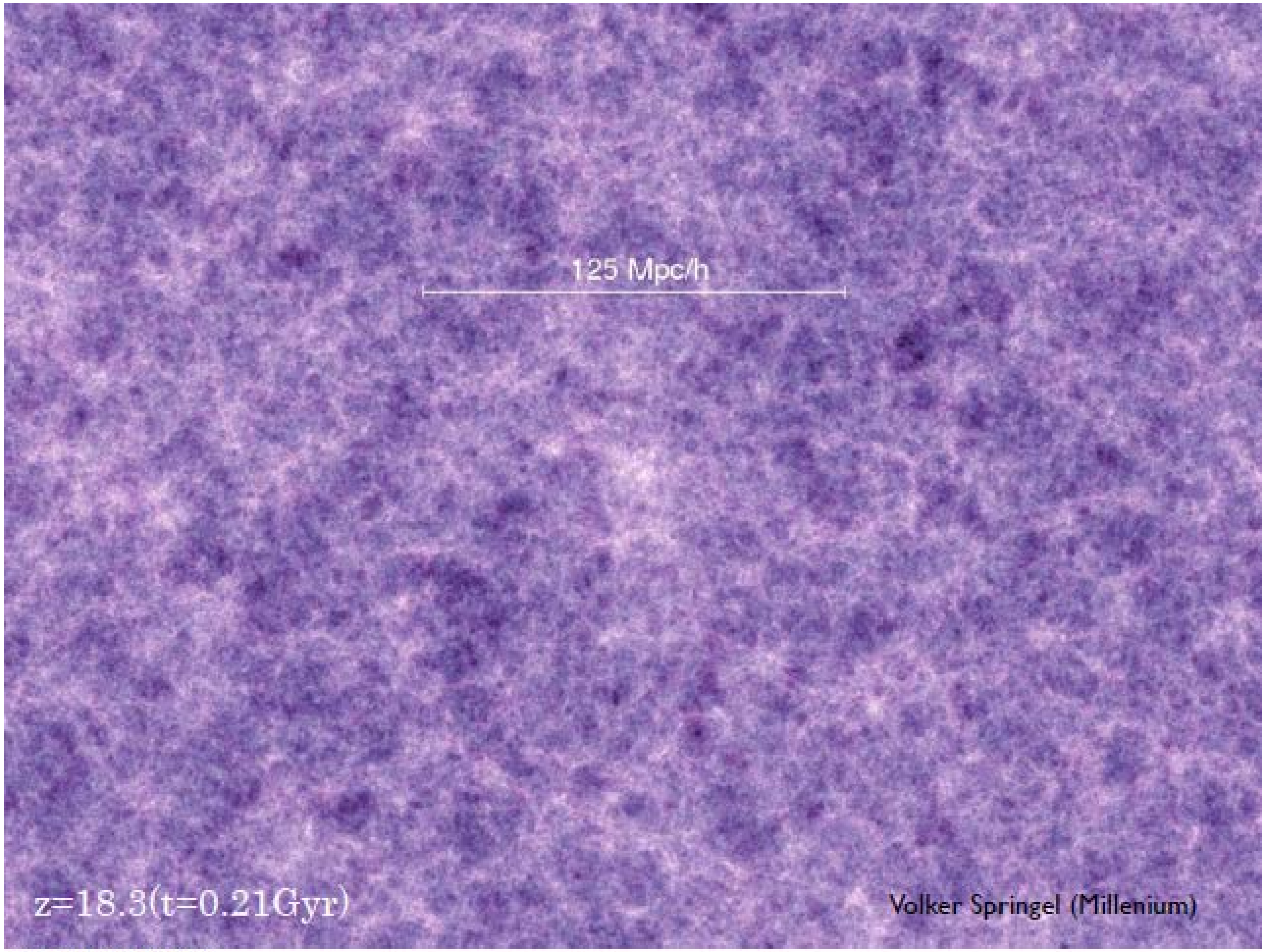
The excursion set approach

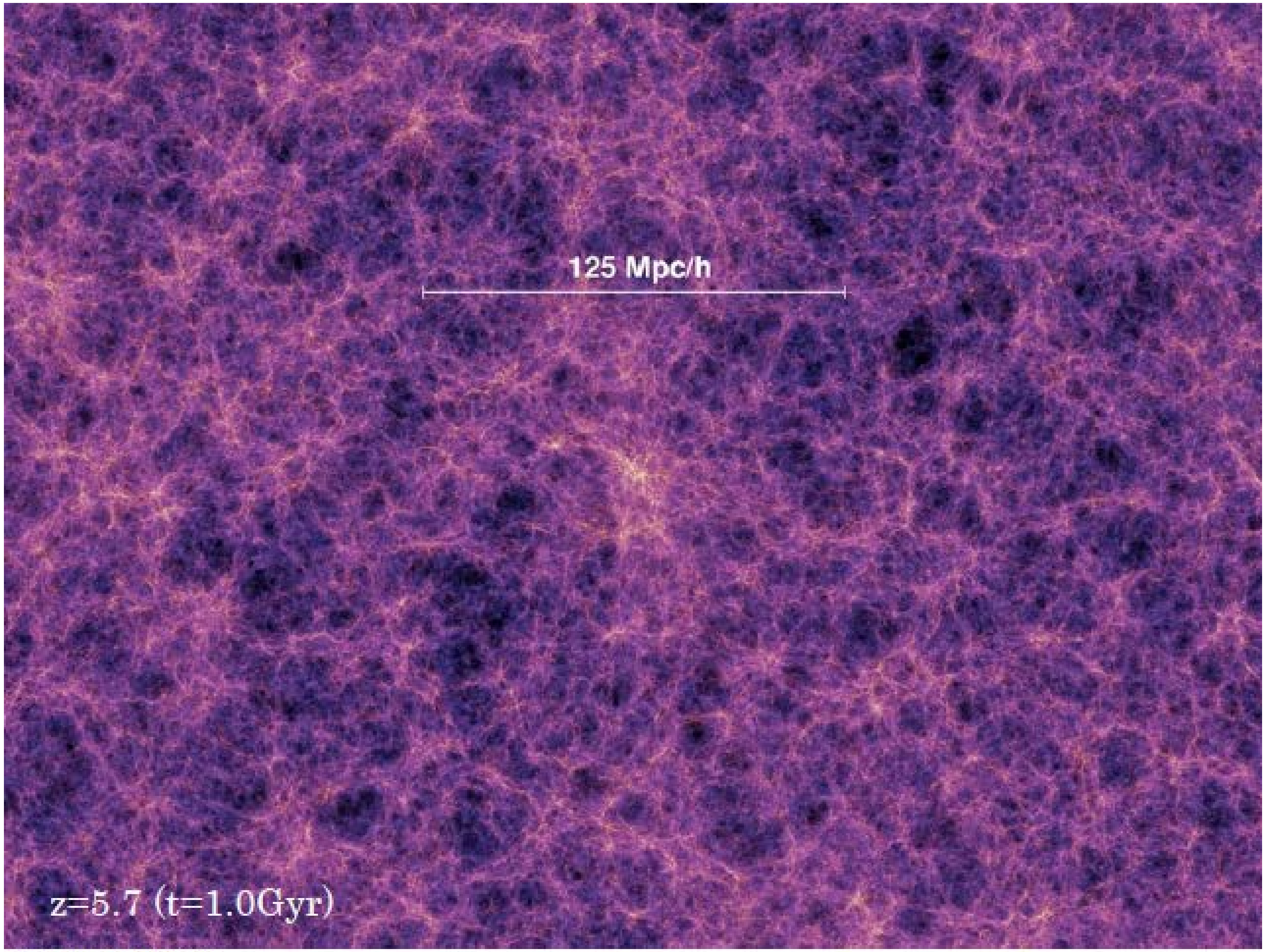
Halo abundances

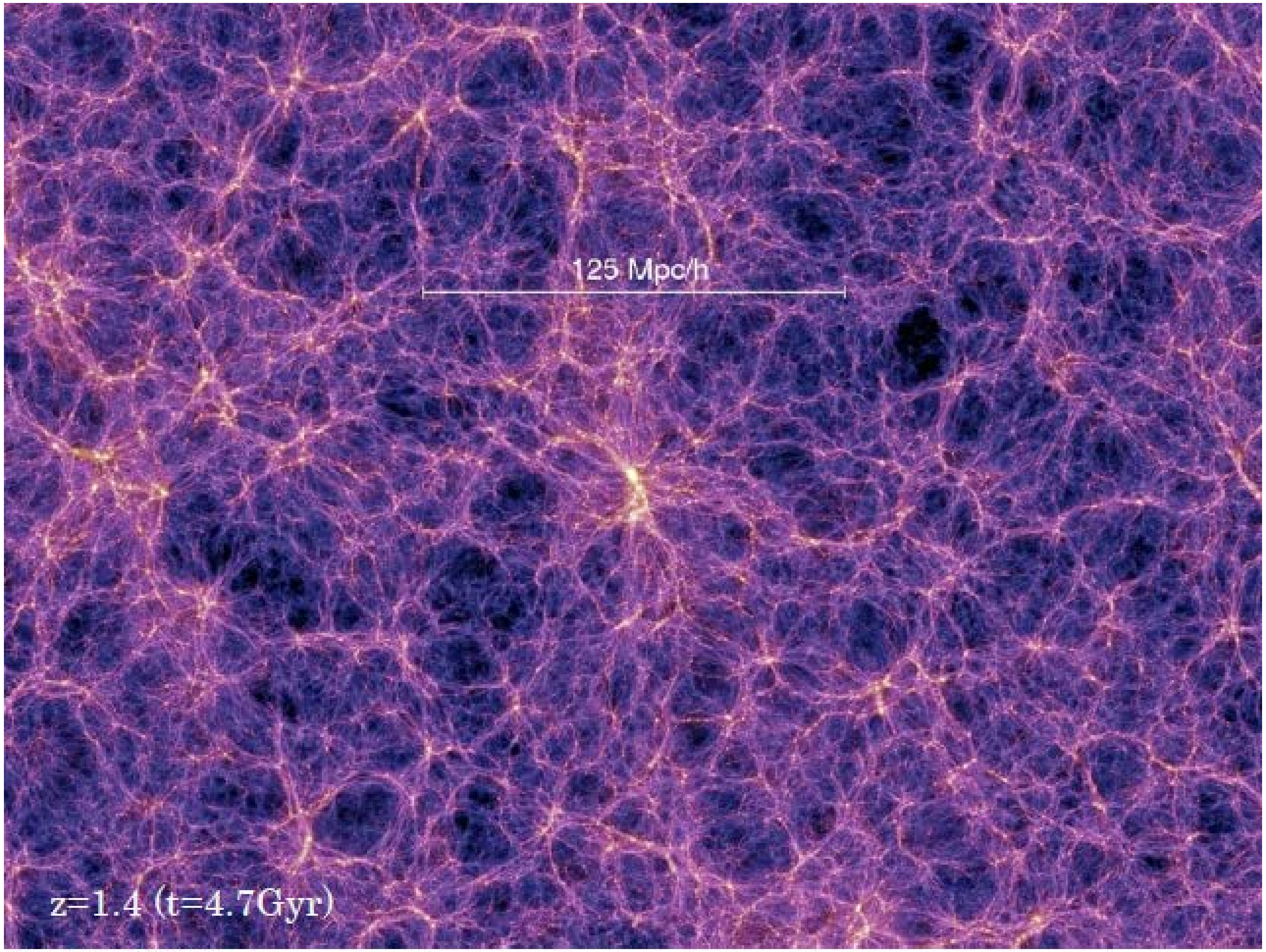
Halo clustering/bias

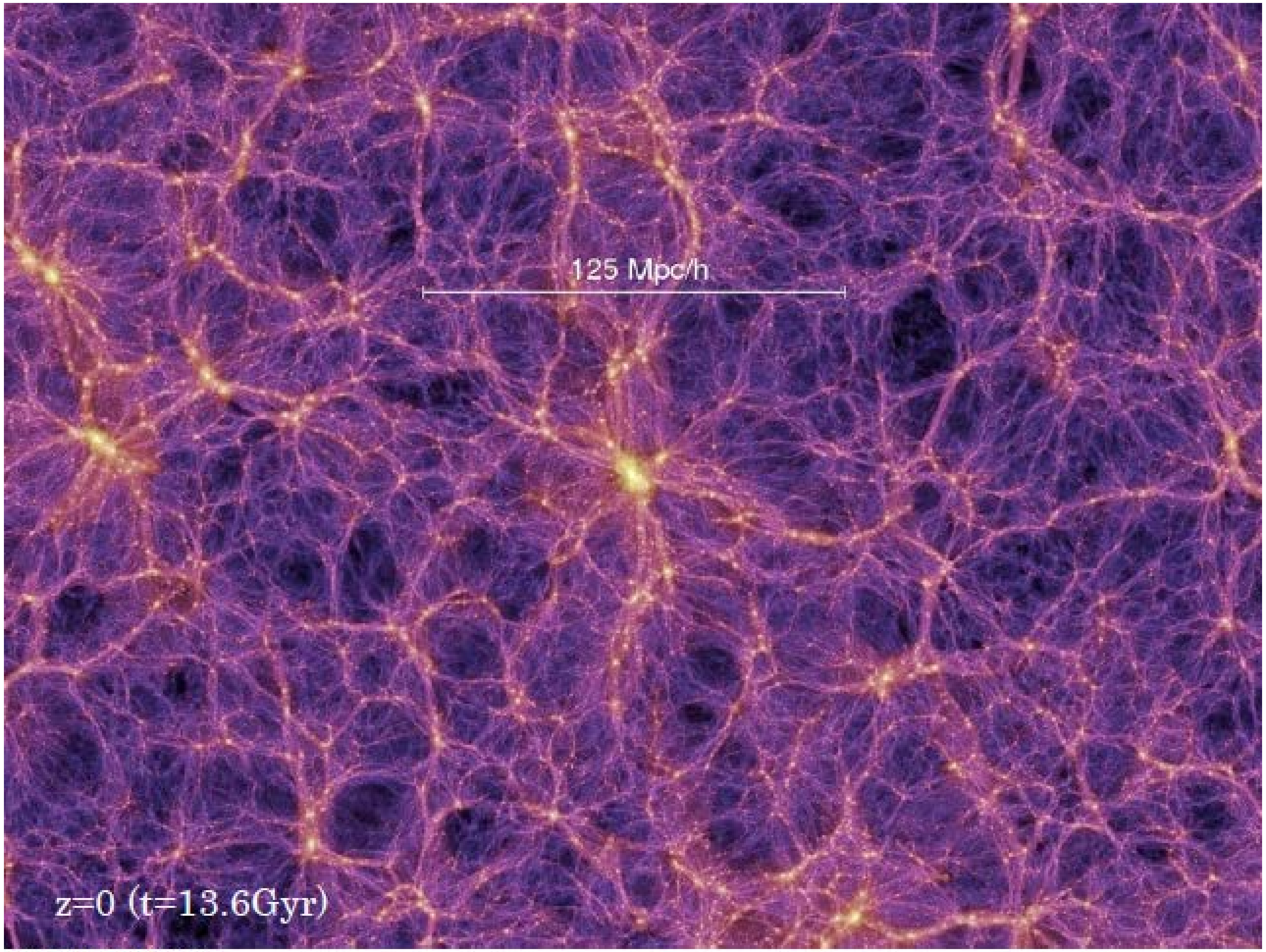
Scale-dependent bias

Connection to Lagrangian EFT

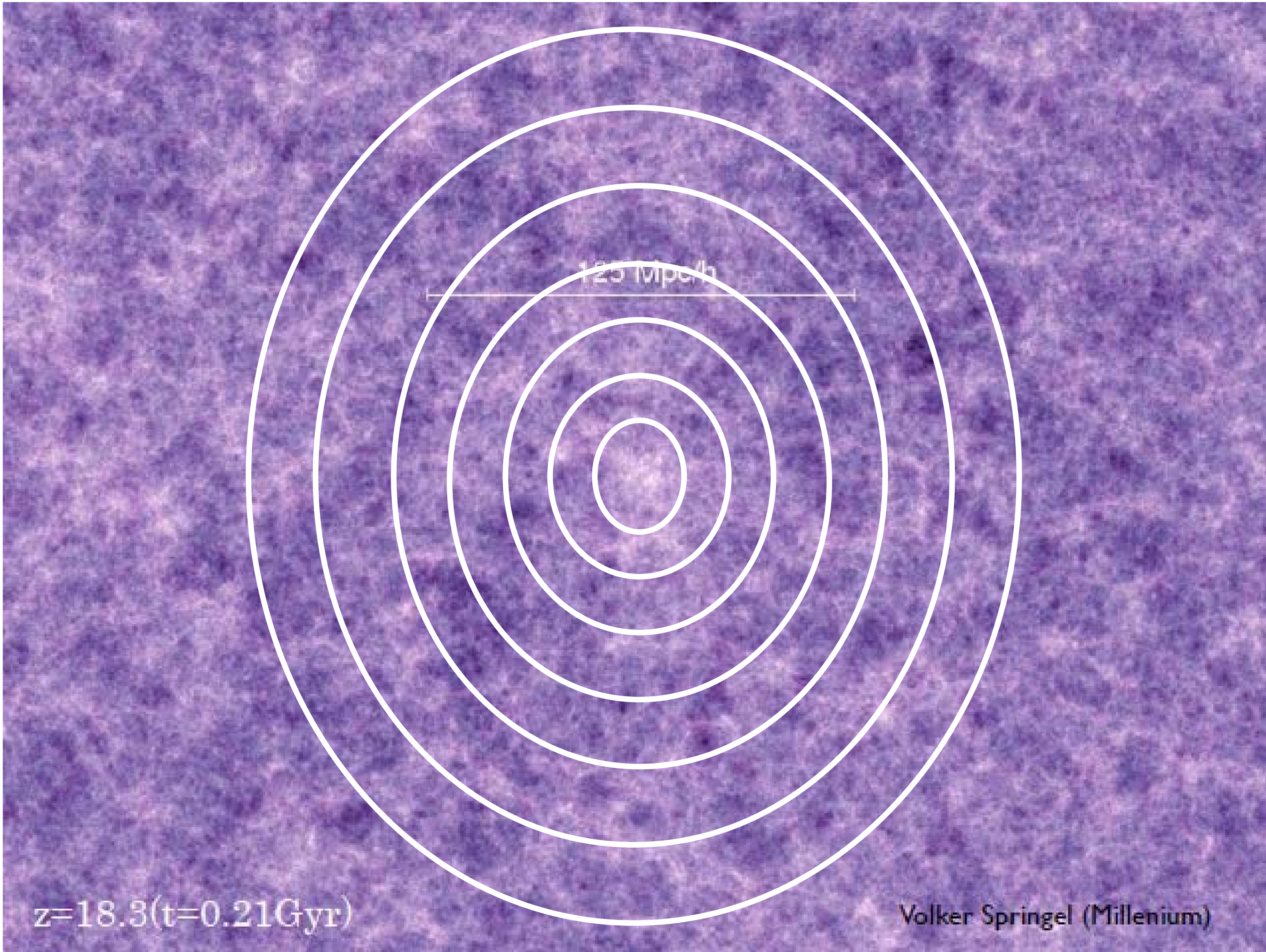

125 Mpc/h
z=18.3(t=0.21Gyr)
Volker Springel (Millenium)


125 Mpc/h
z=5.7 (t=1.0Gyr)

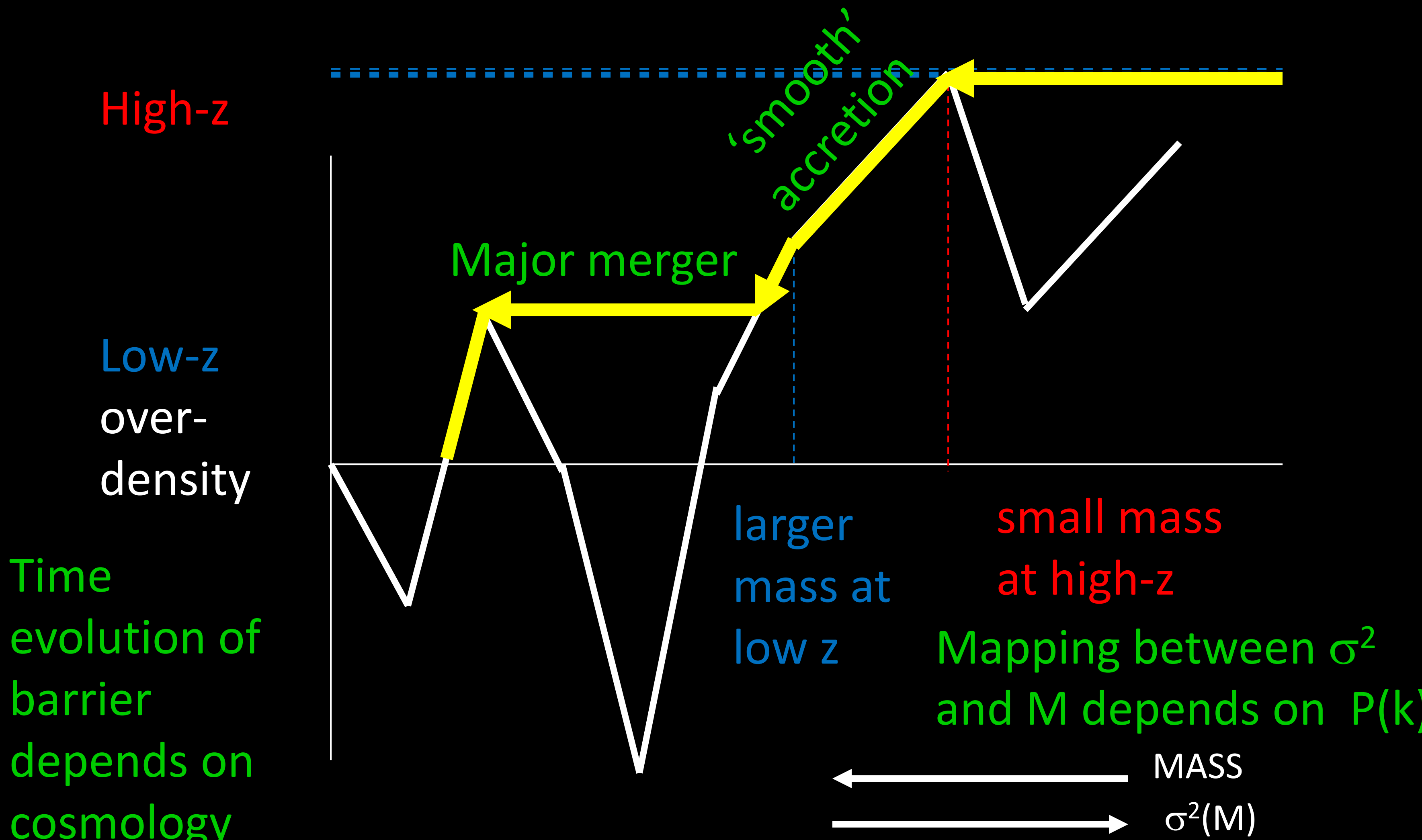

125 Mpc/h
z=1.4 (t=4.7Gyr)


125 Mpc/h
z=0 (t=13.6Gyr)

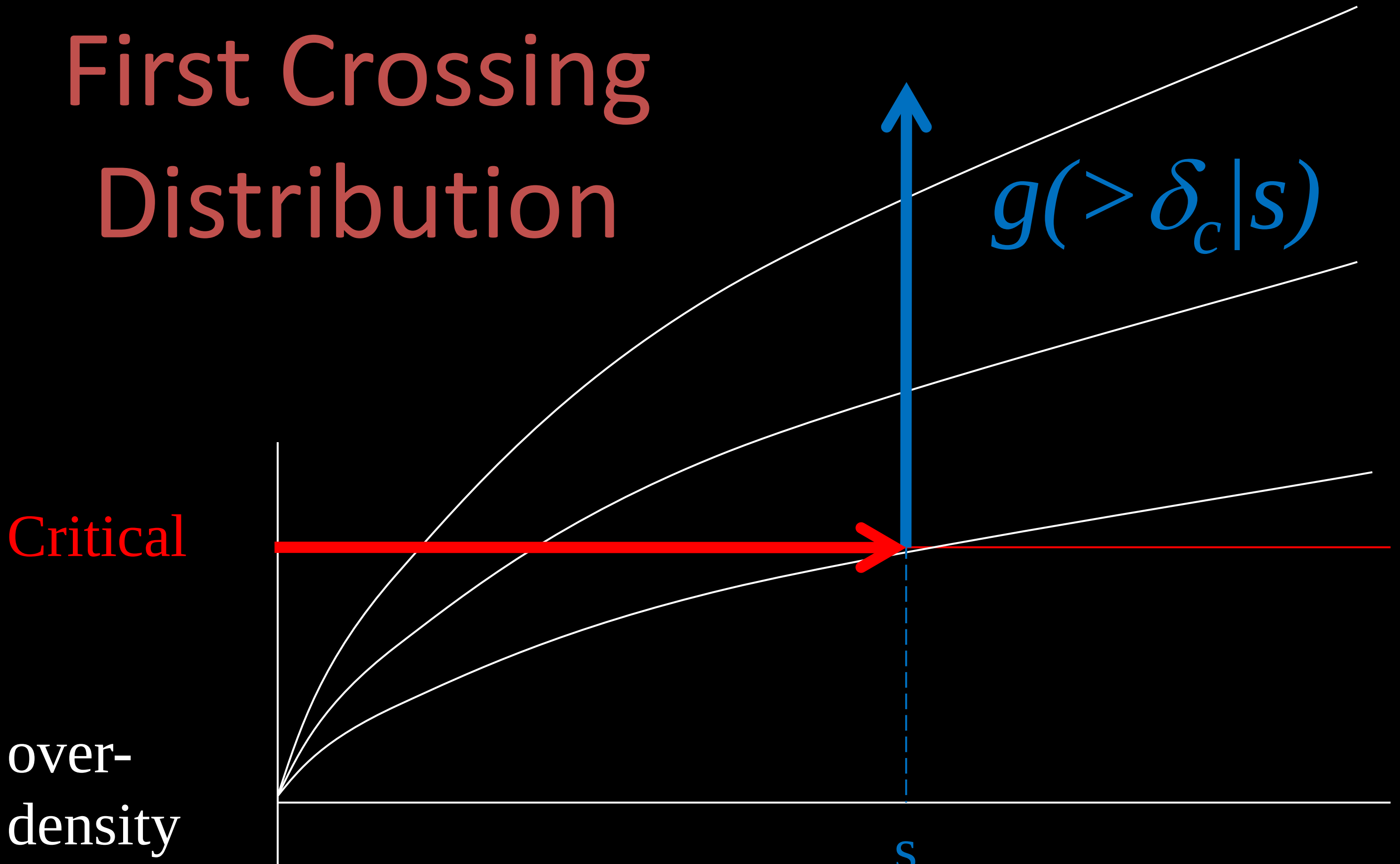

But wait …
We should be doing this in the INITIAL fluctuation field!


125 Mpc/h
z=18.3(t=0.21Gyr)
Volker Springel (Millenium)

The excursion set approach



First Crossing Distribution



$$\int_0^s dS\, f(S)\, p(>\delta_c, s|\text{ first } \delta_c \text{ at } S)$$

$$\approx \int_0^s dS\, f(S)\, p(>\delta_c, s|\,\delta_c, S)$$

First crossing distributions

- Smooth walks: $p(>\delta_c, s | \delta_c, S, \text{first}) = 1$
- Uncorrelated steps: $p(>\delta_c, s | \delta_c, S, \text{first}) = ½$
  - This is the Press-Schechter factor of 2
  - $s\, f(s) = \delta_c \exp(-\delta_c^2/2s) / \sqrt{2\pi s}$
  - Self-similar in units of $\nu = \delta_c/\sqrt{s}$
- Correlated steps somewhere in between
  - NB. Easy if $p(>\delta_c, s | \delta_c, S, \text{first})$ = separable function of s and S

For correlated steps

rather than thinking of a walk

as a list of heights

(i.e. the path integrals of Bond et al 1991),

it is more efficient to think of it

as a curve specified by

its height on one scale and

its derivatives

Correlated steps

Require walk below barrier on scale just larger than s, but above barrier on scale s (Bond et al. 1991):

$$f(s)ds \approx \int d\delta' \int d\delta \; p(\delta,\delta') \quad \text{where}$$

$$\delta_c < \delta < \delta_c + \Delta s\, \delta' \quad \text{and} \quad \delta' > 0$$

$$= \Delta s\; p(\delta_c, s) \int d\delta'\; p(\delta' | \delta_c)\; \delta'$$

Reduces problem from n >>1 dimensions, to just 2

Generalizes trivially to any barrier shape and also to non-Gaussian fields

Correlated steps (constant barrier)

$$\nu f(\nu) = \frac{\nu\, \mathrm{e}^{-\nu^2/2}}{\sqrt{2\pi}} \left[\frac{1 + \mathrm{erf}(\Gamma\nu/\sqrt{2})}{2} + \frac{\mathrm{e}^{-\Gamma^2\nu^2/2}}{\sqrt{2\pi}\Gamma\nu} \right]$$

N.B. Not quite universal because of Γ:

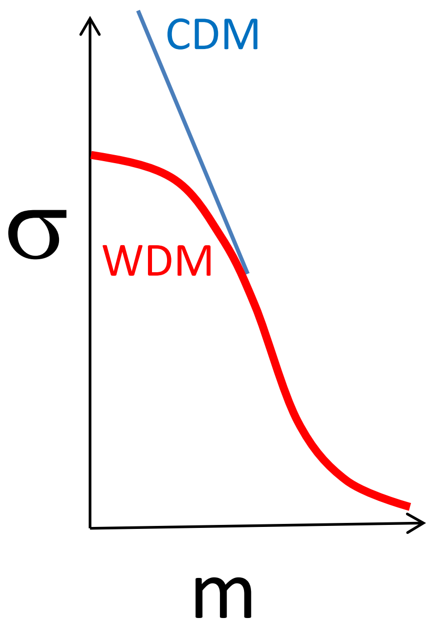
$$\gamma^2 \equiv \frac{\langle \delta\delta' \rangle^2}{\langle \delta^2 \rangle \langle \delta'^2 \rangle} \quad \text{and} \quad \Gamma^2 = \frac{\gamma^2}{1-\gamma^2}$$

From walks to halos

- Assume fraction of walks which cross on scale S = fraction of mass in halos of mass m, where S(m) from $S = \sigma^2(R)$ and $m = \rho\,(4\pi/3)R^3$

For WDM …

- At small enough m, $\sigma(m)$ is flat
- Fraction of walks which didn't cross barrier prior to this σ = non-negligible smooth component which was never bound to anything
- f_{smooth} should be larger at high z
- Fewer halos (progenitors) at high z mean less concentrated halos at low z
- f_{smooth} should be larger in voids = voids are 'emptier' (even more so if $\delta_c(m)$ larger at small m)

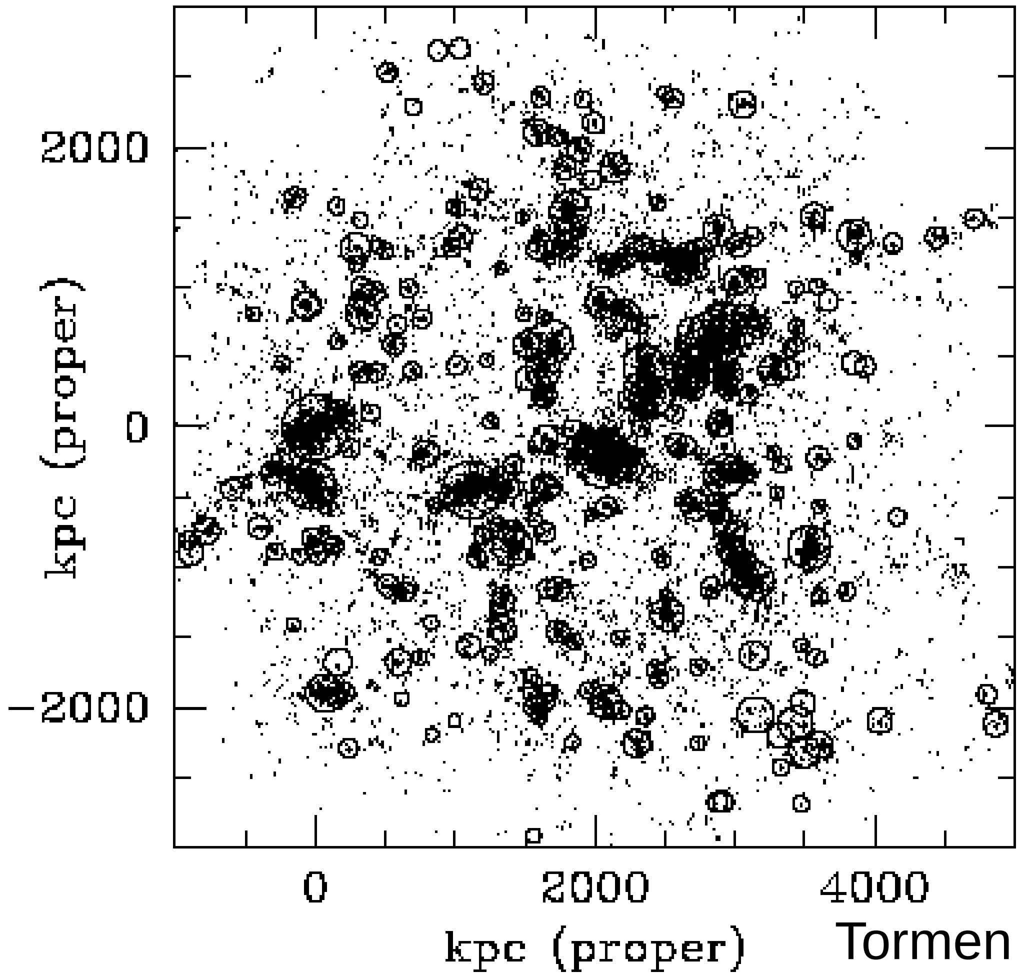


Simplification because…

- Everything local
- Evolution determined by cosmology (competition between gravity and expansion)
- Statistics determined by initial fluctuation field: for Gaussian, specified by initial power-spectrum $P(k)$
- Nearly universal in scaled units: $\delta_c(z)/\sigma(m)$ where $\sigma^2(m) = \langle\delta_m^2\rangle = \int dk/k \;\; k^3 P(k)/2\pi^2 \; W^2(kR_m)$ $\quad m \propto R_m^3$
- Fact that only very fat cows are spherical is a detail (*crucial* for precision cosmology); in excursion set approach, mass-dependent barrier height increases with distance along walk

Spherical evolution model

- 'Collapse' depends on initial over-density Δ_i; same for all initial sizes
- Critical density depends on cosmology
- Final objects all have same density, whatever their initial sizes
- Collapsed objects called halos are ~ 200× denser than critical (background?!), whatever their mass



Tormen 1997

(Figure shows particles at z~2 which, at z~0, are in a cluster)

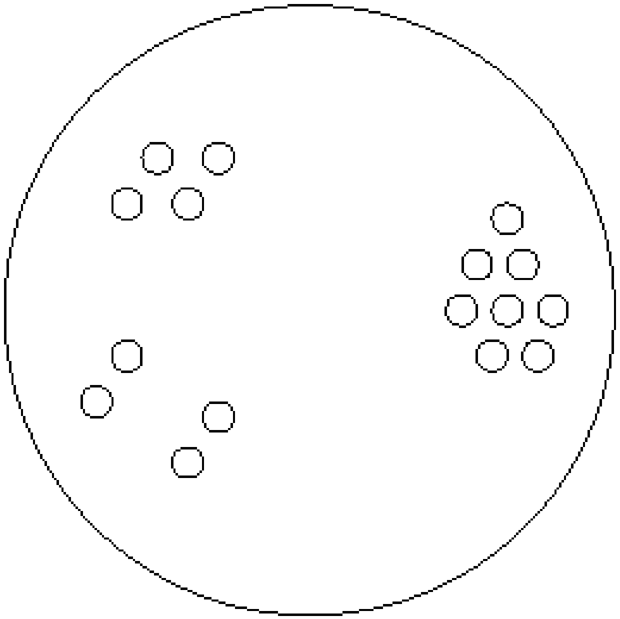


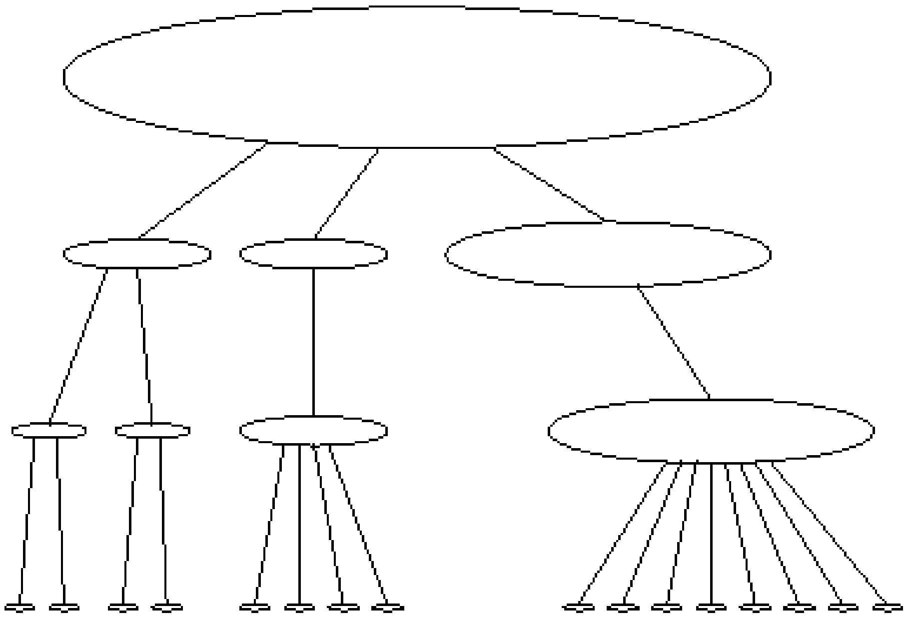




Assume a spherical herd of spherical cows…

Initial spatial distribution within patch (at z~1000)...

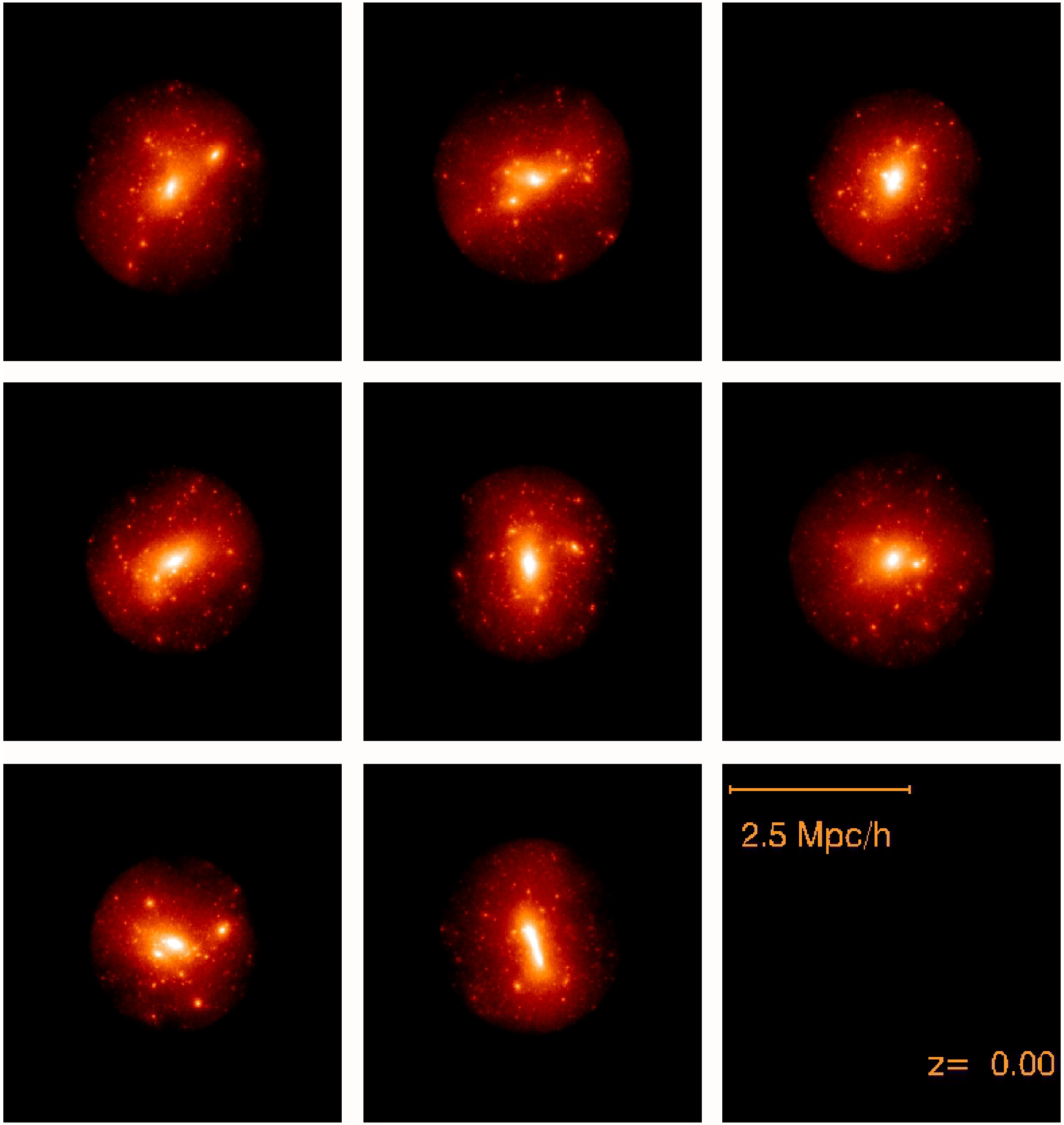




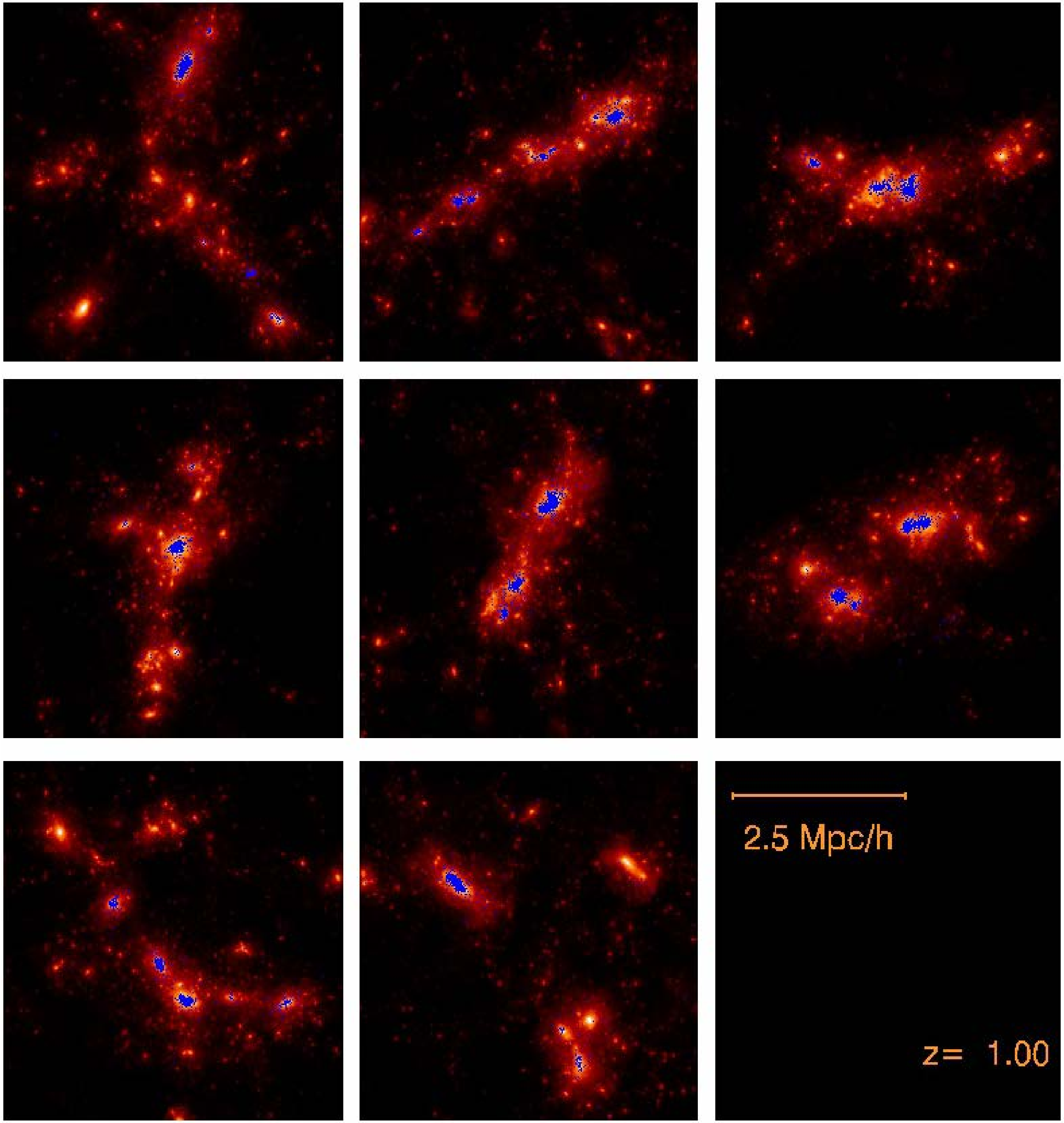
…stochastic (initial conditions Gaussian random field); study 'forest' of merger history 'trees'.

…encodes information about subsequent 'merger history' of object

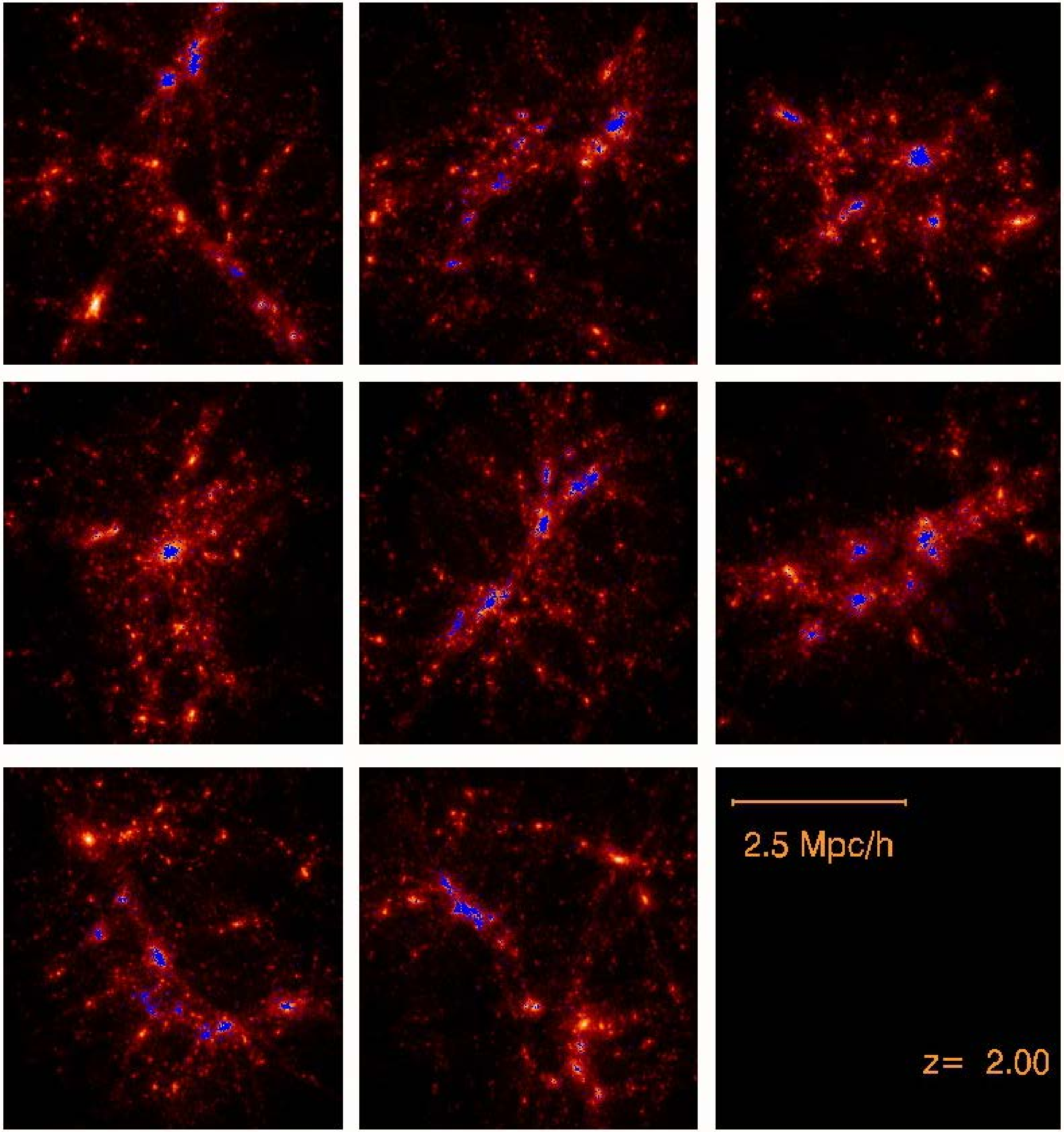
(Mo & White 1996; Sheth 1996)



- 8 halos, $10^{15}M_{sun}$ at z=0 in ΛCDM
- Only dark matter particles within R_{200} shown



- Same objects at z=1
- Blue shows dark matter within 20kpc at z=0



- Same objects at z=2
- Blue shows dark matter within 20kpc at z=0

Spherical evolution mapping …

$(R_{initial}/R)^3 = \mathrm{Mass}/(\rho_{com}\mathrm{Volume}) =$

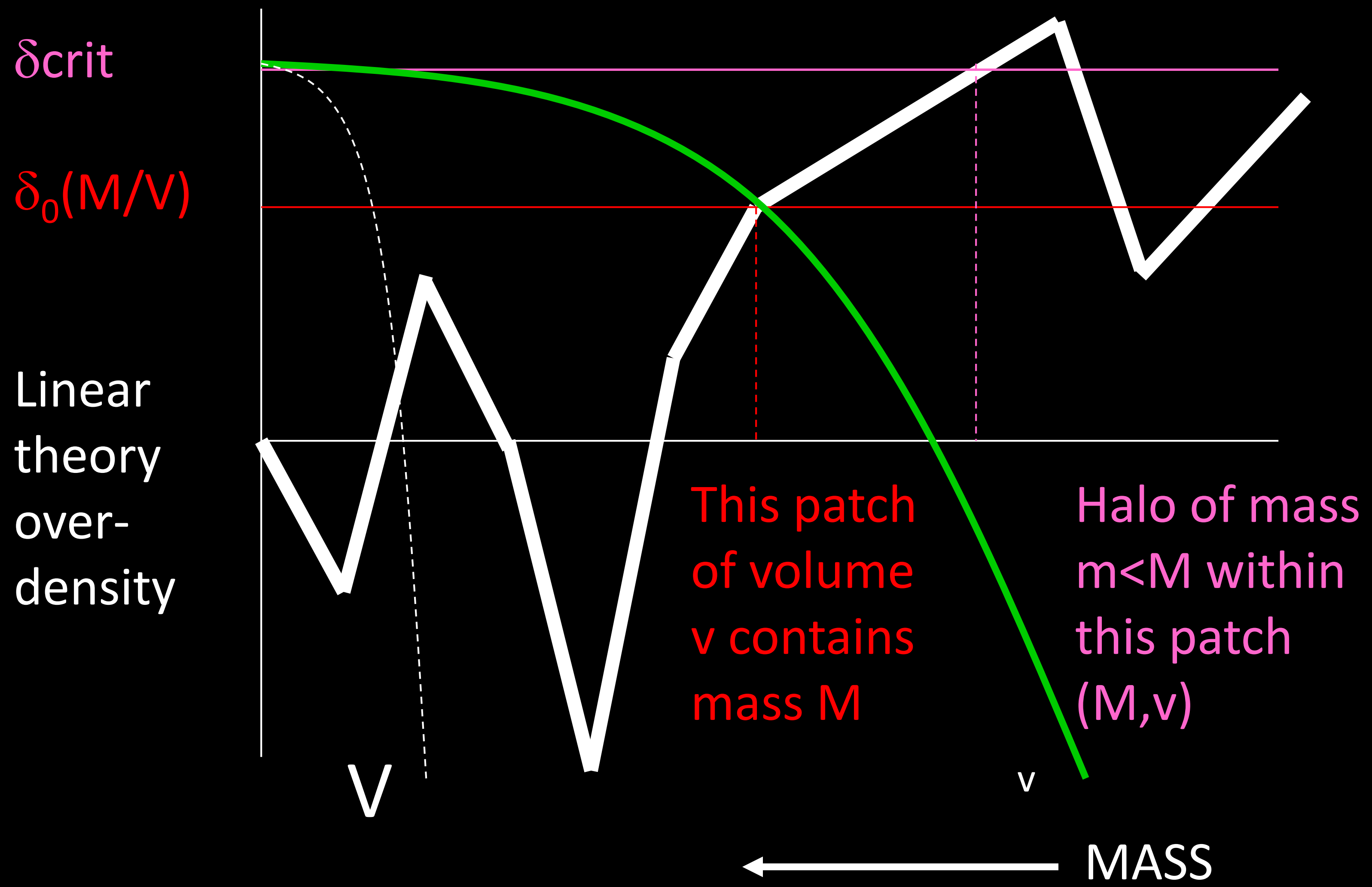
$$1 + \delta \approx (1 - \delta_0/\delta_{sc})^{-\delta sc}$$

… can be inverted:

$$(\delta_0/\delta_{sc}) \approx 1 - (M/\rho_{com}V)^{-1/\delta sc}$$

N.B. For any V, there is a curve $\delta_0(M|V)$.

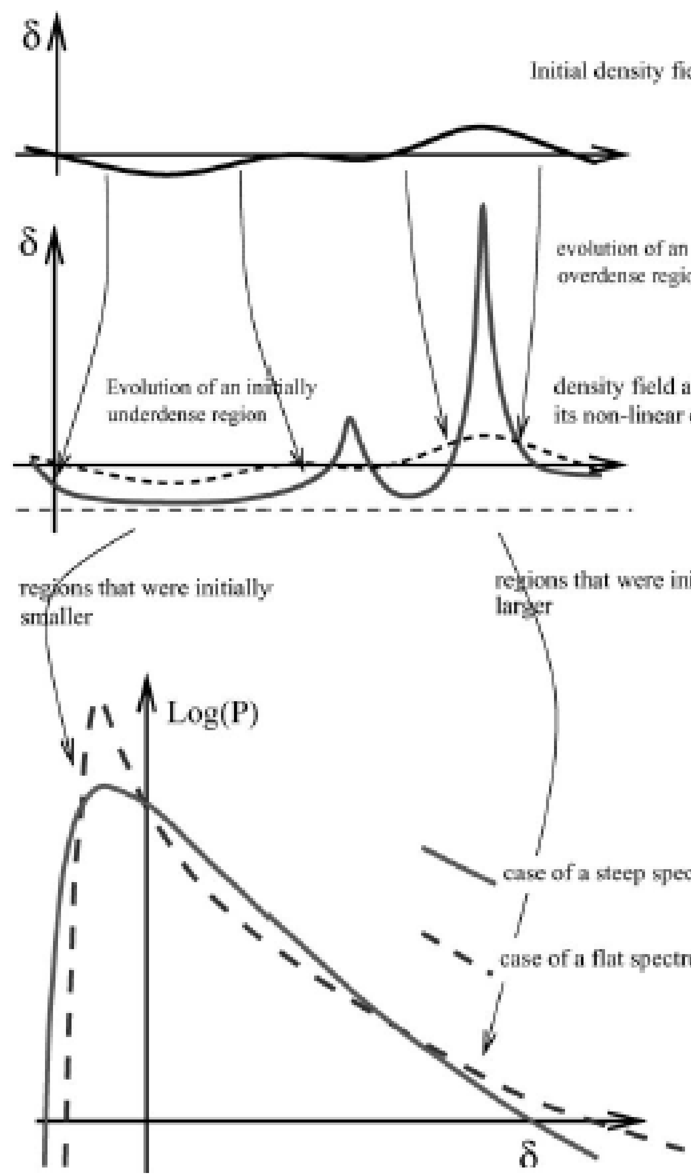
Moving barriers: The Nonlinear PDF



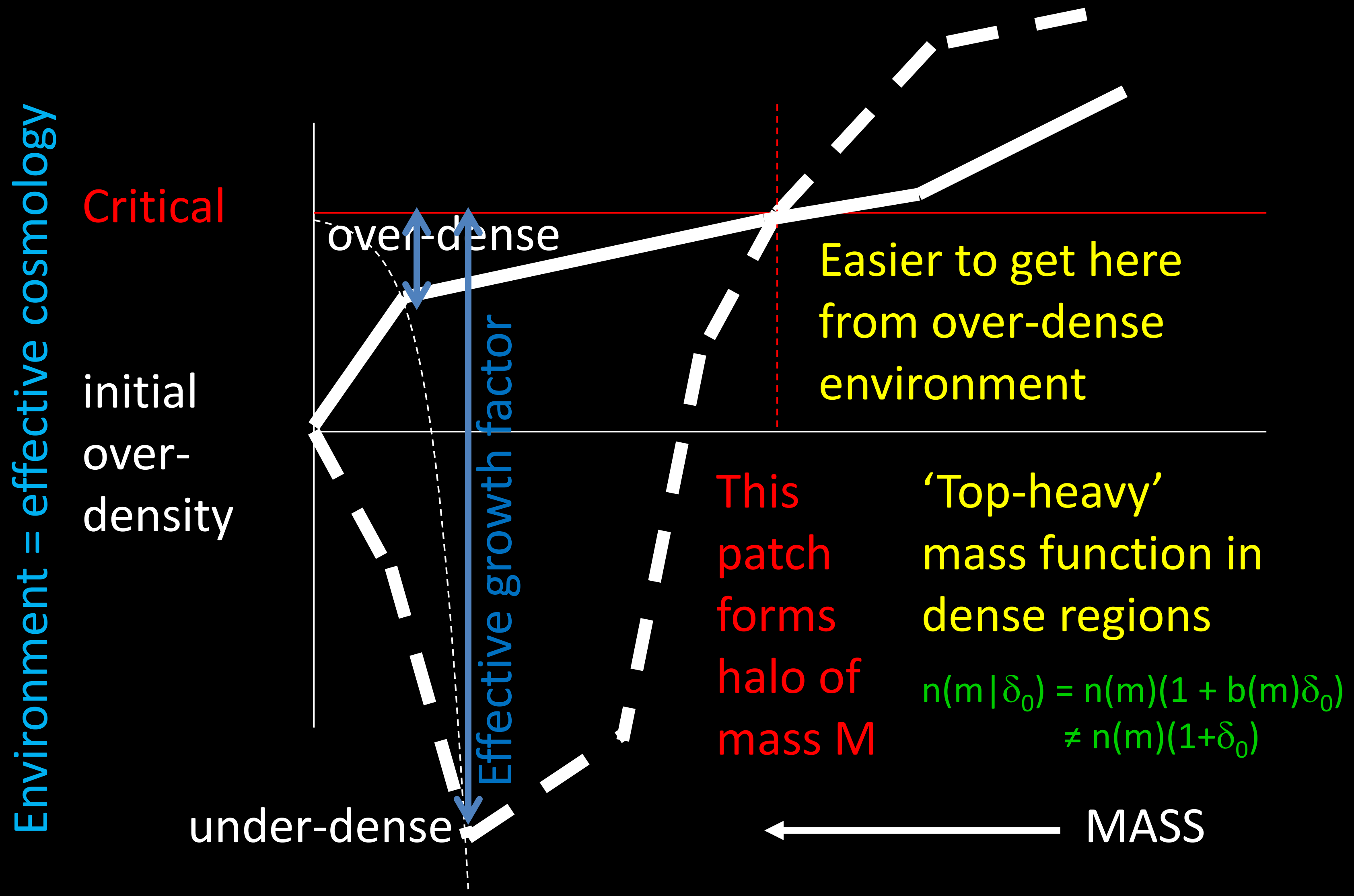
Initially Gaussian fluctuation field becomes very non-Gaussian on small scales

Large scale PDF ~Gaussian even at late times

F. Bernardeau et al. / Physics Reports 367 (2002) 1–248



Correlations with environment



Effective cosmology

- 'Biased' walks will be just like original walk, but with 'shifted' barrier:

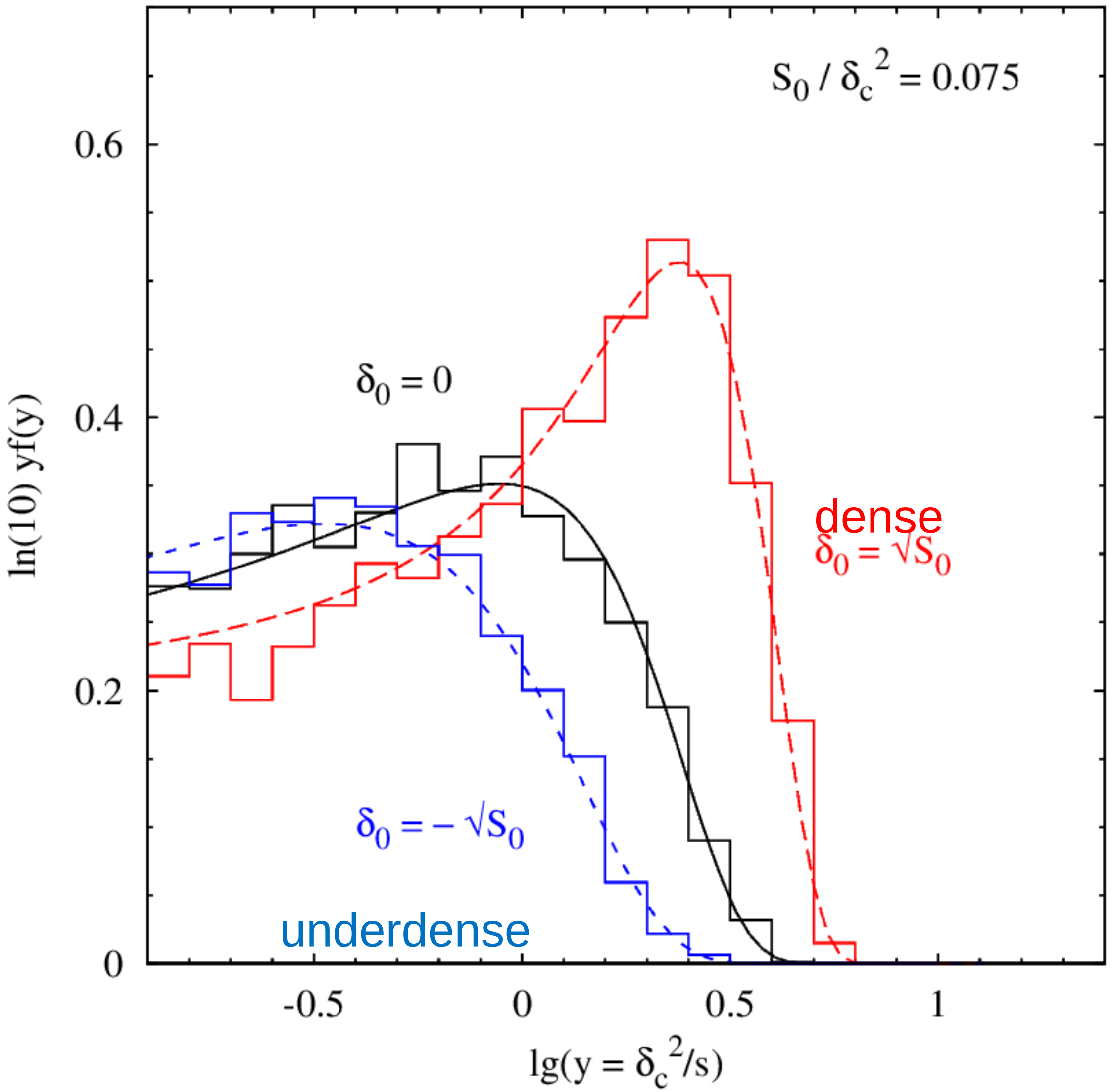
$\delta_c - \delta_0(1+\delta_{NL}) \sim \delta_c\,[1 - \delta_0(1+\delta_{NL})/\delta_c]$

But $1 + \delta \approx (1 - \delta_0/\delta_{sc})^{-\delta sc}$

- $= \delta_c\,(1+\delta_{NL})^{-1/\,\delta c} = \delta_c\, D_{Eff}$
- Basis for 'separate universe' simulation tests of bias.

Conditional first crossing distributions

More massive halos in dense regions = origin of halo bias



Musso, Paranjape, Sheth 2013

Environmental effects

- In hierarchical models, close connection between evolution and environment (dense region ~ dense universe ~ more evolved)
- Gastrophysics determined by formation history of halo
- Observed correlations with environment test hierarchical galaxy formation models – all environmental effects because massive halos populate densest regions

Close connection between abundance and spatial distribution (bias):

- Let δ_R denote δ (smoothed) on scale R
- A halo of mass M forms from a patch where $\delta_R > \delta_c$, $\delta_{R+dR} < \delta_c$, …
- Abundance of halos of mass M from $p(\delta_R > \delta_c, \delta_{R+dR} < \delta_c, \ldots)$
- Bias related to $p(\delta > \delta_c, \delta_{R+dR} < \delta_c, \ldots | \Delta \text{ on } R_\Delta)$
  - Namely, write this as Taylor series in Δ; linear term in expansion is linear bias factor.

Large scale clustering/bias (from the peak-background split)

$$1 + \delta_h(\nu|\delta_0,S_0) = f(\nu|\delta_0,S_0)\,/f(\nu)$$
$$= 1 + b_1(\nu)\delta_0 + \ldots$$

- $b(\nu)$ directly from (derivatives of) $f(\nu)$ means halo abundances predict halo clustering
- $b(\nu)$ increases with ν

  → top-heavy mass function in dense regions:

  $n(m|\delta_0) = n(m)(1 + b(m)\delta_0 + \ldots) \neq n(m)(1+\delta_0)$

  → massive halos (i.e. larger ν) more clustered:

$$<\delta_h\delta_0> = b_1(\nu)\, <\delta_0^2> + \ldots$$

$b_{Lagrangian}$ (ICs) → to $b_{Eulerian}$ (later)

$$1 + \delta_h(m|V_{Eul}) = 1 + b_{Eul}\,\delta_{mEul}$$

$$= N(m|V_{Eul}) / n(m)V_{Eul}$$

$$= (V_{Lag}/V_{Eul})\, N(m|V_{Lag})/n(m)V_{Lag}$$

$$= (1 + \delta_{mEul})\,(1 + b_{Lag}\,\delta_{mLag})$$

$$(\text{for } \delta \ll 1,\ \delta_{NL} = \delta_{Lin})$$

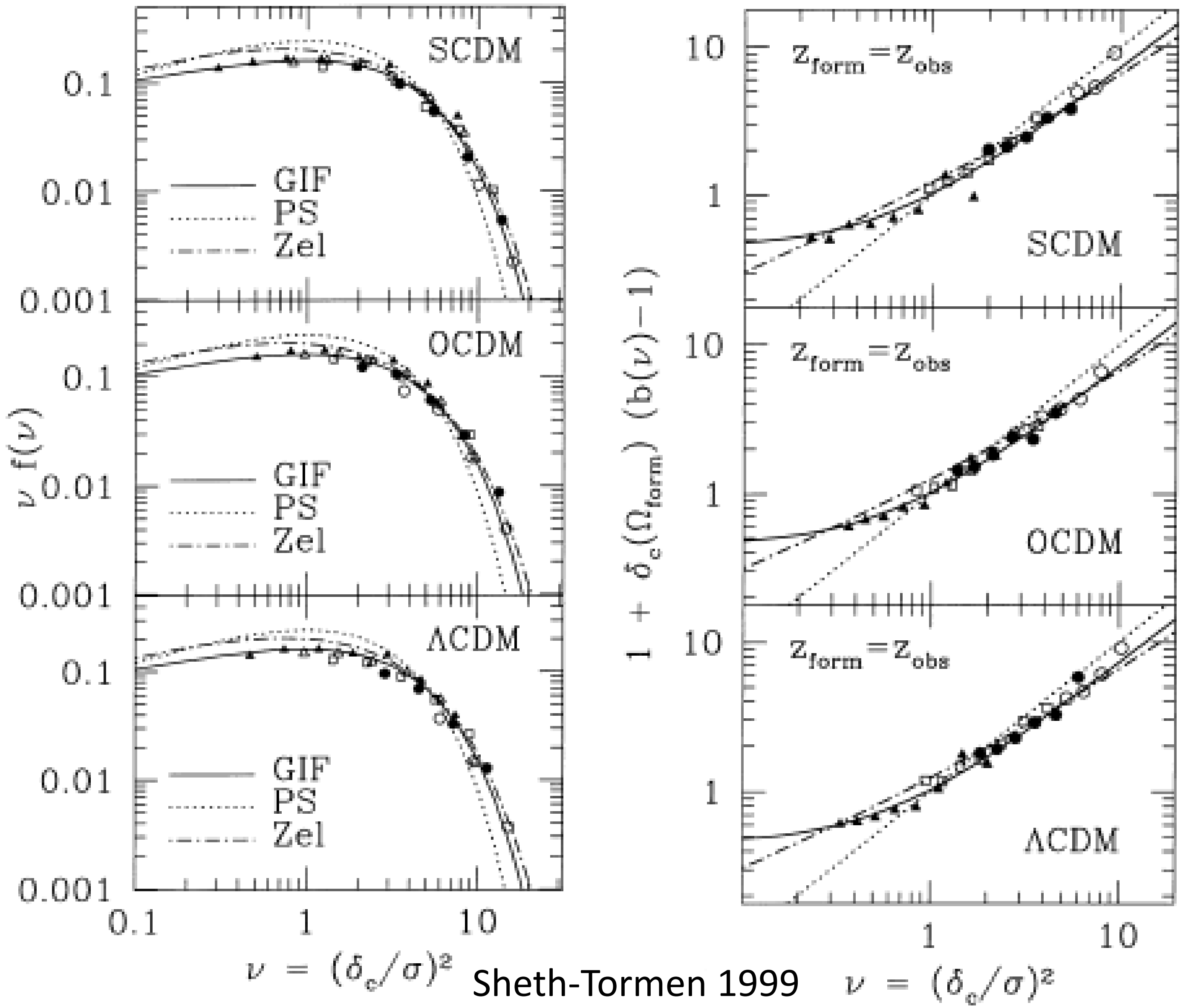
$$= 1 + \delta_m + b_{Lag}\,\delta_m + \ldots$$

$$= 1 + [1 + b_{Lag}]\,\delta_m$$

(Almost) universal mass function and halo bias

See Paranjape et al (2013) for recent progress in modeling this from first principles

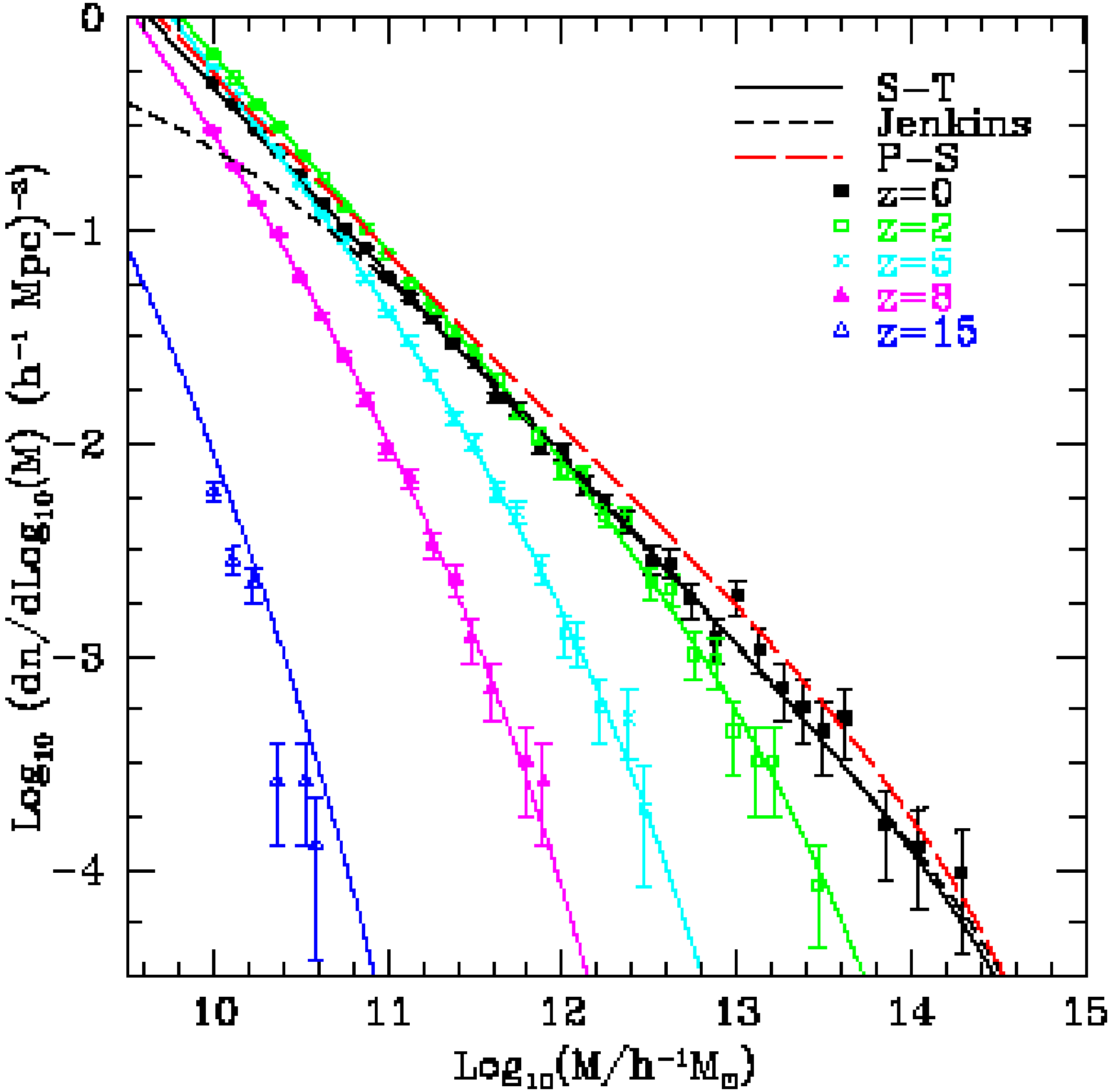
See Castorina et al. (2014) for ν's



Sheth-Tormen 1999

The Halo Mass Function

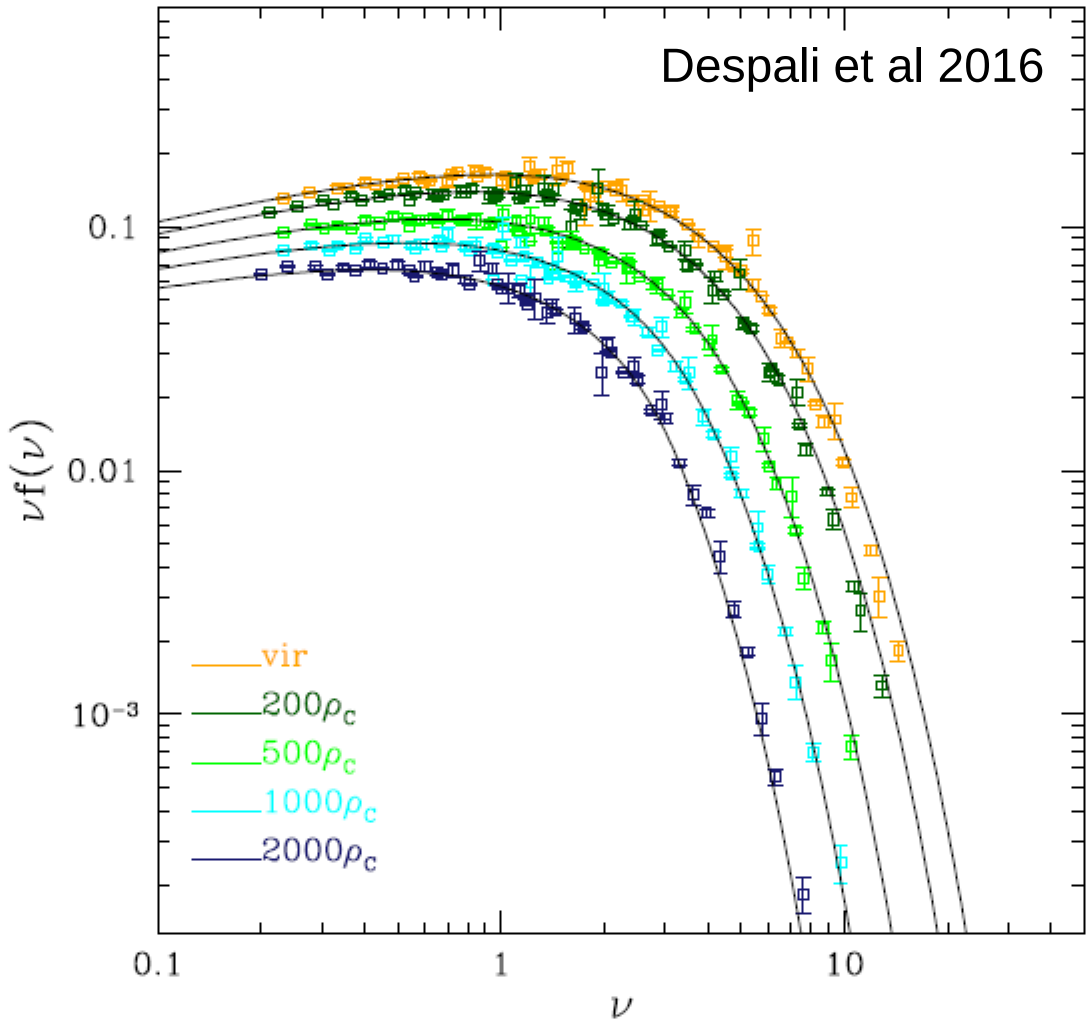
(Reed et al. 2003)

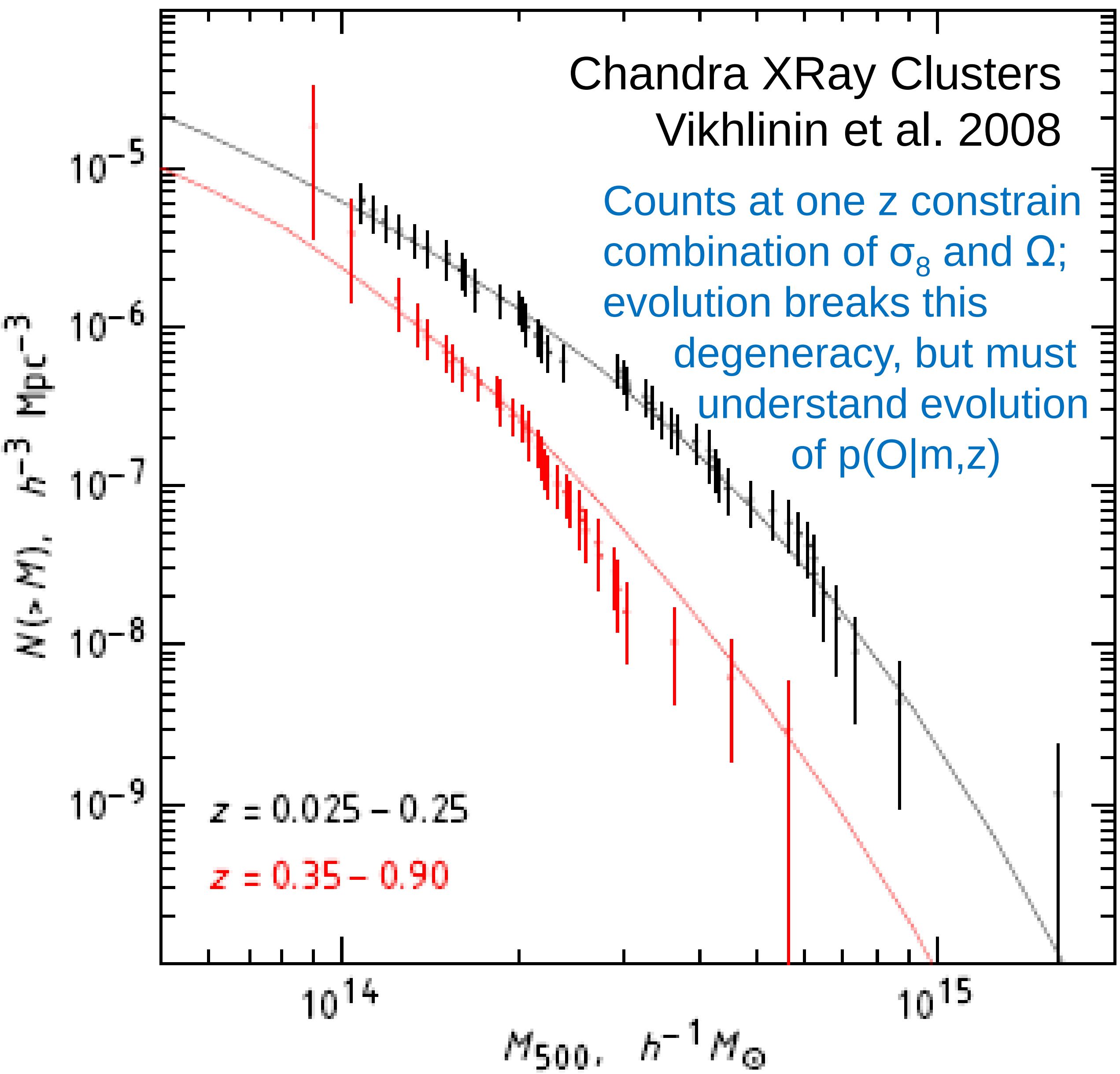


- Small halos collapse/virialize first
- Can also model halo spatial distribution
- Massive halos more strongly clustered

Aside:

Universal mass function + universal profile shape = easy to translate between different halo definitions





Chandra XRay Clusters
Vikhlinin et al. 2008
Counts at one z constrain combination of σ_8 and Ω; evolution breaks this degeneracy, but must understand evolution of p(O|m,z)
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
$N(>M)$, h^{-3} Mpc^{-3}
$z = 0.025 - 0.25$
$z = 0.35 - 0.90$
10^{14}
10^{15}
M_{500}, $h^{-1} M_\odot$

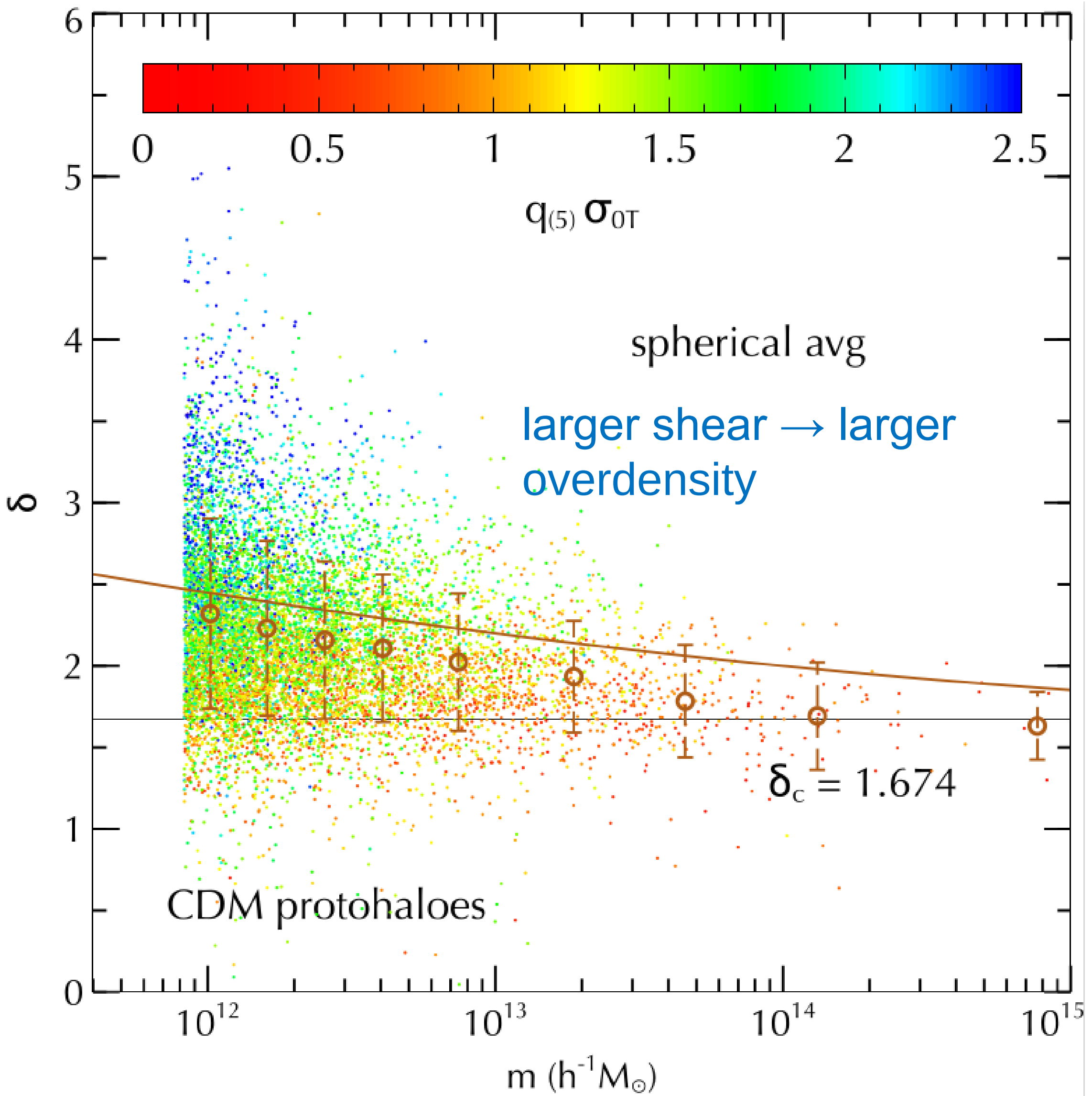
Halo formation more complicated than simple spherical

- Halos may be closely related to peaks in the initial field

  (BBKS 1986; Paranjape et al. 2013)

- Shear must also matter

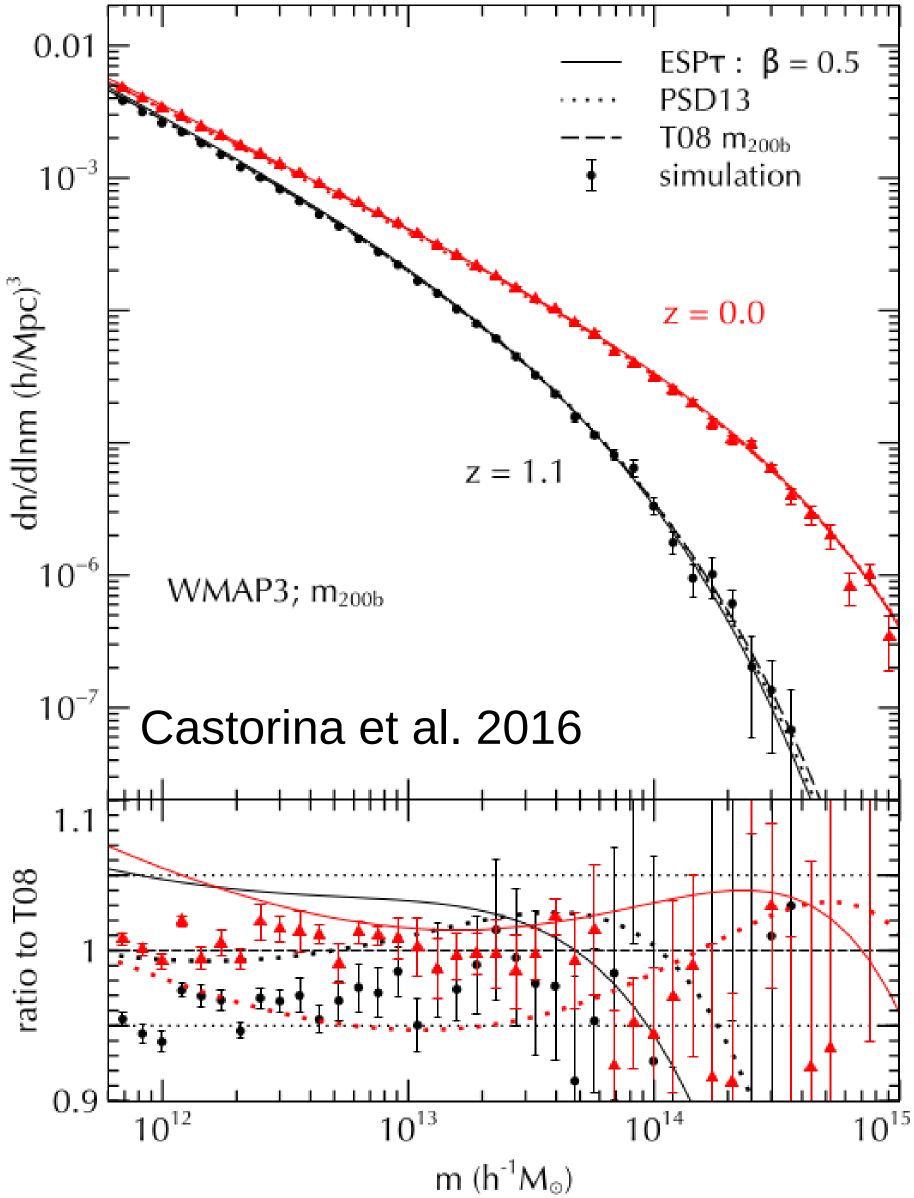
  (Bond, Myers 1996; Sheth, Mo, Tormen 2001)



Sheth, Chan, Scoccimarro 2013; Castorina et al. 2016

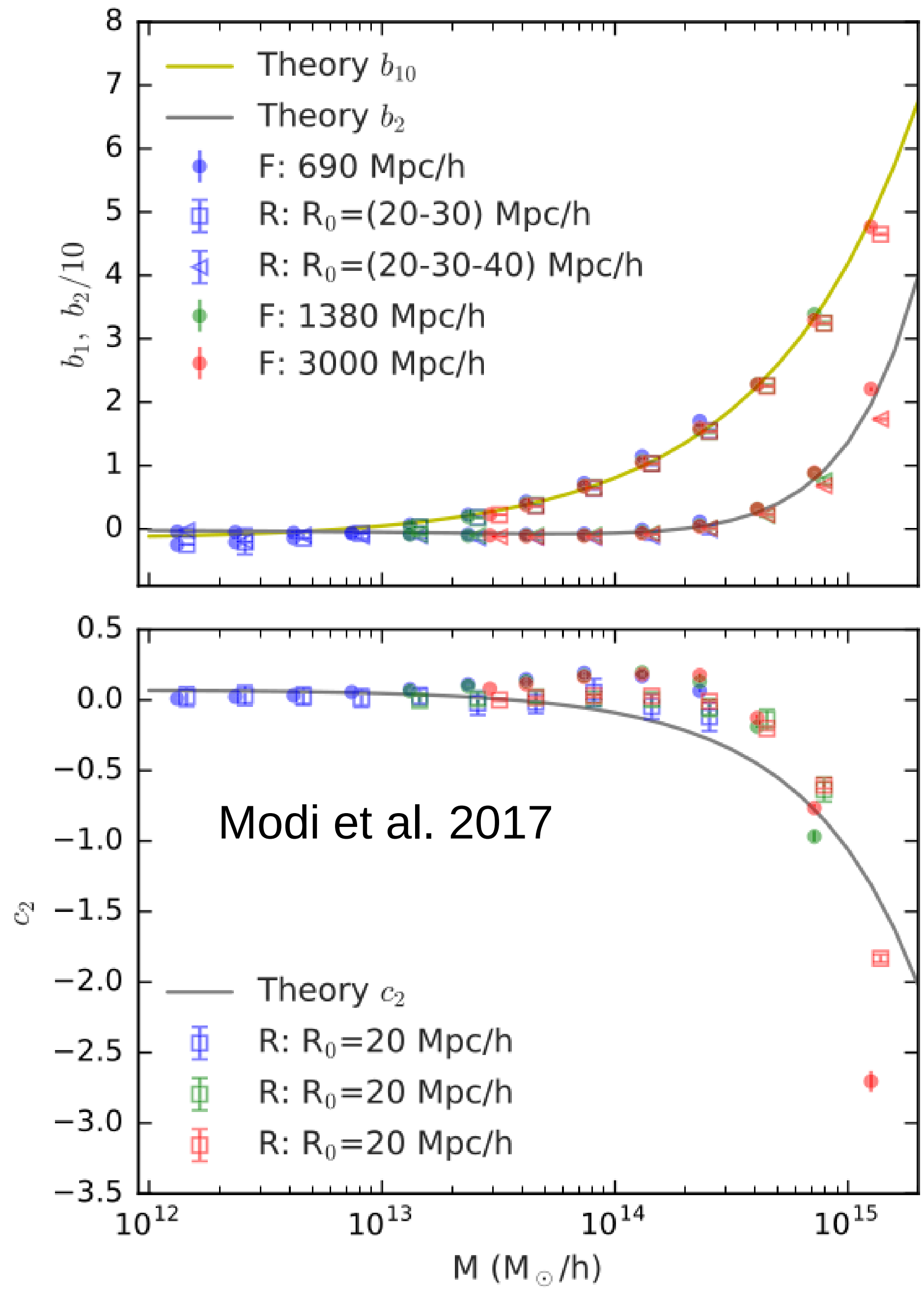
Excursion set peaks + shear works quite well

(Paranjape et al. 2013; Castorina et al. 2016)



Density bias quite well understood

Room for improvement in tidal bias models



Tracer $n(m) = \int d\delta \ldots g(\delta,\delta',\delta'',\text{shear},\ldots)$

Bias from $n(m|\Delta,\Sigma)/n(m)$

$= \int d\delta \ldots g(\delta,\delta',\ldots|\Delta,\Sigma)/n(m)$

But $\langle\Delta|\text{halo}\rangle$

$= \int d\delta \ldots g(\delta,\delta',\ldots) \langle\Delta|\delta,\delta',\ldots\rangle/n(m)$

so close connection between bias and profile around bias tracers

δ': velocity bias (Desjacques, Sheth 2009)

δ'': scale dependent bias (Musso, Paranjape, Sheth 2013)

N.B. Environment = effective cosmology built-in

(e.g.Martino-Sheth 2009 for density; Desjacques 2013 for shear)

Account for additional nonlocality from contribution of tidal term to nonlinear evolution $\delta(\delta_0, q_0)$ to get Eulerian bias.

$$1 + \delta_{\mathrm{h}}^{\mathrm{E}}(\delta, \mathrm{q}^2) = (1+\delta)(1+\delta_{\mathrm{h}}^{\mathrm{L}})$$

$$\begin{aligned}
&= (1+\delta)\left(1 + b_1^{\mathrm{L}}\,\delta_0 + b_2^{\mathrm{L}}\,\frac{\delta_0^2}{2} + c_2^{\mathrm{L}}\,\frac{q_0^2}{2} + \ldots\right) \\
&= 1 + b_1^{\mathrm{L}}\,\delta_0 + b_2^{\mathrm{L}}\,\frac{\delta_0^2}{2} + c_2^{\mathrm{L}}\,\frac{q_0^2}{2} + \delta + b_1^{\mathrm{L}}\,\delta_0\delta \\
&= 1 + \delta\,(b_1^{\mathrm{L}} + 1) + \frac{\delta^2}{2}(8b_1^{\mathrm{L}}/21 + b_2^{\mathrm{L}}) \\
&\quad + \frac{q_0^2}{2}(c_2^{\mathrm{L}} - 8b_1^{\mathrm{L}}/21).
\end{aligned}$$

Scale dependence of bias depends on the properties of a proto-halo patch which determine halo formation

E.g., if protohalo is (i) a sufficiently overdense initial patch which is (ii) a local maximum, and which is (iii) less dense when smoothed on a larger scale, then

$$\mathrm{bias}(k) = [b_{100} + b_{010}\, k^2 R_h^{\,2} + b_{001}\, d\ln W(kR_h)/d\ln R_h]\, W(kR_h)$$

Coefficients depend on halo mass (R_h), density (i), steepness (iii), isolation (ii); Common to ‘marginalize’ over (ii) and (iii)

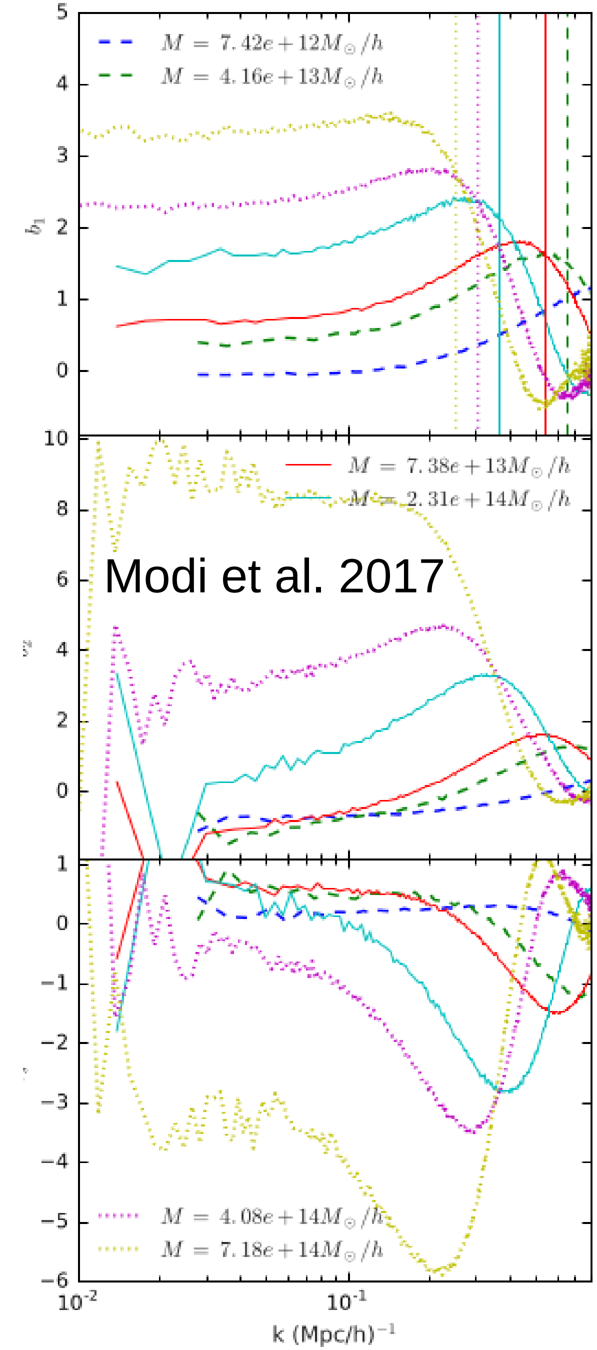
Woe to any approach which assumes W is sharp in k!

N.B. This is just linear bias; there are even more coefficients for quadratic and higher order bias …

Scale-dependent bias at all levels

Generically:

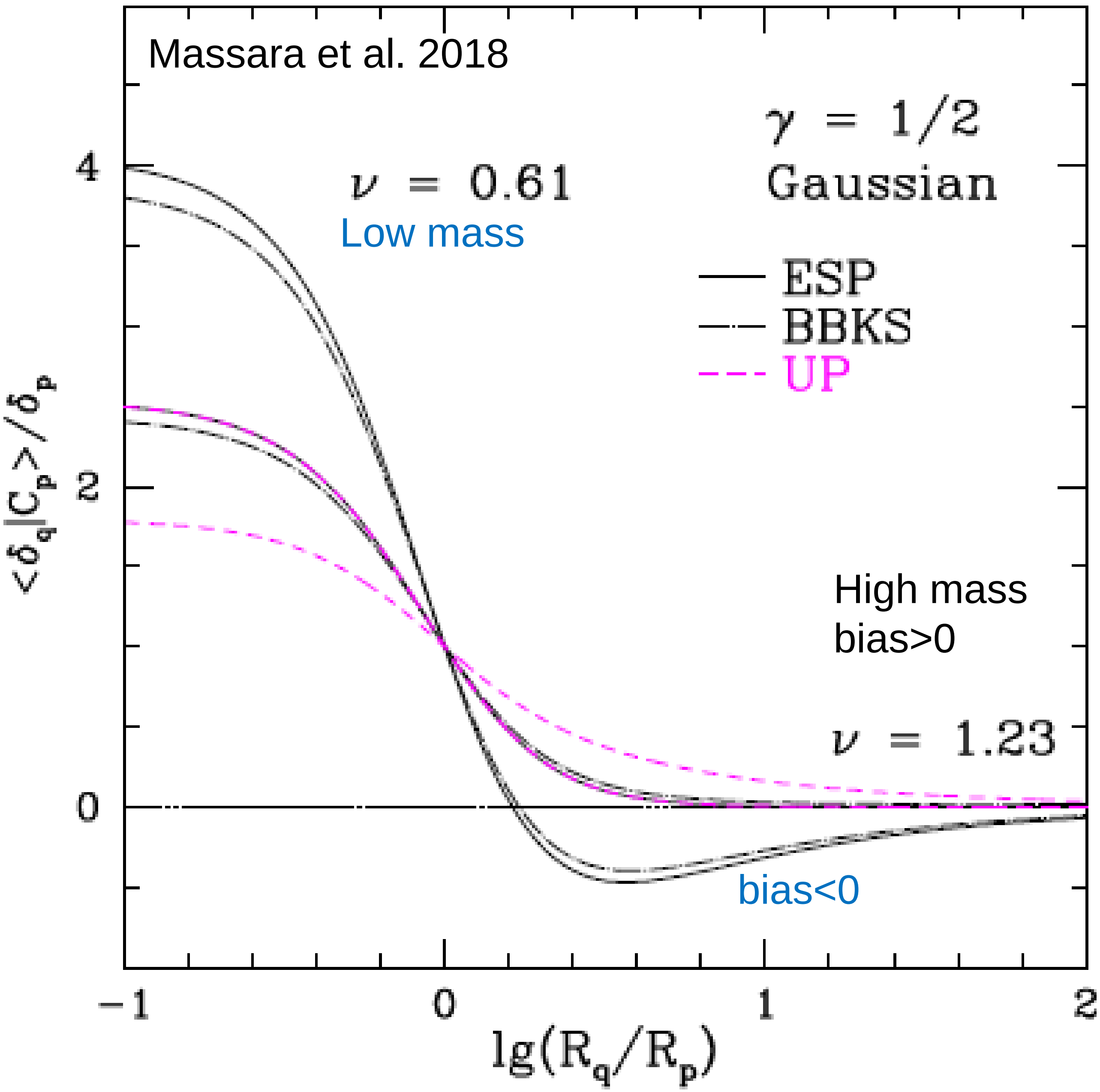
- constant at small k
- k^2 at intermediate k
- cutoff at high k
  (because halos are not point particles)



Density profile = cross correlation between peak and mass

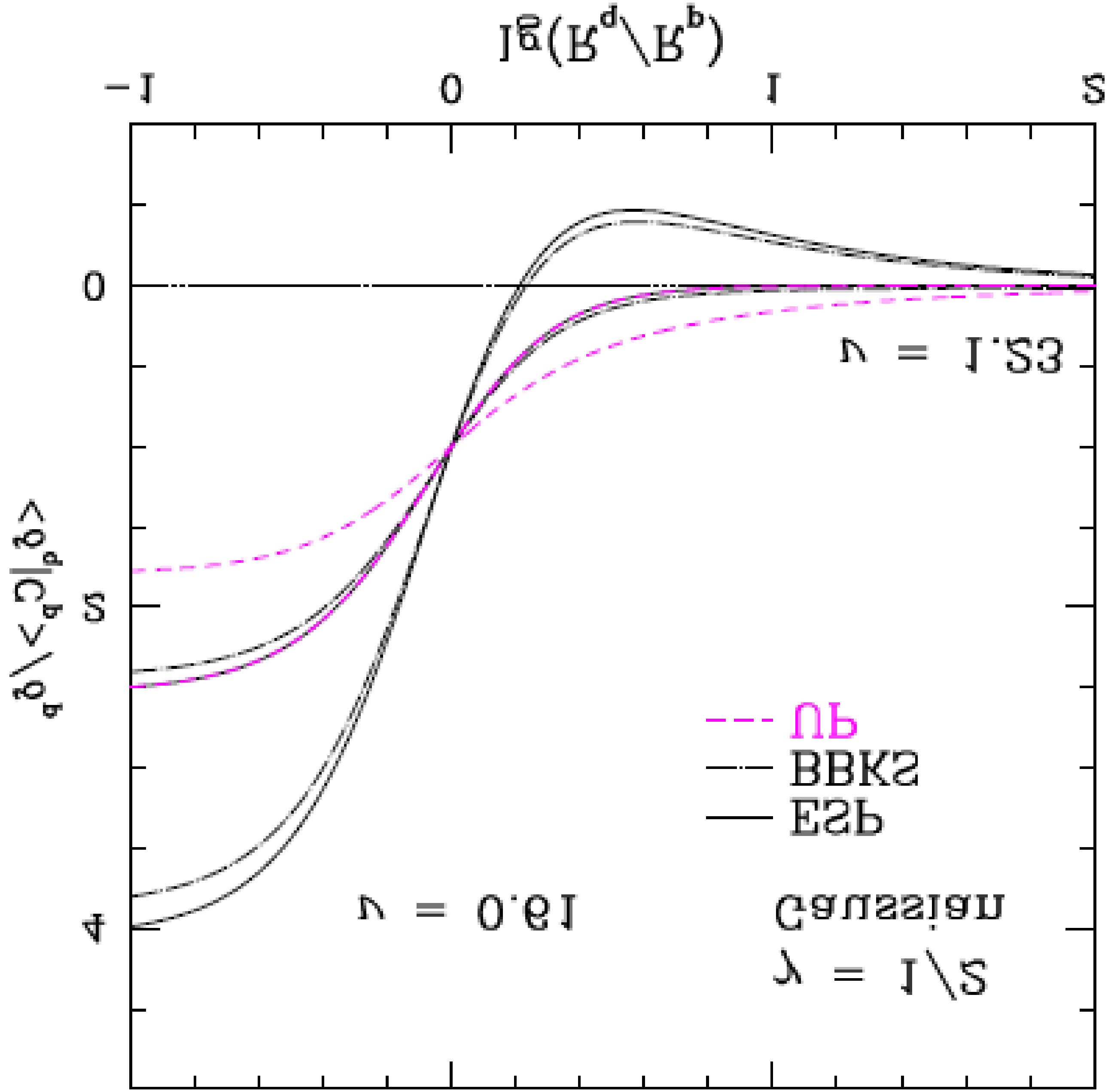
Generic: Low mass = more concentrated

Lagrangian bias is scale dependent

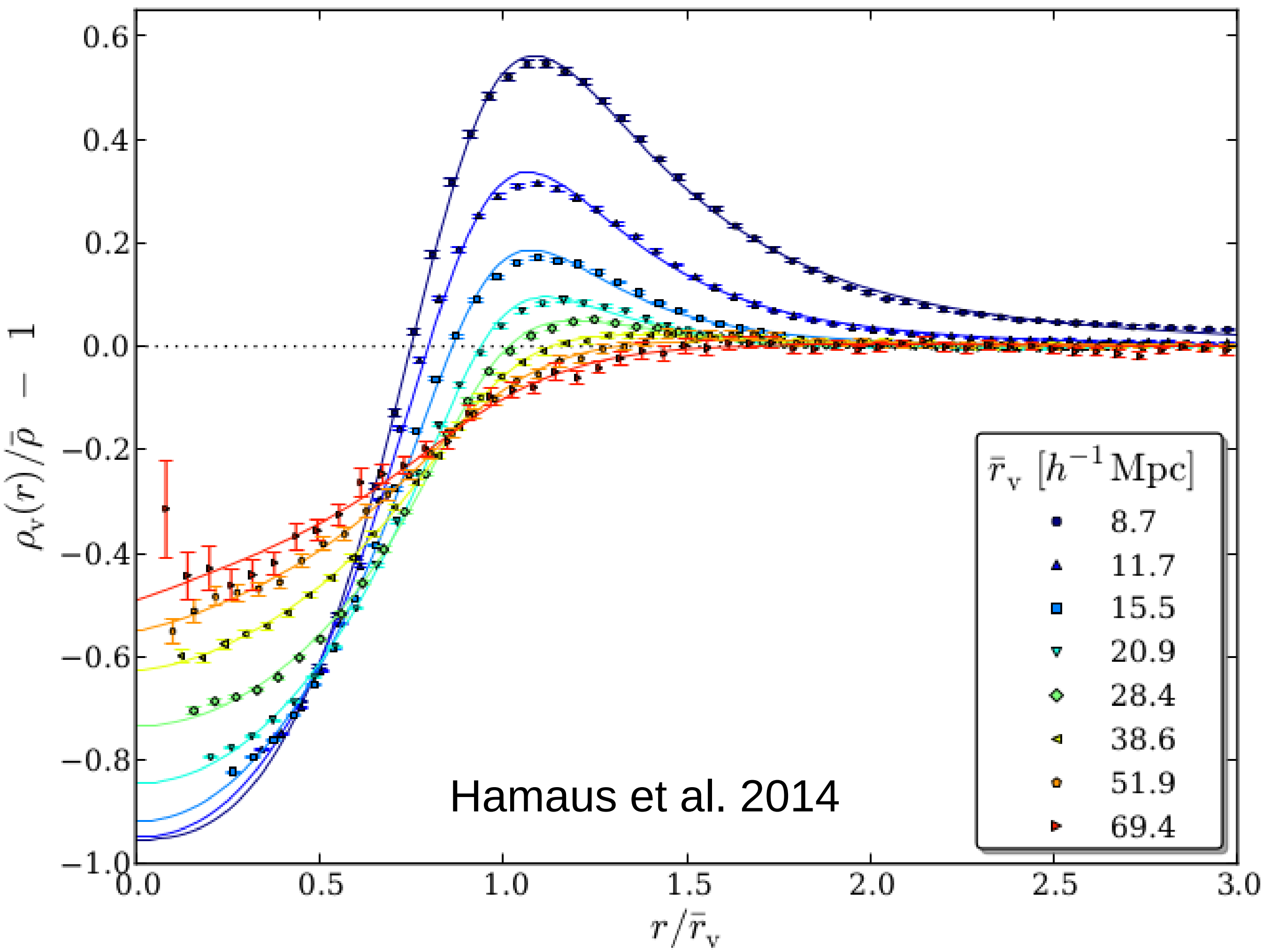


Density profile = cross correlation with mass

Generic: Small void = obvious wall



In simulations, small voids indeed surrounded by walls



Summary

- Getting closer to a model which includes nonlocal, nonspherical effects, and reconciles peaks/halos
- These generate k-dependent bias (monopole), as well as anisotropic bias (e.g. quadrupole), even in real-space
- Nonlocal bias matters at high mass
- Useful for making physically motivated ‘fitting formulae’ which simplify data analysis

Study of random walks with correlated steps

=

Cosmological constraints from large scale structures

The other half of phase space: Non-Maxwellian Velocities

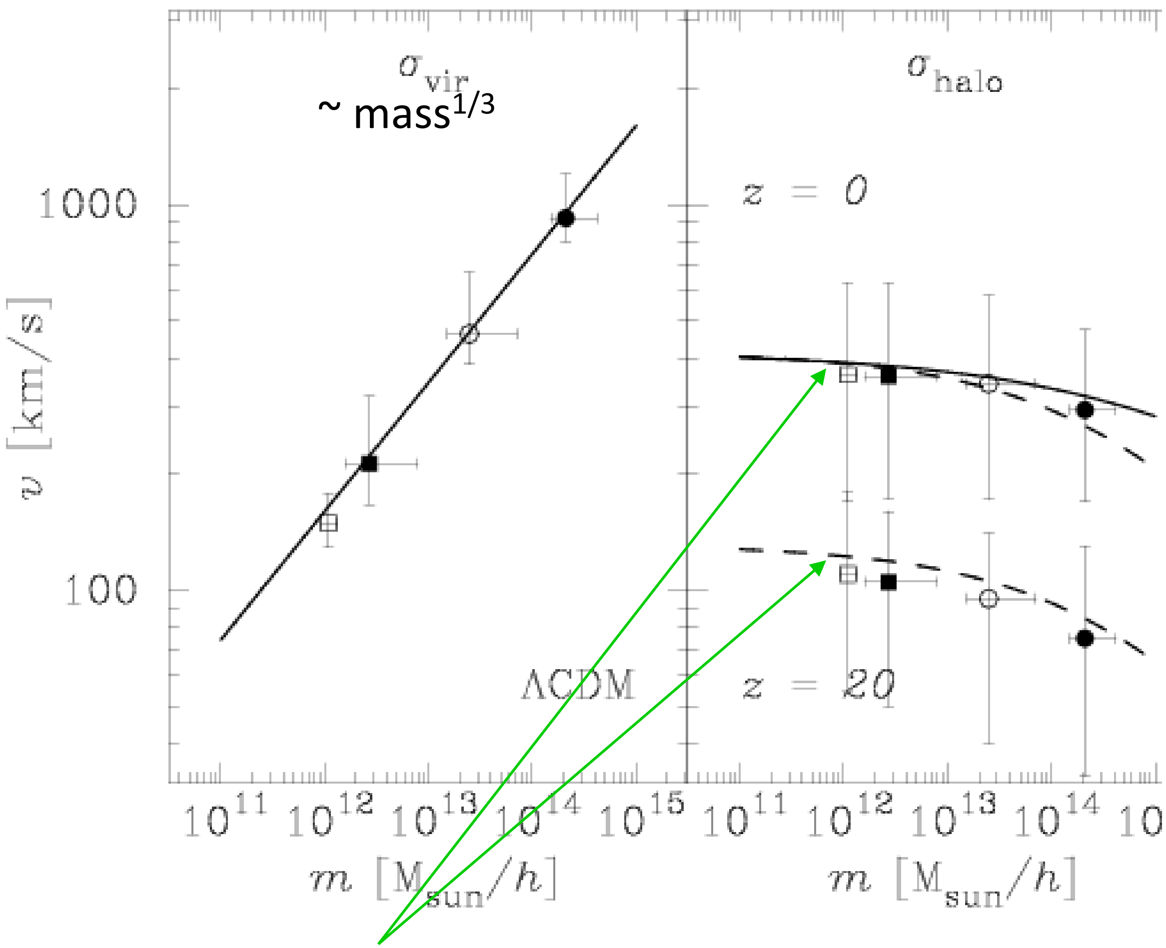
- $v = v_{vir} + v_{halo}$
- Maxwellian/Gaussian velocity within halo (dispersion depends on parent halo mass, because $v^2 \sim GM/r_{\rm vir} \sim M^{2/3}$)

  + Gaussian velocity of parent halo (from linear theory ≈ independent of *m*)
- Hence, at fixed *m*, distribution of v is convolution of two Gaussians, i.e.,

$p(v|m)$ is Gaussian, with dispersion

$$\sigma_{\rm vir}^2(m) + \sigma_{\rm Lin}^2 = (m/m_*)^{2/3}\sigma_{\rm vir}^2(m_*) + \sigma_{\rm Lin}^2$$

Two contributions to velocities



- Virial motions (i.e., nonlinear theory terms) dominate for particles in massive halos
- Halo motions (linear theory) dominate for particles in low mass halos

Growth rate of halo motions ~ consistent with linear theory;
Zeldovich should be good approximation for halo motions

Exponential tails are generic

- $p(\mathrm{v}) = \int dm\, mn(m)\, \mathrm{G}(\mathrm{v}|m)$

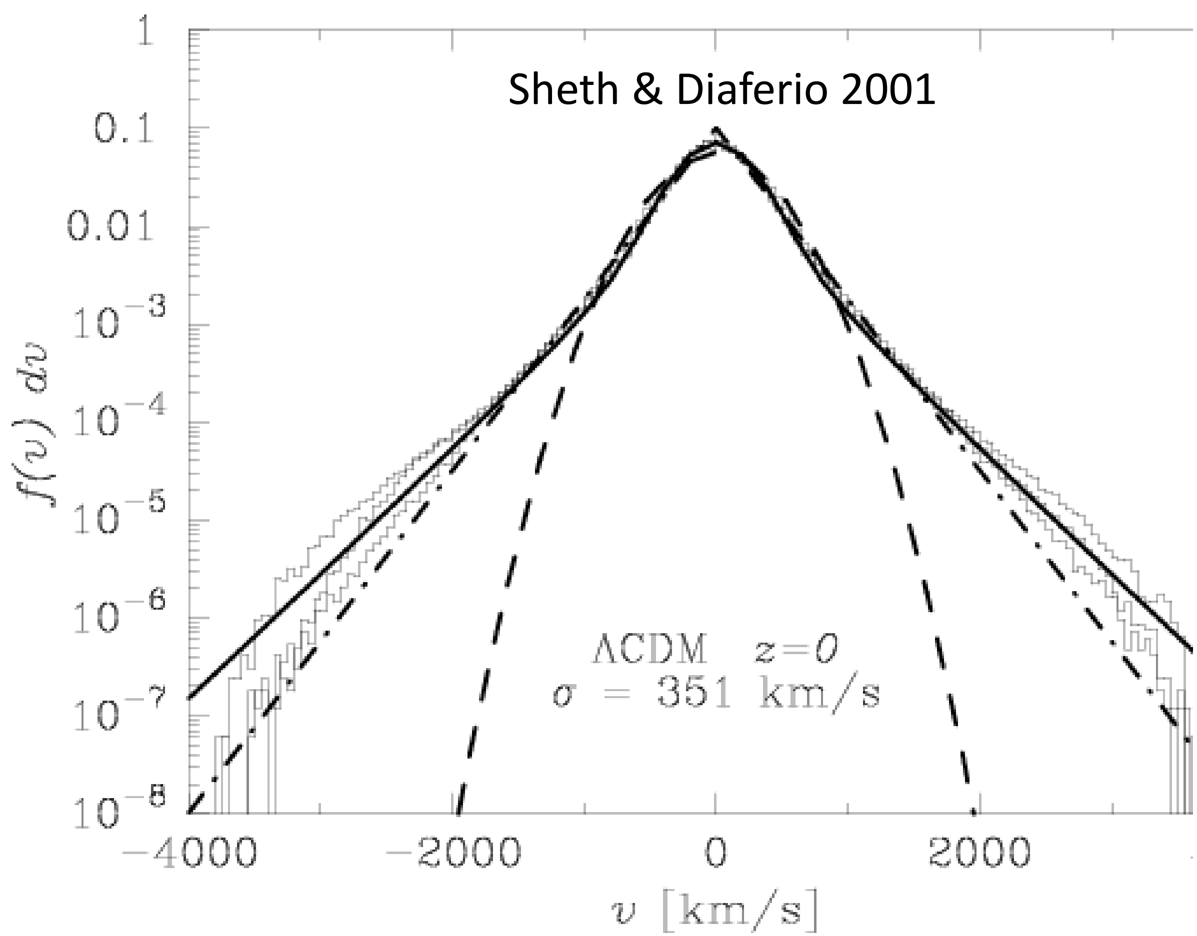
  $\mathcal{F}(t) = \int d\mathrm{v}\, e^{i\mathrm{v}t}\, p(\mathrm{v}) = \int dm\, n(m)m\, \mathrm{e}^{-t^2\sigma_{\mathrm{vir}}^2(m)/2}\, \mathrm{e}^{-t^2\sigma_{\mathrm{Lin}}^2/2}$

- For $P(k) \sim k^{-1}$, mass function $n(m)$ ~ power-law times $\exp[-(m/m_*)^{2/3}/2]$, so integral is:

  $\mathcal{F}(t) = \mathrm{e}^{-t^2\sigma_{\mathrm{Lin}}^2/2}\, [1 + t^2\sigma_{\mathrm{vir}}^2(m_*)]^{-1/2}$

- Fourier transform is product of Gaussian and FT of K_0 Bessel function, so $p(\mathrm{v})$ is convolution of $\mathrm{G}(\mathrm{v})$ with $K_0(\mathrm{v})$
- Since $\sigma_{\mathrm{vir}}(m_*) \sim \sigma_{\mathrm{Lin}}$, $p(\mathrm{v})$ ~ Gaussian at $|\mathrm{v}|<\sigma_{\mathrm{Lin}}$ but exponential-like tails extend to large v

Comparison with simulations



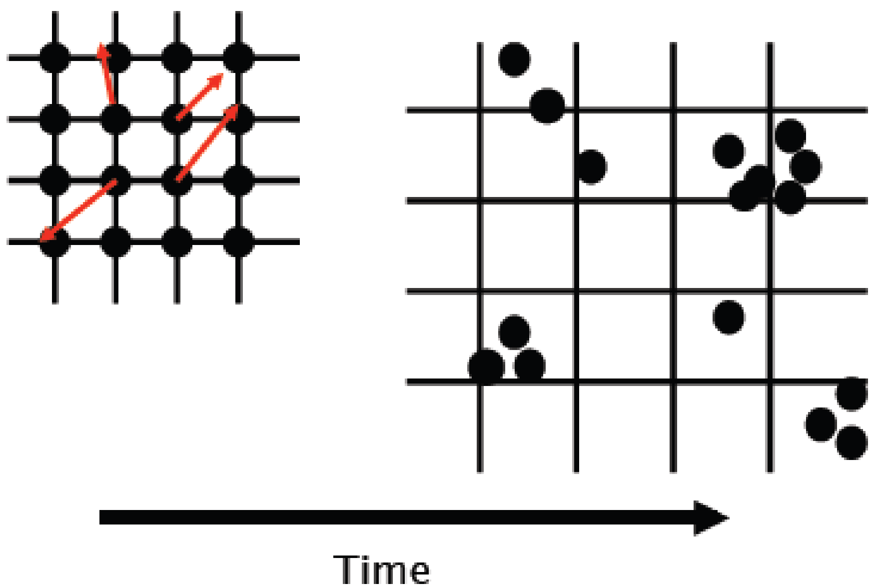
Gaussian core with exponential tails as expected

Similarly p(tSZ) and p(kSZ) should be non-Gaussian.

Structure grows because of perturbations in the initial <u>velocity</u> field

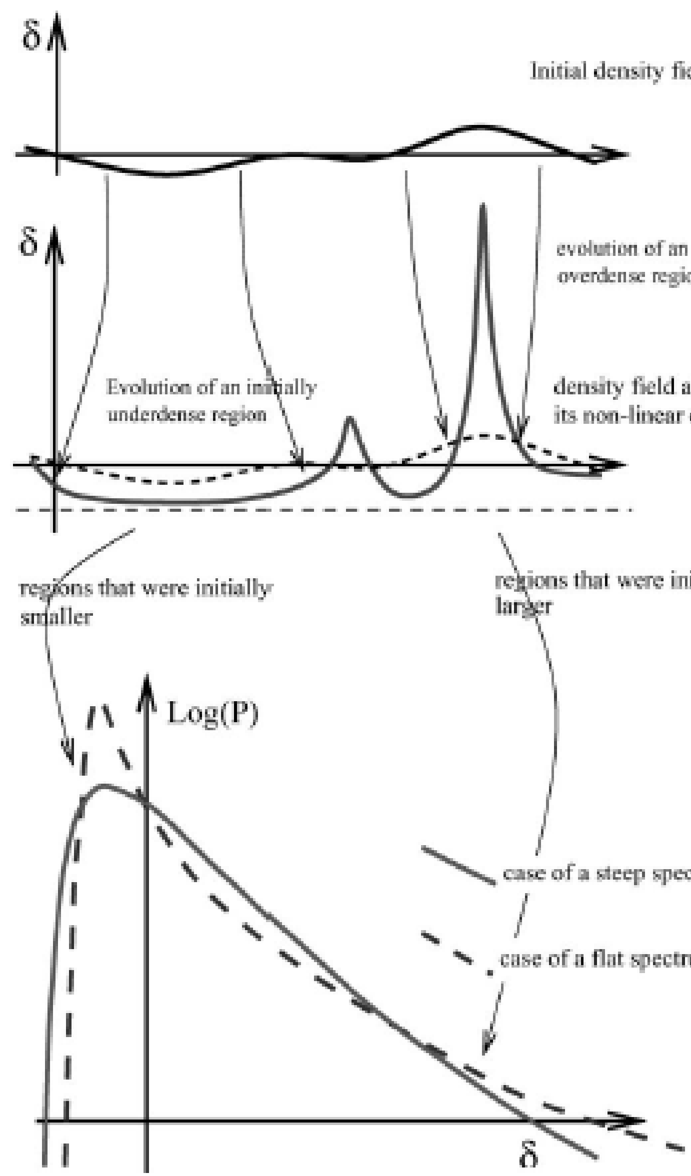
Initially distribution of matter is approximately homogeneous (δ is small)

Final distribution is clustered



Because of these motions, the fluctuation field can become very non-Gaussian (even though the displacements themselves are Gaussian)

F. Bernardeau et al. / Physics Reports 367 (2002) 1–248



The Zeldovich Approximation I.

$$\mathbf{x} = \mathbf{q} + D(t)\,\mathbf{u}(\mathbf{q})/(fH) = \mathbf{q} + D(t)\,\mathbf{S}(\mathbf{q})$$

How are Zeldovich displacements S (for shift) related to density?

$$d\mathbf{x}_i/d\mathbf{q}_j = \delta_{ij} + D(t)\, d\mathbf{S}_i/d\mathbf{q}_j$$
$$= \delta_{ij} - D(t)\, d[d\Phi/d\mathbf{q}_i]/d\mathbf{q}_j$$

- Displacements are related to one derivative of potential so Jacobian of x-q transformation involves second derivatives of potential: a 3x3 matrix.
- The 3 eigenvalues of Φ_{ij}, say λ_1, λ_2, λ_3, describe the principal axes of an ellipsoid (not a sphere!): in this respect, Zeldovich is more general than spherical.

Zeldovich approximation II.

In principal axis frame:

$$d\mathbf{x}_i/d\mathbf{q}_i = 1 - D(t)\,\lambda_i$$

Thus D(t) λ describes how the axis shrinks (or expands).

Hence, the density is

$$1 + \delta(t) = \ \Pi_{i=1}^{3}\,(1 - D(t)\lambda_i)^{-1}$$

To lowest order this is

$$1 + \delta(t) = 1 + D(t)\sum \lambda_i + D^2(t)\,(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\,\lambda_3) + \ldots$$
$$= 1 + D(t)\,\delta_{initial} + D^2(t)\,(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\,\lambda_3) + \ldots$$

Evidently, δ_{Linear} is just the trace of Φ_{ij}. This is why it can be arbitrarily negative, and even when it is, the true overdensity is still sensible.

Second order is combination of δ^2_{Linear} and tidal effects.

III. Zeldovich sphere

Expansion/contraction same in all directions means $\lambda_1 = \lambda_2 = \lambda_3$

$$1 + \delta(t) = [1 - D(t)\,\lambda]^{-3}$$
$$= [1 - D(t)\,\delta_L/3]^{-3}$$

This has $\delta_c = 3$ (because it ignores accelerated collapse as object shrinks)

Lagrangian EFT philosophy

$$\mathbf{x} = \mathbf{q} + \mathbf{S}_{PT}(\mathbf{q},\mathbf{t}) + \mathbf{S}_{NL}(\mathbf{q},\mathbf{t})$$

- S_{NL} is correction to the displacement predicted by PT (where PT can mean Zeldovich, or higher order).
  - One could think of it as a sum of terms, each associated with a ‘turnaround’ in the ‘multi-stream’, ‘shell-crossed’ regime
- In halo model, expect halo motions to be well-approximated by PT, but virial motions will not be, so think of $S_{NL} \sim S_{vir}$.

Halos and Lagrangian EFT

$$\mathbf{x} = \mathbf{q} + \mathbf{S}(\mathbf{q},t)$$

$$\mathbf{x}-\mathbf{x}_{halo} + \mathbf{x}_{halo} = \mathbf{q}-\mathbf{q}_{halo} + \mathbf{q}_{halo}+\mathbf{S}_{\mathbf{halo}}(\mathbf{q},t) + \mathbf{S}(\mathbf{q},t)-\mathbf{S}_{halo}(\mathbf{q},t)$$

$$\mathbf{x} - \mathbf{x}_{halo} = \mathbf{q} - \mathbf{q}_{halo} + \mathbf{S}(\mathbf{q},t) - \mathbf{S}_{halo}(\mathbf{q},t)$$

$$\mathbf{S}(\mathbf{q},t) = \mathbf{x} - \mathbf{x}_{halo} - (\mathbf{q} - \mathbf{q}_{halo}) + \mathbf{S}_{halo}(\mathbf{q},t)$$

$$\mathbf{x} - \mathbf{q} = \mathbf{x} - \mathbf{x}_{halo} - (\mathbf{q} - \mathbf{q}_{halo}) + \mathbf{S}_{PThalo}(\mathbf{q},t) + \mathbf{E}_{FT}(\mathbf{q},t)$$

virial profile	protohalo profile	center-of-mass motion in PT	Error .