks, moreover, in order to be able to “restart” the flow after some “collapse” (change of topological structure of the network) we really need an existence theorem also for *non-regular* networks.

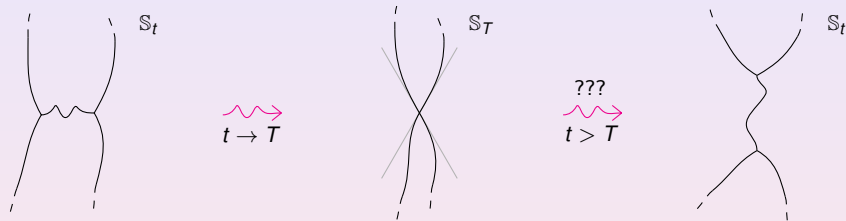


### Theorem (Ilmanen–Neves–Schulze, 2014)

For any initial network of *non-intersecting* curves there exists a (possibly non-unique) *Brakke flow by curvature* in a positive maximal time interval such that for every positive time the evolving network is smooth and regular.

## Short time existence for non-regular networks

All these results and the method work exclusively for *regular* networks, moreover, in order to be able to “restart” the flow after some “collapse” (change of topological structure of the network) we really need an existence theorem also for *non-regular* networks.



### Theorem (Ilmanen–Neves–Schulze, 2014)

For any initial network of **non-intersecting** curves there exists a (possibly non-unique) **Brakke flow by curvature** in a positive maximal time interval such that for every positive time the evolving network is smooth and regular.

No hope for uniqueness. Conjecturally, the flow is unique for “generic” initial networks.

## Long time behavior

After discussing the existence for short time of the curvature flow, our aim is then to analyse the global (in time) behavior of the evolving networks, in particular the formation of “singularities”.

## Long time behavior

After discussing the existence for short time of the curvature flow, our aim is then to analyse the global (in time) behavior of the evolving networks, in particular the formation of “singularities”.

The previous results provide a smooth flow of regular networks (satisfying the compatibility conditions of every order) in a maximal time interval  $[0, T)$ . We will discuss later on the case  $T = +\infty$ , now we assume that  $T$  is finite and investigate what is happening to the moving network as  $t \rightarrow T$ .



## Long time behavior

After discussing the existence for short time of the curvature flow, our aim is then to analyse the global (in time) behavior of the evolving networks, in particular the formation of “singularities”.

The previous results provide a smooth flow of regular networks (satisfying the compatibility conditions of every order) in a maximal time interval  $[0, T)$ . We will discuss later on the case  $T = +\infty$ , now we assume that  $T$  is finite and investigate what is happening to the moving network as  $t \rightarrow T$ .

### Theorem

*If  $T < +\infty$  is the maximal time of smooth existence of the curvature flow of a network  $\mathbb{S}_t$ , then at least one of the following holds:*

- 1. the length of at least one curve of  $\mathbb{S}_t$  goes to zero on a sequence  $t_n \rightarrow T$ ,*
- 2. the curvature is not bounded as  $t \rightarrow T$ .*

## Gagliardo–Nirenberg estimates

**Theorem (Nirenberg, "On elliptic partial...", Ann. SNS 13, 1959  
Section 3, pp. 257–263)**

*Let  $\gamma$  be a  $C^\infty$ , regular curve in  $\mathbb{R}^2$  with finite length  $L$ . If  $u$  is a  $C^\infty$  function defined on  $\gamma$  and  $m \geq 1$ ,  $p \in [2, +\infty]$ , we have the estimates*

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{L^{m\sigma}} \|u\|_{L^2}$$

*for every  $n \in \{0, \dots, m-1\}$  where*

$$\sigma = \frac{n + 1/2 - 1/p}{m}$$

*and the constants  $C_{n,m,p}$  and  $B_{n,m,p}$  are independent of  $\gamma$ .  
In particular, if  $p = +\infty$ ,*

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^\sigma \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{L^{m\sigma}} \|u\|_{L^2} \quad \text{with} \quad \sigma = \frac{n + 1/2}{m}$$

## Evolution of a triod

### Theorem

*If none of the lengths of the three curves of an evolving triod goes to zero, the flow is smooth for all times and the triod converges (asymptotically) to the Steiner configuration connecting the three endpoints.*

# Evolution of a triod

## Theorem

*If none of the lengths of the three curves of an evolving triod goes to zero, the flow is smooth for all times and the triod converges (asymptotically) to the Steiner configuration connecting the three endpoints.*

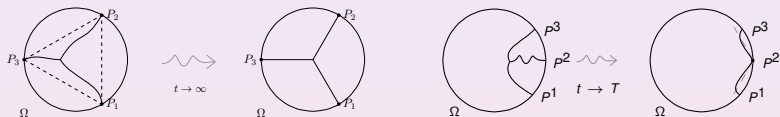


If all the angles of the triangle with vertices the three end-points on the boundary are less than 120 degrees and the initial triod is contained in the triangle, then the triod converges in infinite time to the Steiner configuration connecting the three points.

# Evolution of a triod

## Theorem

*If none of the lengths of the three curves of an evolving triod goes to zero, the flow is smooth for all times and the triod converges (asymptotically) to the Steiner configuration connecting the three endpoints.*



If all the angles of the triangle with vertices the three end-points on the boundary are less than 120 degrees and the initial triod is contained in the triangle, then the triod converges in infinite time to the Steiner configuration connecting the three points.

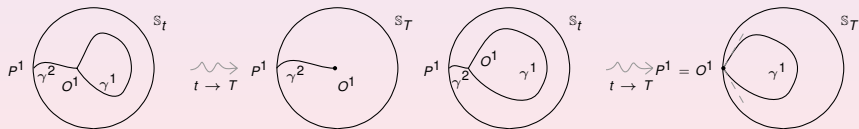
If the fixed end-points on the boundary form a triangle with an angle of more than 120 degrees, then the length of a curve goes to zero in finite time.

# Evolution of a spoon

## Theorem

*The maximal time of existence of a smooth evolution of a spoon is finite and one of the following situations occurs:*

- the closed loop shrinks to a point in a finite time and the maximum of the curvature goes to  $+\infty$ , as  $t \rightarrow T$ ;*
- the “open” curve vanishes and there is a 2–point formation on the boundary of the domain of evolution, but the curvature remains bounded.*

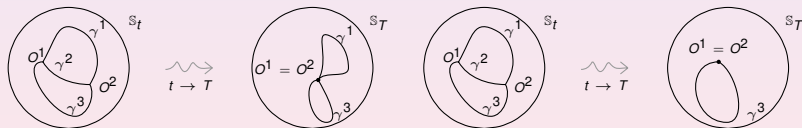


## Evolution of a “theta” (double cell)

### Theorem

The maximal time of existence of a smooth evolution of a “theta” is finite and one of the following situations occurs:

1. the length of a curve that connects the two 3–points goes to zero as  $t \rightarrow T$  and the curvature remains bounded;
2. the length of the curves composing one of the loops goes to zero as  $t \rightarrow T$  and the maximum of the curvature goes to  $+\infty$ .



In any case the network cannot completely vanish shrinking to a point as  $t \rightarrow T$  (not easy to show).

## Long time behavior

We have seen examples in which the length of at least a curve goes to zero and examples in which at the same time the length of a curve goes to zero and the curvature of the network is unbounded.



## Long time behavior

We have seen examples in which the length of at least a curve goes to zero and examples in which at the same time the length of a curve goes to zero and the curvature of the network is unbounded.

But NO examples in which the lengths of all the curves of the network remain uniformly bounded away from zero during the evolution and the curvature is unbounded, as  $t \rightarrow T$ .

## Long time behavior

We have seen examples in which the length of at least a curve goes to zero and examples in which at the same time the length of a curve goes to zero and the curvature of the network is unbounded.

But NO examples in which the lengths of all the curves of the network remain uniformly bounded away from zero during the evolution and the curvature is unbounded, as  $t \rightarrow T$ .

### Conjecture

*If no length of the curves of the network goes to zero on any sequence of times  $t_n \rightarrow T$ , then  $T$  is not a singular time (maximal time of smooth existence).*

## Long time behavior

We are actually able to show the following:

### Theorem

*If no length goes to zero on any sequence of times  $t_n \rightarrow T$  and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (M1)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

## Long time behavior

We are actually able to show the following:

### Theorem

*If no length goes to zero on any sequence of times  $t_n \rightarrow T$  and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (M1)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

To show this theorem and in general to understand the nature of the singularities of the flow, we will employ a blow–up technique where the validity of such conjecture will play a key role.

## Huisken's dynamical rescaling – Blow-up technique

We rescale the smooth flow  $\mathbb{S}_t$  of regular networks in its maximal time interval  $[0, T)$  of smooth existence as follows:

### Definition

Fixed  $x_0 \in \mathbb{R}^2$ , we define the “rescaled” flow as

$$\tilde{\mathbb{S}}_{x_0, \tau} = \frac{\mathbb{S}_{t(\tau)} - x_0}{\sqrt{2(T - t(\tau))}}$$

where  $\tau \in [-1/2 \log T, +\infty)$  and  $\tau(t) = -\frac{1}{2} \log(T - t)$ .

We denote with  $\sigma$  the arclength measure on the rescaled networks.

## Rescaled monotonicity formula

## Proposition

Let  $x_0 \in \mathbb{R}^2$ , for every  $\tau \in [-1/2 \log T, +\infty)$  the following identity holds

$$\frac{d}{d\tau} \int_{\tilde{S}_{x_0, \tau}} e^{-\frac{|x|^2}{2}} d\sigma = - \int_{\tilde{S}_{x_0, \tau}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma + \widetilde{BT}(\tau).$$

## Rescaled monotonicity formula

## Proposition

Let  $x_0 \in \mathbb{R}^2$ , for every  $\tau \in [-1/2 \log T, +\infty)$  the following identity holds

$$\frac{d}{d\tau} \int_{\tilde{S}_{x_0, \tau}} e^{-\frac{|x|^2}{2}} d\sigma = - \int_{\tilde{S}_{x_0, \tau}} |\tilde{k} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma + \widetilde{BT}(\tau).$$

Integrating between  $\tau_1$  and  $\tau_2$  with  $-1/2 \log T \leq \tau_1 \leq \tau_2 < +\infty$  we get

$$\int_{\tau_1}^{\tau_2} \int_{\tilde{S}_{x_0, \tau}} |\tilde{k} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau = \int_{\tilde{S}_{x_0, \tau_1}} e^{-\frac{|x|^2}{2}} d\sigma - \int_{\tilde{S}_{x_0, \tau_2}} e^{-\frac{|x|^2}{2}} d\sigma + \int_{\tau_1}^{\tau_2} \widetilde{BT}(\tau) d\tau$$

moreover, the last integral is uniformly bounded independently of  $\tau_1, \tau_2$ .

## Rescaled monotonicity formula

## Proposition

Let  $x_0 \in \mathbb{R}^2$ , for every  $\tau \in [-1/2 \log T, +\infty)$  the following identity holds

$$\frac{d}{d\tau} \int_{\tilde{S}_{x_0, \tau}} e^{-\frac{|x|^2}{2}} d\sigma = - \int_{\tilde{S}_{x_0, \tau}} |\tilde{k} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma + \widetilde{BT}(\tau).$$

Integrating between  $\tau_1$  and  $\tau_2$  with  $-1/2 \log T \leq \tau_1 \leq \tau_2 < +\infty$  we get

$$\int_{\tau_1}^{\tau_2} \int_{\tilde{S}_{x_0, \tau}} |\tilde{k} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau = \int_{\tilde{S}_{x_0, \tau_1}} e^{-\frac{|x|^2}{2}} d\sigma - \int_{\tilde{S}_{x_0, \tau_2}} e^{-\frac{|x|^2}{2}} d\sigma + \int_{\tau_1}^{\tau_2} \widetilde{BT}(\tau) d\tau$$

moreover, the last integral is uniformly bounded independently of  $\tau_1, \tau_2$ .

Putting  $\tau_1 = -1/2 \log T$  and sending  $\tau_2$  to  $+\infty$ , we conclude

$$\int_{-1/2 \log T}^{+\infty} \int_{\tilde{S}_{x_0, \tau}} |\tilde{k} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau \leq \int_{\tilde{S}_{x_0, -1/2 \log T}} e^{-\frac{|x|^2}{2}} d\sigma + C < +\infty.$$



## Blow-up limits – Skrinkers

As

$$\int_{-1/2 \log T}^{+\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau}} |\tilde{\mathbf{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau < +\infty,$$

for a subsequence of rescaled times  $\tau_i \rightarrow +\infty$ , we have

$$\lim_{i \rightarrow +\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau_i}} |\tilde{\mathbf{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma = 0.$$

## Blow-up limits – Skrinkers

As

$$\int_{-1/2 \log T}^{+\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau < +\infty,$$

for a subsequence of rescaled times  $\tau_i \rightarrow +\infty$ , we have

$$\lim_{i \rightarrow +\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau_i}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma = 0.$$

Then, by a standard (geometric) compactness argument, based on the previous estimates, the sequence of rescaled networks  $\tilde{\mathbb{S}}_{x_0, \tau_n}$  (possibly after reparametrization) converges (up to a subsequence) weakly in  $W_{loc}^{2,2}$  and strongly in  $C_{loc}^{1,\alpha}$ , to a (possibly empty) limit network  $\tilde{\mathbb{S}}_\infty$  (possibly with multiplicity higher than one) satisfying distributionally the *shrinkers equation*

$$\tilde{\underline{k}} + x^\perp = 0$$

at every point  $x \in \tilde{\mathbb{S}}_\infty$ .

## Blow-up limits – Skrinkers

As

$$\int_{-1/2 \log T}^{+\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau < +\infty,$$

for a subsequence of rescaled times  $\tau_i \rightarrow +\infty$ , we have

$$\lim_{i \rightarrow +\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau_i}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma = 0.$$

Then, by a standard (geometric) compactness argument, based on the previous estimates, the sequence of rescaled networks  $\tilde{\mathbb{S}}_{x_0, \tau_n}$  (possibly after reparametrization) converges (up to a subsequence) weakly in  $W_{loc}^{2,2}$  and strongly in  $C_{loc}^{1,\alpha}$ , to a (possibly empty) limit network  $\tilde{\mathbb{S}}_\infty$  (possibly with multiplicity higher than one) satisfying distributionally the *shrinkers equation*

$$\tilde{\underline{k}} + x^\perp = 0$$

at every point  $x \in \tilde{\mathbb{S}}_\infty$ .

By a “bootstrap” argument a shrinker is actually smooth and this equation holds classically.

## Blow-up limits – Skrinkers

As

$$\int_{-1/2 \log T}^{+\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma d\tau < +\infty,$$

for a subsequence of rescaled times  $\tau_i \rightarrow +\infty$ , we have

$$\lim_{i \rightarrow +\infty} \int_{\tilde{\mathbb{S}}_{x_0, \tau_i}} |\tilde{\underline{k}} + x^\perp|^2 e^{-\frac{|x|^2}{2}} d\sigma = 0.$$

Then, by a standard (geometric) compactness argument, based on the previous estimates, the sequence of rescaled networks  $\tilde{\mathbb{S}}_{x_0, \tau_n}$  (possibly after reparametrization) converges (up to a subsequence) weakly in  $W_{loc}^{2,2}$  and strongly in  $C_{loc}^{1,\alpha}$ , to a (possibly empty) limit network  $\tilde{\mathbb{S}}_\infty$  (possibly with multiplicity higher than one) satisfying distributionally the *shrinkers equation*

$$\tilde{\underline{k}} + x^\perp = 0$$

at every point  $x \in \tilde{\mathbb{S}}_\infty$ .

By a “bootstrap” argument a shrinker is actually smooth and this equation holds classically.

**The limit shrinker is not unique!!!**

## Regular shrinkers

### Definition

A smooth regular open (no boundary points) network  $\mathbb{S}$  is called a regular shrinker if at every point  $x \in \mathbb{S}$  there holds

$$\underline{k} + x^\perp = 0.$$

The name comes from the fact that the evolution given by  $\mathbb{S}_t = \sqrt{-2t}\mathbb{S}$  is a self-similarly shrinking curvature flow in the time interval  $(-\infty, 0)$  with  $\mathbb{S} = \mathbb{S}_{-1/2}$ .

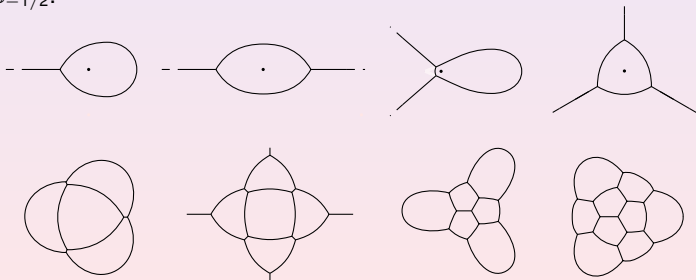
## Regular shrinkers

### Definition

A smooth regular open (no boundary points) network  $\mathbb{S}$  is called a regular shrinker if at every point  $x \in \mathbb{S}$  there holds

$$\underline{k} + x^\perp = 0.$$

The name comes from the fact that the evolution given by  $\mathbb{S}_t = \sqrt{-2t}\mathbb{S}$  is a self-similarly shrinking curvature flow in the time interval  $(-\infty, 0)$  with  $\mathbb{S} = \mathbb{S}_{-1/2}$ .



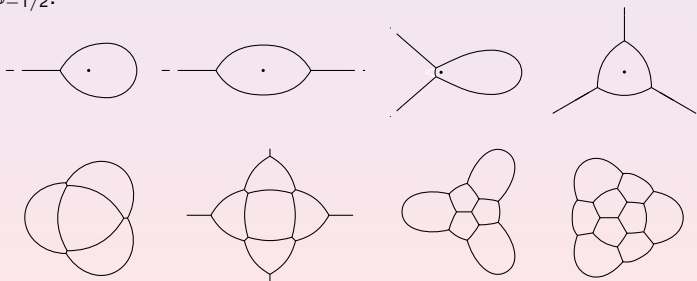
## Regular shrinkers

### Definition

A smooth regular open (no boundary points) network  $\mathbb{S}$  is called a regular shrinker if at every point  $x \in \mathbb{S}$  there holds

$$\underline{k} + x^\perp = 0.$$

The name comes from the fact that the evolution given by  $\mathbb{S}_t = \sqrt{-2t}\mathbb{S}$  is a self-similarly shrinking curvature flow in the time interval  $(-\infty, 0)$  with  $\mathbb{S} = \mathbb{S}_{-1/2}$ .



The unit circle, a straight line or an infinite regular flat triod are regular shrinkers also.

## Sketch of the proof

### Theorem

*If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (M1)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*



## Sketch of the proof

### Theorem

*If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (**M1**)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

*Proof.* Assuming that the shrinking limit network  $\tilde{\mathbb{S}}_\infty$  is not empty, a first consequence of **M1** is that it is embedded, it has no self–intersections.

## Sketch of the proof

### Theorem

*If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (**M1**)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

*Proof.* Assuming that the shrinking limit network  $\tilde{\mathbb{S}}_\infty$  is not empty, a first consequence of **M1** is that it is embedded, it has no self–intersections. Then, since all the lengths are uniformly bounded below away from zero, the lengths of all the curves of  $\tilde{\mathbb{S}}_\infty$  are not finite.

## Sketch of the proof

### Theorem

*If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (M1)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

*Proof.* Assuming that the shrinking limit network  $\tilde{\mathbb{S}}_\infty$  is not empty, a first consequence of **M1** is that it is embedded, it has no self–intersections. Then, since all the lengths are uniformly bounded below away from zero, the lengths of all the curves of  $\tilde{\mathbb{S}}_\infty$  are not finite. If any of such curves is bounded, it must have self–intersections (by the analysis of the ODE satisfied by such curves, which is implied by the shrinkers equation – Abresch & Langer).

## Sketch of the proof

### Theorem

*If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (M1)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

*Proof.* Assuming that the shrinking limit network  $\tilde{\mathbb{S}}_\infty$  is not empty, a first consequence of **M1** is that it is embedded, it has no self–intersections. Then, since all the lengths are uniformly bounded below away from zero, the lengths of all the curves of  $\tilde{\mathbb{S}}_\infty$  are not finite. If any of such curves is bounded, it must have self–intersections (by the analysis of the ODE satisfied by such curves, which is implied by the shrinkers equation – Abresch & Langer). Moreover, in general, any unbounded curve of a shrinker must be a line or a halfline (again using the ODE).

## Sketch of the proof

### Theorem

*If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence,  $T$  is not a singular time and the flow is smooth.*

### "Multiplicity–One Conjecture" (M1)

*Every possible limit of rescaled networks of the flow is a network with multiplicity one.*

*Proof.* Assuming that the shrinking limit network  $\tilde{\mathbb{S}}_\infty$  is not empty, a first consequence of **M1** is that it is embedded, it has no self–intersections. Then, since all the lengths are uniformly bounded below away from zero, the lengths of all the curves of  $\tilde{\mathbb{S}}_\infty$  are not finite. If any of such curves is bounded, it must have self–intersections (by the analysis of the ODE satisfied by such curves, which is implied by the shrinkers equation – Abresch & Langer). Moreover, in general, any unbounded curve of a shrinker must be a line or a halfline (again using the ODE).

We conclude that  $\tilde{\mathbb{S}}_\infty$  is composed only by lines and halflines for the origin of  $\mathbb{R}^2$ .

## Sketch of the proof

Since the lengths of the converging sequence to  $\tilde{\mathbb{S}}_\infty$  are going to  $+\infty$ , two triple junctions cannot go to collide, then, by the  $C_{loc}^1$ -convergence, there are only three possibilities for the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$ :

- ▶ the empty set
- ▶ a straight line
- ▶ a halfline
- ▶ an infinite regular flat triod

all with multiplicity one (notice they all have zero curvature).

## Sketch of the proof

Since the lengths of the converging sequence to  $\tilde{\mathbb{S}}_\infty$  are going to  $+\infty$ , two triple junctions cannot go to collide, then, by the  $C_{loc}^1$ -convergence, there are only three possibilities for the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$ :

- ▶ the empty set
- ▶ a straight line
- ▶ a halfline
- ▶ an infinite regular flat triod

all with multiplicity one (notice they all have zero curvature).

If we get an empty set, it means that the evolving network  $\mathbb{S}_t$  “stays far” from the point  $x_0$ .

## Sketch of the proof

Since the lengths of the converging sequence to  $\tilde{\mathbb{S}}_\infty$  are going to  $+\infty$ , two triple junctions cannot go to collide, then, by the  $C_{loc}^1$ -convergence, there are only three possibilities for the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$ :

- ▶ the empty set
- ▶ a straight line
- ▶ a halfline
- ▶ an infinite regular flat triod

all with multiplicity one (notice they all have zero curvature).

If we get an empty set, it means that the evolving network  $\mathbb{S}_t$  “stays far” from the point  $x_0$ . In the other cases, we can then apply

- ▶ White’s theorem in the case of a straight line



## Sketch of the proof

Since the lengths of the converging sequence to  $\tilde{\mathbb{S}}_\infty$  are going to  $+\infty$ , two triple junctions cannot go to collide, then, by the  $C_{loc}^1$ -convergence, there are only three possibilities for the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$ :

- ▶ the empty set
- ▶ a straight line
- ▶ a halfline
- ▶ an infinite regular flat triod

all with multiplicity one (notice they all have zero curvature).

If we get an empty set, it means that the evolving network  $\mathbb{S}_t$  “stays far” from the point  $x_0$ . In the other cases, we can then apply

- ▶ White’s theorem in the case of a straight line
- ▶ a reflection argument and then White’s theorem in the case of a halfline

## Sketch of the proof

Since the lengths of the converging sequence to  $\tilde{\mathbb{S}}_\infty$  are going to  $+\infty$ , two triple junctions cannot go to collide, then, by the  $C_{loc}^1$ -convergence, there are only three possibilities for the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$ :

- ▶ the empty set
- ▶ a straight line
- ▶ a halfline
- ▶ an infinite regular flat triod

all with multiplicity one (notice they all have zero curvature).

If we get an empty set, it means that the evolving network  $\mathbb{S}_t$  “stays far” from the point  $x_0$ . In the other cases, we can then apply

- ▶ White’s theorem in the case of a straight line
- ▶ a reflection argument and then White’s theorem in the case of a halfline
- ▶ a result by Magni–M–Novaga or Ilmanen–Neves–Schulze for an infinite regular flat triod

## Sketch of the proof

Since the lengths of the converging sequence to  $\tilde{\mathbb{S}}_\infty$  are going to  $+\infty$ , two triple junctions cannot go to collide, then, by the  $C_{loc}^1$ -convergence, there are only three possibilities for the blow-up limit shrinker  $\tilde{\mathbb{S}}_\infty$ :

- ▶ the empty set
- ▶ a straight line
- ▶ a halfline
- ▶ an infinite regular flat triod

all with multiplicity one (notice they all have zero curvature).

If we get an empty set, it means that the evolving network  $\mathbb{S}_t$  “stays far” from the point  $x_0$ . In the other cases, we can then apply

- ▶ White’s theorem in the case of a straight line
- ▶ a reflection argument and then White’s theorem in the case of a halfline
- ▶ a result by Magni–M–Novaga or Ilmanen–Neves–Schulze for an infinite regular flat triod

to conclude that in a neighbourhood of  $x_0$  the curvature of the networks  $\mathbb{S}_t$  is bounded uniformly in time, for  $t \in [0, T)$ .

## Sketch of the proof

Since the previous conclusion holds for all  $x_0 \in \mathbb{R}^2$ , it follows that the curvature is globally bounded along the flow, which is a contradiction.

Remember indeed that

### Theorem

*If  $T < +\infty$  is the maximal time of smooth existence of the curvature flow  $\mathbb{S}_t$ , then:*

- 1. either the length of at least one curve of  $\mathbb{S}_t$  goes to zero when  $t \rightarrow T$ ,*
- 2. or the curvature is not bounded as  $t \rightarrow T$ .*



## Long time behavior

Hence, assuming (from now on) the truth of the

**“Multiplicity–One Conjecture”**

*Every possible limit of rescaled networks is a network with multiplicity one.*

## Long time behavior

Hence, assuming (from now on) the truth of the

### “Multiplicity–One Conjecture”

*Every possible limit of rescaled networks is a network with multiplicity one.*

at a singular time  $T < +\infty$  (the maximal time interval of smooth existence of the curvature flow  $\mathbb{S}_t$ ) the length of at least one curve of  $\mathbb{S}_t$  goes to zero, on a sequence of times  $t_n \rightarrow T$ .

## Long time behavior

Hence, assuming (from now on) the truth of the

### “Multiplicity–One Conjecture”

*Every possible limit of rescaled networks is a network with multiplicity one.*

at a singular time  $T < +\infty$  (the maximal time interval of smooth existence of the curvature flow  $\mathbb{S}_t$ ) the length of at least one curve of  $\mathbb{S}_t$  goes to zero, on a sequence of times  $t_n \rightarrow T$ .

Then, there are two possible situations:

- ▶ The curvature stays bounded along the flow
- ▶ The curvature is unbounded as  $t \rightarrow T$

## Singularity types

In the smooth case of a closed curve evolving by curvature there are no singularities with bounded curvature. Indeed the maximum of the modulus of the curvature always satisfies

$$|k|_{\max}(t) \geq 1/\sqrt{2(T-t)}$$

(the analogous statement for a network is an open problem).



## Singularity types

In the smooth case of a closed curve evolving by curvature there are no singularities with bounded curvature. Indeed the maximum of the modulus of the curvature always satisfies

$$|k|_{\max}(t) \geq 1/\sqrt{2(T-t)}$$

(the analogous statement for a network is an open problem).

Singularities are then “classified” by the rate the curvature goes to  $+\infty$ , as  $t \rightarrow T$ :

- ▶ *Type I* – The maximum of the modulus of the curvature is of order  $1/\sqrt{T-t}$ , as  $t \rightarrow T$ .
- ▶ *Type II* – The maximum of the modulus of the curvature is of higher order (on a sequence of times  $t_i \rightarrow T$ ).

## Singularity types

In the smooth case of a closed curve evolving by curvature there are no singularities with bounded curvature. Indeed the maximum of the modulus of the curvature always satisfies

$$|k|_{\max}(t) \geq 1/\sqrt{2(T-t)}$$

(the analogous statement for a network is an open problem).

Singularities are then “classified” by the rate the curvature goes to  $+\infty$ , as  $t \rightarrow T$ :

- ▶ *Type I* – The maximum of the modulus of the curvature is of order  $1/\sqrt{T-t}$ , as  $t \rightarrow T$ .
- ▶ *Type II* – The maximum of the modulus of the curvature is of higher order (on a sequence of times  $t_i \rightarrow T$ ).

For the flow of networks there are actually singularities (collapse of curves) with bounded curvature, that we can call *Type 0* singularities (copyright of Tom Ilmanen).

## Analysis of singularities – Collapse with bounded curvature

As a first remark, regions cannot collapse in this situation, otherwise the curvature goes to  $+\infty$ . Moreover, only “isolated” curves can vanish, by the *Multiplicity–One Conjecture*, that is, it is not possible that more than two triple junctions collide together at a single point. Indeed, in such case there must be a region collapsing.

## Analysis of singularities – Collapse with bounded curvature

As a first remark, regions cannot collapse in this situation, otherwise the curvature goes to  $+\infty$ . Moreover, only “isolated” curves can vanish, by the *Multiplicity–One Conjecture*, that is, it is not possible that more than two triple junctions collide together at a single point. Indeed, in such case there must be a region collapsing.

Then, the analysis consists in understanding the possible networks  $\mathbb{S}_T$  that we get as limits of the flow  $\mathbb{S}_t$ , when  $t \rightarrow T$ .

## Analysis of singularities – Collapse with bounded curvature

As a first remark, regions cannot collapse in this situation, otherwise the curvature goes to  $+\infty$ . Moreover, only “isolated” curves can vanish, by the *Multiplicity–One Conjecture*, that is, it is not possible that more than two triple junctions collide together at a single point. Indeed, in such case there must be a region collapsing.

Then, the analysis consists in understanding the possible networks  $\mathbb{S}_T$  that we get as limits of the flow  $\mathbb{S}_t$ , when  $t \rightarrow T$ .

After reparametrizing every curve proportional to arclength, the bound on the curvature implies the convergence in  $C^1 \cap W^{2,\infty}$  of  $\mathbb{S}_t$  to a limit network  $\mathbb{S}_T$ , as  $t \rightarrow T$ . Moreover, such limit network is unique and all its curves are of class  $W^{2,\infty}$ , that is, with bounded curvature. Anyway,  $\mathbb{S}_t$  is *non-regular* since a 4-point appears for every vanishing curve, but the sum of the unit tangent vectors of the four curves must be zero, by the  $C^1/W^{2,\infty}$ -convergence.

## Analysis of singularities – Collapse with bounded curvature

As a first remark, regions cannot collapse in this situation, otherwise the curvature goes to  $+\infty$ . Moreover, only “isolated” curves can vanish, by the *Multiplicity–One Conjecture*, that is, it is not possible that more than two triple junctions collide together at a single point. Indeed, in such case there must be a region collapsing.

Then, the analysis consists in understanding the possible networks  $\mathbb{S}_T$  that we get as limits of the flow  $\mathbb{S}_t$ , when  $t \rightarrow T$ .

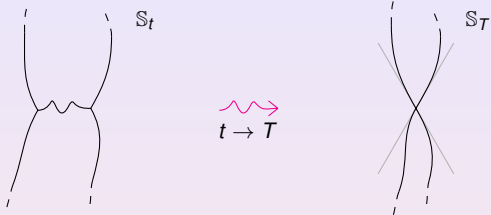
After reparametrizing every curve proportional to arclength, the bound on the curvature implies the convergence in  $C^1 \cap W^{2,\infty}$  of  $\mathbb{S}_t$  to a limit network  $\mathbb{S}_T$ , as  $t \rightarrow T$ . Moreover, such limit network is unique and all its curves are of class  $W^{2,\infty}$ , that is, with bounded curvature. Anyway,  $\mathbb{S}_t$  is *non-regular* since a 4–point appears for every vanishing curve, but the sum of the unit tangent vectors of the four curves must be zero, by the  $C^1/W^{2,\infty}$ –convergence.

### Theorem

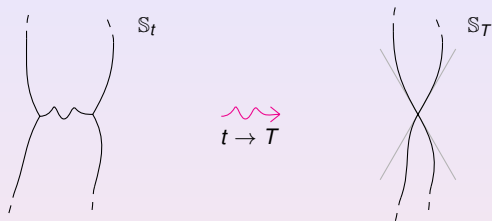
*If M1 is true, every interior vertex of such limit network either is a regular triple junction or it is a 4–point where the four concurring curves have opposite unit tangents in pairs and form angles of 120/60 degrees among them.*

This is exactly what we saw in the simulation when a single curve vanishes.

## Analysis of singularities – Collapse with bounded curvature



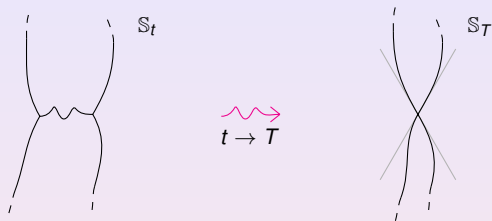
## Analysis of singularities – Collapse with bounded curvature



This analysis can be extended to the curves containing the fixed boundary points (*boundary curves*). If any of them vanishes, we get two curves forming an angle of 120 degrees (remember the example of the *spoon network*). In such case, the flow stops (we really “have to decide” whether/how to continue the flow).



## Analysis of singularities – Collapse with bounded curvature



This analysis can be extended to the curves containing the fixed boundary points (*boundary curves*). If any of them vanishes, we get two curves forming an angle of 120 degrees (remember the example of the *spoon network*). In such case, the flow stops (we really “have to decide” whether/how to continue the flow).

As the limit network  $S_T$  contains at least one 4-junction, in order to restart the flow we need the short time existence theorem of Ilmanen, Neves and Schulze.

## Analysis of singularities – Tree-like networks

### Theorem

*If **M1** holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

## Analysis of singularities – Tree-like networks

### Theorem

*If **M1** holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

### Remarks:

- ▶ *Type 0* – singularities actually exist

## Analysis of singularities – Tree-like networks

### Theorem

*If **M1** holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

### Remarks:

- ▶ *Type 0* – singularities actually exist
- ▶ This result can be localized (if the network is locally a tree around a point, the curvature is locally bounded)

## Analysis of singularities – Tree-like networks

### Theorem

*If **M1** holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

### Remarks:

- ▶ *Type 0* – singularities actually exist
- ▶ This result can be localized (if the network is locally a tree around a point, the curvature is locally bounded)
- ▶ If locally no region is collapsing, then the network is locally a tree, hence, locally the curvature is bounded

## Analysis of singularities – Tree-like networks

### Theorem

*If **M1** holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

### Remarks:

- ▶ *Type 0* – singularities actually exist
- ▶ This result can be localized (if the network is locally a tree around a point, the curvature is locally bounded)
- ▶ If locally no region is collapsing, then the network is locally a tree, hence, locally the curvature is bounded

### Corollary

*If **M1** holds, the curvature is unbounded if and only if a region is collapsing. Hence, the singularities with bounded curvature coincide with the vanishing of isolated curves with formation of 4–points, in the way we have seen before.*

## Analysis of singularities – Tree-like networks

### Theorem

*If **M1** holds and the network is a tree (no loops), the curvature is uniformly bounded during the flow, hence the only singularities are given by the collapse of a curve with only two triple junctions going to collide.*

### Remarks:

- ▶ *Type 0* – singularities actually exist
- ▶ This result can be localized (if the network is locally a tree around a point, the curvature is locally bounded)
- ▶ If locally no region is collapsing, then the network is locally a tree, hence, locally the curvature is bounded

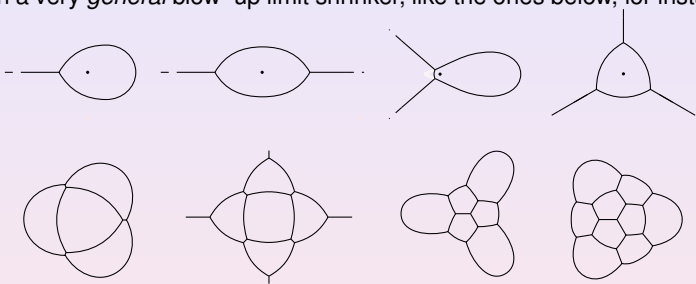
### Corollary

*If **M1** holds, the curvature is unbounded if and only if a region is collapsing. Hence, the singularities with bounded curvature coincide with the vanishing of isolated curves with formation of 4–points, in the way we have seen before.*

Also this corollary can be localized.

## Analysis of singularities – Collapse of a region/Unbounded curvature

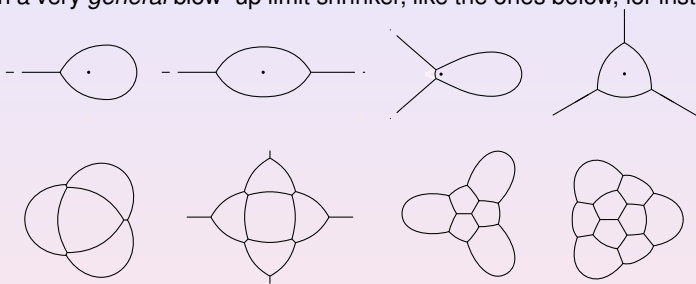
In the situation when the curvature is unbounded, that is, at least one region is collapsing, even if **M1** is true, using again the blow-up procedure we can obtain a very *general* blow-up limit shrinker, like the ones below, for instance:





## Analysis of singularities – Collapse of a region/Unbounded curvature

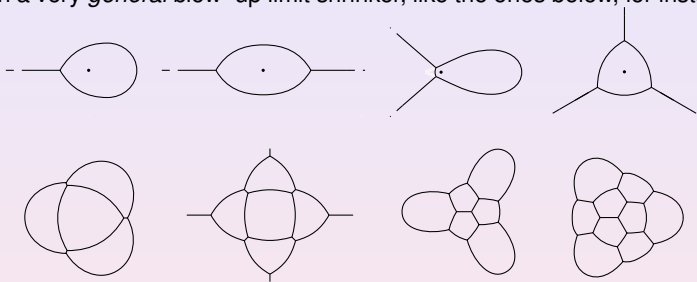
In the situation when the curvature is unbounded, that is, at least one region is collapsing, even if **M1** is true, using again the blow-up procedure we can obtain a very *general* blow-up limit shrinker, like the ones below, for instance:



Then, the (local) structure (topology) of the evolving network plays an important role in the analysis/classification of the possible limit shrinkers (as the topology “simplifies” taking a limit).

## Analysis of singularities – Collapse of a region/Unbounded curvature

In the situation when the curvature is unbounded, that is, at least one region is collapsing, even if **M1** is true, using again the blow-up procedure we can obtain a very *general* blow-up limit shrinker, like the ones below, for instance:



Then, the (local) structure (topology) of the evolving network plays an important role in the analysis/classification of the possible limit shrinkers (as the topology “simplifies” taking a limit).

In this case the blow-up limit is a shrinker *with regions*. These regions are the “memory” of the collapsing regions in the flow  $\mathbb{S}_t$ , as  $t \rightarrow T$ , and the unbounded halflines of the shrinker give the *limit tangents* of the *non-vanishing* curves of  $\mathbb{S}_t$  arriving at the group of collapsing regions.

## Analysis of singularities – Collapse of a region/Unbounded curvature

If, as in the case of bounded curvature, the networks  $\mathbb{S}_t$  converge to some limit  $\mathbb{S}_T$ , as  $t \rightarrow T$ , then the point of collapse will be a multi-point of  $\mathbb{S}_T$  and the unbounded halflines of the blow-up limit shrinker give the tangents of the curves of  $\mathbb{S}_T$  concurring at such point.

## Analysis of singularities – Collapse of a region/Unbounded curvature

If, as in the case of bounded curvature, the networks  $\mathbb{S}_t$  converge to some limit  $\mathbb{S}_T$ , as  $t \rightarrow T$ , then the point of collapse will be a multi-point of  $\mathbb{S}_T$  and the unbounded halflines of the blow-up limit shrinker give the tangents of the curves of  $\mathbb{S}_T$  concurring at such point.

Unfortunately, in this case with unbounded curvature, we are not able to show that we have a *unique* limit network  $\mathbb{S}_T$  (also not a *unique* blow-up limit shrinker).

## Analysis of singularities – Collapse of a region/Unbounded curvature

If, as in the case of bounded curvature, the networks  $\mathbb{S}_t$  converge to some limit  $\mathbb{S}_T$ , as  $t \rightarrow T$ , then the point of collapse will be a multi-point of  $\mathbb{S}_T$  and the unbounded halflines of the blow-up limit shrinker give the tangents of the curves of  $\mathbb{S}_T$  concurring at such point.

Unfortunately, in this case with unbounded curvature, we are not able to show that we have a *unique* limit network  $\mathbb{S}_T$  (also not a *unique* blow-up limit shrinker).

### Conjecture

*As  $t \rightarrow T$ , there exists a unique limit network  $\mathbb{S}_T$ , possibly non-regular with multiple points and/or with triple junctions not satisfying 120 degrees condition.*

## Analysis of singularities – Collapse of a region/Unbounded curvature

If, as in the case of bounded curvature, the networks  $\mathbb{S}_t$  converge to some limit  $\mathbb{S}_T$ , as  $t \rightarrow T$ , then the point of collapse will be a multi-point of  $\mathbb{S}_T$  and the unbounded halflines of the blow-up limit shrinker give the tangents of the curves of  $\mathbb{S}_T$  concurring at such point.

Unfortunately, in this case with unbounded curvature, we are not able to show that we have a *unique* limit network  $\mathbb{S}_T$  (also not a *unique* blow-up limit shrinker).

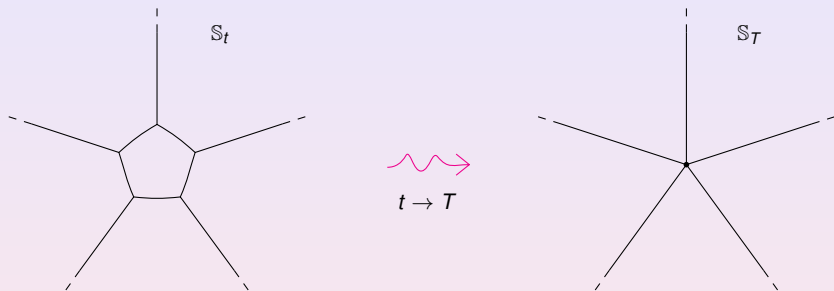
### Conjecture

*As  $t \rightarrow T$ , there exists a unique limit network  $\mathbb{S}_T$ , possibly non-regular with multiple points and/or with triple junctions not satisfying 120 degrees condition.*

Assuming such uniqueness, one can prove that (possibly after reparametrization) the *vanishing* part of  $\mathbb{S}_t$  collapse to a point and the *non-vanishing* part converge in  $C^1$  to a limit network  $\mathbb{S}_T$ . The point of collapse is then a multi-point of  $\mathbb{S}_T$  and the curves of  $\mathbb{S}_T$  concurring at such point are of class  $C^\infty$  far from the multi-point.

Moreover, the curvature of each of such curves goes like  $k = o(1/d)$ , where  $d$  is the arclength distance to the multi-point along the curve.

# Analysis of singularities – Collapse of a region/Unbounded curvature



Example of a (homothetic) collapse of a (symmetric) pentagonal region of  $S_t$  (five-ray star).



## Restarting the flow after a singularity

In the case of singularities with bounded curvature the (unique) limit network  $\mathbb{S}_T$  satisfies the hypotheses of the INS Theorem

**Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)**

*For any initial network of **non-intersecting** curves there exists a (possibly non-unique) **Brakke flow by curvature** in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.*



## Restarting the flow after a singularity

In the case of singularities with bounded curvature the (unique) limit network  $\mathbb{S}_T$  satisfies the hypotheses of the INS Theorem

### Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

*For any initial network of **non-intersecting** curves there exists a (possibly non-unique) **Brakke flow by curvature** in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.*

If instead a region is collapsing, but the previous “uniqueness of the limit network” conjecture holds, we only have that the curvature of the limit network  $\mathbb{S}_T$  is  $o(1/d)$ , where  $d$  is the arclength distance to the multi-points of the network.

## Restarting the flow after a singularity

In the case of singularities with bounded curvature the (unique) limit network  $\mathbb{S}_T$  satisfies the hypotheses of the INS Theorem

### Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

*For any initial network of **non-intersecting** curves there exists a (possibly non-unique) **Brakke flow by curvature** in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.*

If instead a region is collapsing, but the previous “uniqueness of the limit network” conjecture holds, we only have that the curvature of the limit network  $\mathbb{S}_T$  is  $o(1/d)$ , where  $d$  is the arclength distance to the multi-points of the network. However, the INS Theorem can be extended (unpublished) to cover also this case.

## Restarting the flow after a singularity

In the case of singularities with bounded curvature the (unique) limit network  $\mathbb{S}_T$  satisfies the hypotheses of the INS Theorem

### Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

*For any initial network of **non-intersecting** curves there exists a (possibly non-unique) **Brakke flow by curvature** in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.*

If instead a region is collapsing, but the previous “uniqueness of the limit network” conjecture holds, we only have that the curvature of the limit network  $\mathbb{S}_T$  is  $o(1/d)$ , where  $d$  is the arclength distance to the multi-points of the network. However, the INS Theorem can be extended (unpublished) to cover also this case.

Hence, we are able to restart the flow after a singularity, possibly losing the uniqueness.

## Restarting the flow after a singularity

In the case of singularities with bounded curvature the (unique) limit network  $\mathbb{S}_T$  satisfies the hypotheses of the INS Theorem

### Theorem (T. Ilmanen, A. Neves, F. Schulze – 2014)

*For any initial network of **non-intersecting** curves there exists a (possibly non-unique) **Brakke flow by curvature** in a positive maximal time interval such that for every positive time the evolving network is connected, smooth and regular.*

If instead a region is collapsing, but the previous “uniqueness of the limit network” conjecture holds, we only have that the curvature of the limit network  $\mathbb{S}_T$  is  $o(1/d)$ , where  $d$  is the arclength distance to the multi-points of the network. However, the INS Theorem can be extended (unpublished) to cover also this case.

Hence, we are able to restart the flow after a singularity, possibly losing the uniqueness.

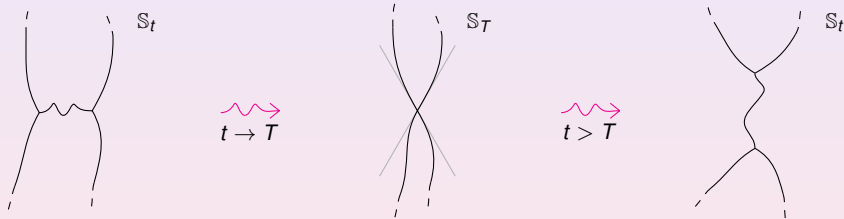
We anyway state the following conjecture:

### Conjecture

*The limit network  $\mathbb{S}_T$  has always bounded curvature.*

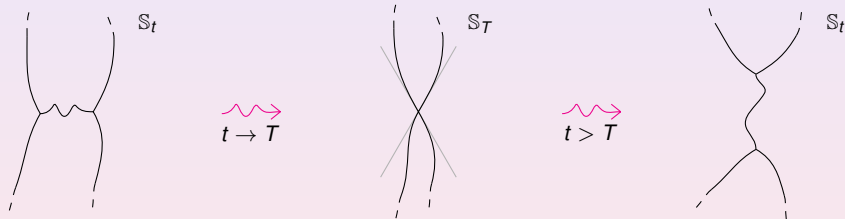
## Restarting the flow after a singularity – The special case of bounded curvature/Standard transition

In the case of collapse of an isolated curve with bounded curvature, the INS Theorem produces the curvature flow that we observed in the simulation.



## Restarting the flow after a singularity – The special case of bounded curvature/Standard transition

In the case of collapse of an isolated curve with bounded curvature, the INS Theorem produces the curvature flow that we observed in the simulation.



There is actually hope for uniqueness in this case, that we call *standard transition*.

## Asymptotic behavior

If the whole network does not vanish at some time (it can happen) or a boundary curve collapses, we can then define a flow “passing through” singularities.

## Asymptotic behavior

If the whole network does not vanish at some time (it can happen) or a boundary curve collapses, we can then define a flow “passing through” singularities. In order to study its asymptotic behavior, we need that it is defined for every positive time and this requires that the singular times do not “accumulate” (that is, they are a discrete set).

### Conjecture

*The singular times do not “accumulate”.*



## Asymptotic behavior

If the whole network does not vanish at some time (it can happen) or a boundary curve collapses, we can then define a flow “passing through” singularities. In order to study its asymptotic behavior, we need that it is defined for every positive time and this requires that the singular times do not “accumulate” (that is, they are a discrete set).

### Conjecture

*The singular times do not “accumulate”.*

### Stronger Conjecture

*The number of singular times is finite (at least for a tree-like network).*

## Asymptotic behavior

If the whole network does not vanish at some time (it can happen) or a boundary curve collapses, we can then define a flow “passing through” singularities. In order to study its asymptotic behavior, we need that it is defined for every positive time and this requires that the singular times do not “accumulate” (that is, they are a discrete set).

### Conjecture

*The singular times do not “accumulate”.*

### Stronger Conjecture

*The number of singular times is finite (at least for a tree-like network).*

In such case the flow is definitely smooth and the evolving network converges (asymptotically and up to a subsequence) to a flat regular network connecting the fixed endpoints, “critical” for the length functional (Steiner configuration).

## Asymptotic behavior

If the whole network does not vanish at some time (it can happen) or a boundary curve collapses, we can then define a flow “passing through” singularities. In order to study its asymptotic behavior, we need that it is defined for every positive time and this requires that the singular times do not “accumulate” (that is, they are a discrete set).

### Conjecture

*The singular times do not “accumulate”.*

### Stronger Conjecture

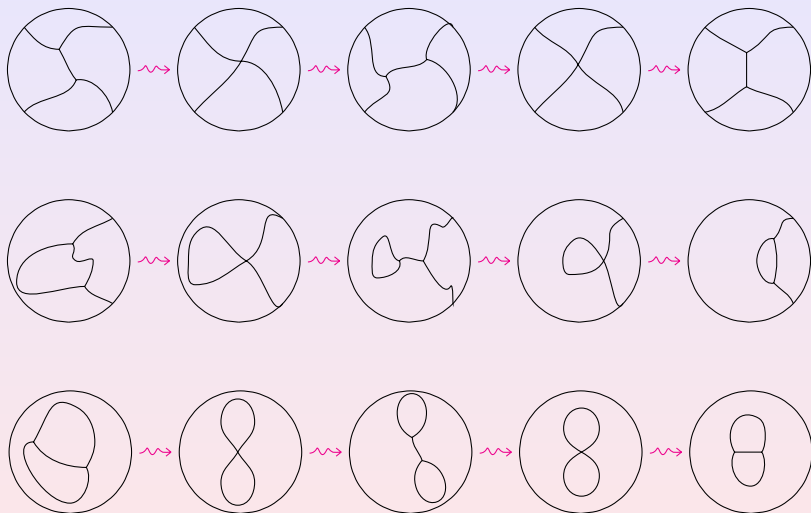
*The number of singular times is finite (at least for a tree-like network).*

In such case the flow is definitely smooth and the evolving network converges (asymptotically and up to a subsequence) to a flat regular network connecting the fixed endpoints, “critical” for the length functional (Steiner configuration).

### Conjecture

*The limit of  $\mathbb{S}_t$  as  $t \rightarrow +\infty$ , is unique (the full sequence of networks converges).*

## Possible “oscillation of shape” phenomenon



Possible infinite “oscillations” via standard transitions from a shape to another and viceversa.

# Open problems and research directions

## Open problems and research directions

**Main Open Problem – “Multiplicity–One Conjecture” (M1)**

*Every possible limit of rescaled networks is a network with multiplicity one.*

## Open problems and research directions

### Main Open Problem – “Multiplicity–One Conjecture” (**M1**)

*Every possible limit of rescaled networks is a network with multiplicity one.*

### Theorem (CM, M. Novaga, A. Pluda – 2015)

- ▶ *If during the flow the triple–junctions stay uniformly far each other, then **M1** is true.*
- ▶ *If the initial network has at most **two** triple junctions, then **M1** is true.*

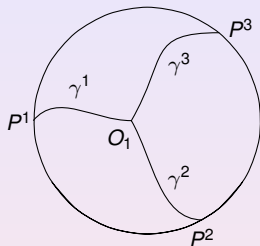
Analysis of singularity formation for some flows of networks with “few” triple junctions can then be made rigorous (under uniqueness of the limit hypothesis in some cases).

Only 1 triple junction



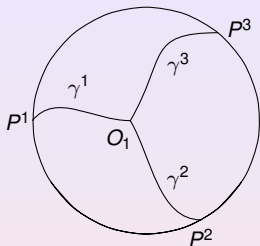
## Only 1 triple junction

The Triod – A. Magni, CM, M. Novaga, V. Tortorelli

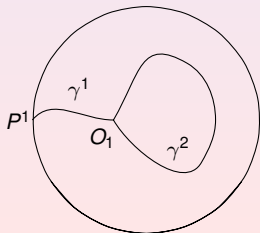


## Only 1 triple junction

The Triod – A. Magni, CM, M. Novaga, V. Tortorelli

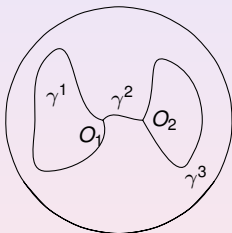


The Spoon – A. Pluda



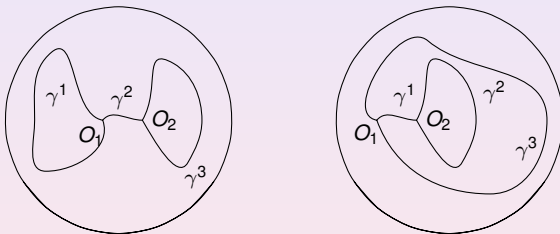
## 2 triple junctions – CM, M. Novaga, A. Pluda

### The Eyeglasses



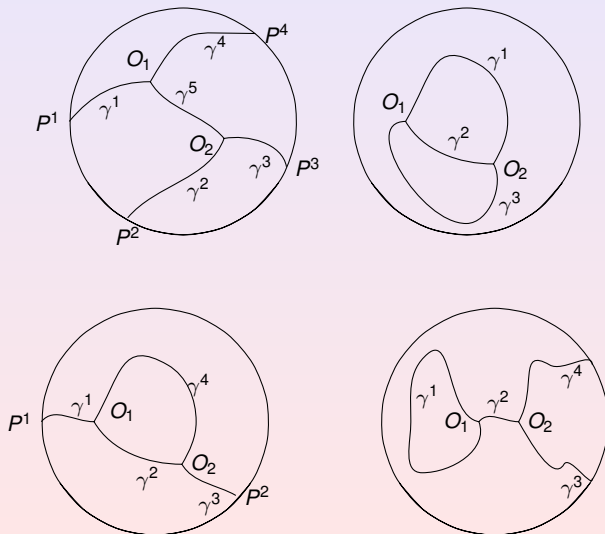
## 2 triple junctions – CM, M. Novaga, A. Pluda

### The Eyeglasses and... the Broken Eyeglasses



## 2 triple junctions – CM, M. Novaga, A. Pluda

### The “Steiner”, Theta, Lens and Island



# Open problems and research directions

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits



## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

Uniqueness of the limit networks at the singular times

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

Uniqueness of the limit networks at the singular times

Existence and non-existence/properties/classification of shrinkers

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

Uniqueness of the limit networks at the singular times

Existence and non-existence/properties/classification of shrinkers

Better estimates in the restarting theorem – more precise quantitative control on the curvature for  $t > T$

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

Uniqueness of the limit networks at the singular times

Existence and non-existence/properties/classification of shrinkers

Better estimates in the restarting theorem – more precise quantitative control on the curvature for  $t > T$

Generic (stable) singularities

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

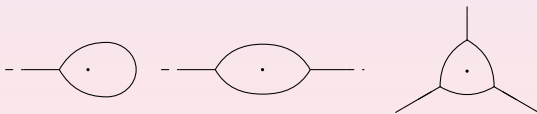
Uniqueness of the limit networks at the singular times

Existence and non-existence/properties/classification of shrinkers

Better estimates in the restarting theorem – more precise quantitative control on the curvature for  $t > T$

Generic (stable) singularities

- Line, cross, circle (a very special network) or



are the only dynamically stable shrinkers?

## Open problems and research directions

Uniqueness issues (regular and non-regular networks)

Uniqueness of the blow-up limits

Existence of Type II singularities/Bound from below on the blow-up rate of the curvature

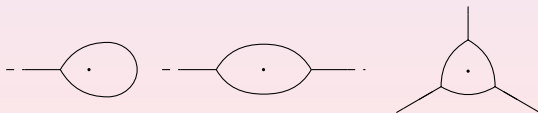
Uniqueness of the limit networks at the singular times

Existence and non-existence/properties/classification of shrinkers

Better estimates in the restarting theorem – more precise quantitative control on the curvature for  $t > T$

Generic (stable) singularities

- Line, cross, circle (a very special network) or



are the only dynamically stable shrinkers?

- *Generically, only regions with at most three edges can collapse*

## Open problems and research directions – Higher dimensions

Study of the motion by mean curvature of 2–dimensional interfaces in  $\mathbb{R}^3$  (*double–bubble*), for instance.



## Open problems and research directions – Higher dimensions

Study of the motion by mean curvature of 2–dimensional interfaces in  $\mathbb{R}^3$  (*double–bubble*), for instance.

- ▶ Short time existence/uniqueness for special initial interfaces by Depner – Garcke – Kohsaka

## Open problems and research directions – Higher dimensions

Study of the motion by mean curvature of 2–dimensional interfaces in  $\mathbb{R}^3$  (*double–bubble*), for instance.

- ▶ Short time existence/uniqueness for special initial interfaces by Depner – Garcke – Kohsaka
- ▶ Short time existence/estimates for special initial interfaces by Schulze – White

Thanks for your attention