# ICTP Summer School on Dynamical Systems
# Rotations of the circle and renormalization
# Homework 2

Let $G : [0,1] \to [0,1]$ be the Gauss map. Let $x \in [0,1]$, let $[a_0, a_1, \dots] = [a_0(x), a_1(x), \dots]$ denote the (a) continued fraction expansion of $x$ (infinite and uniquely defined if $x$ is irrational, one of the finite continued fraction expansions otherwise).

**Exercise 2.1.** (a) Complete the proof by induction that if $x \notin \mathbb{Q}$ and $(a_i)_{i=0}^{+\infty}$ is the itinerary of $\mathcal{O}_G^+(x)$ then $x = [a_0, a_1, \dots, a_n, \dots]$.

(b) Assume that $x \in (3/4, 4/5)$ and find $a_0(x)$ and $a_1(x)$. Now suppose also that $G^2(x) = x$ and find $x$.

(c) Construct an example of a point $x \in [0,1)$ such that the orbit $\mathcal{O}_G^+(x)$ accumulates to $\frac{1}{3}$, that is there is an increasing subsequence $n_k$ such that

$$\lim_{k\to\infty} G^{n_k}(z) = \frac{1}{3}.$$

*** Exercise 2.2.** [Leading digits of powers of two.] Consider the sequence $(2^n)_{n\in\mathbb{N}}$ of powers of two:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \dots$$

The *leading digit* of a number is simply the first digit in the decimal expansion. For example, the leading digit of 512 is 5. The leading digits of the previous sequence are written in bold font:

**1**, **2**, **4**, **8**, **1**6, **3**2, **6**4, **1**28, **2**56, **5**12, **1**024, **2**048, ...

Consider the sequence of the leading digits:

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, \dots$$

Compute the frequency of the digit 1 in the sequence of leading digits of $(2^n)_{n\in\mathbb{N}}$, i.e. what is the limit

$$\lim_{N\to\infty} \frac{Card\{0 \le n < N \quad \text{s.t. the leading digit of } 2^n \text{ is } k\}}{N}$$

(Here Card denotes cardinality)

[*Answer:* $\log_{10}(k+1) - \log_{10} k = \log_{10}\left(1 + \frac{1}{k}\right)$.]

# More Exercises

**Exercise 2.3.** Let $\sigma : \Sigma \to \Sigma$ be the shift map on the space $\Sigma = \mathbb{N}^{\mathbb{N}}$ of sequences $\underline{a} = (a_n)_{n\in\mathbb{N}}$ with $a_i$ positive integers. Let $X = [0,1) - \mathbb{Q}$ the set of irrational points in $[0,1)$ and consider the map $\psi : \Sigma \to X$ given by

$$\psi(\underline{a}) = [a_1, a_2, \dots, ] = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}.$$

(a) Show that $\psi$ is a conjugacy between the Gauss map $G : X \to X$ and $\sigma : \Sigma \to \Sigma$.

(b) Describe all fixed points of $G$ in terms of their continued fraction expansion.

(c) Is $\psi : \Sigma \to [0,1)$ a conjugacy between $G : [0,1) \to [0,1)$ and $\sigma : \Sigma \to \Sigma$? Explain.

**Exercise 2.4.** Using itineraries of the Gauss map:

(a) write the continued fraction expansions of the number $0 \le \alpha \le 1$ such that

$$\alpha = \frac{1}{5 + \frac{1}{4+\alpha}}.$$

(b) Assume that $x \in [0,1]$ is such that $G^k(x) \in (1/3, 1/2)$ for all $k \in \mathbb{N}$. Find $x$.

(c) Show that if $\lim_{n\to\infty} G^n(x) = 0$ then $\lim_{n\to\infty} a_n(x) = \infty$.

(d) construct an example of a point $x \in [0,1)$ which is non periodic under $G$ and each even iterate (that is, each iterate of the form $G^{2n}(x)$ for $n \in \mathbb{N}$) belongs to the left half of the unit interval.

* **Exercise 2.5.** Let $G : [0,1] \to [0,1]$ be the Gauss map. Show that if $\alpha$ is a periodic point of period $n$ for $G$ then it is a quadratic irrational.

[*Hint*: Try first for $n = 2$.]