Measure preserving transformations - Exercises

Exercise 1 Let $(X, \mathscr{B}, \mu)$ a measure space. Prove the following statements.

(a) Let $T\colon X \to X$ invertible (injective and surjective) and $A \subset \mathscr{B}$. Then $T(T^{-1}(A)) = A$ and $T^{-1}(T(A)) = A$. If T is not invertible, what can go wrong? Which inclusion holds if T is not surjective? And if it is not injective?

(b) Using point (a), prove that if T is invertible, then T is measure preserving for a measure μ if and only if $\mu(T(A)) = \mu(A)$, for all $A \in \mathscr{B}$. [In other words, if T is invertible we can use images instead of preimages in the definition of measure preserving.]

(c) The map $T_*\mu\colon X \to \mathbb{R} \cup \{+\infty\}$ is a measure.

Exercise 2 Decide and prove whether the following transformations are measure preserving or not.

(a) Let $(X, \mathscr{B}, \lambda)$ with $X = [0, 1]$, $\mathscr{B}$ the Borel σ-algebra and λ the Lebesgue measure. Let $T\colon X \to X$ the *tent map*, defined by

$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Does T preserve λ?

*(b) Let $(X, \mathscr{B}, \lambda)$ with $X = [0, 1]$, $\mathscr{B}$ the Borel σ-algebra and λ the Lebesgue measure. Let $G\colon X \to X$ the *Gauss map*, defined by

$$G(x) = \begin{cases} 0 & x = 0 \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} - n & x \in P_n = \left(\frac{1}{n+1}, \frac{1}{n}\right] \end{cases}$$

Does G preserve λ?

*(c) Let X, $\mathscr{B}$ and G as above. Now consider the *Gauss measure* μ defined by the density $\frac{1}{C(1+x)}$. Find the value of C needed to have $\mu([0, 1]) = 1$. For the measure given by the value of C found, is G measure preserving?