ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Solutions 2

Solutions to Exercise 2.1

Part (a) See Lecture Notes

Part (b) Let us recall that $a_0(x)$ is the integer part of $1/x$. Thus, since

$$\frac{3}{4} < x < \frac{4}{5} \quad \Leftrightarrow \quad \frac{4}{3} < \frac{1}{x} < \frac{5}{4},$$

we have that $a_0(x) = [1/x] = 1$. Then $G(x) = 1/x - 1$ and since from the previous inequality

$$\frac{4}{3} - 1 < \frac{1}{x} - 1 < \frac{5}{4} - 1 \quad \Leftrightarrow \quad \frac{1}{3} < G(x) < \frac{1}{4},$$

so that $3 < 1/G(x) < 4$. Thus $a_1(x) = [1/G(x)] = 3$.

Recall from the lecture that $a_k(x) = [1/G^k(x)]$. Thus, since $G^2(x) = x$, the continued fraction entries of x are periodic with period 2. Since the first two entries by part (a) are $a_0(x) = 1$ and $a_1(x) = 3$, we have that $a_n(x) = 1$ for any n even and $a_n(x) = 3$ for any n odd. From the periodic expression for the continued fraction $x = [1, 3, 1, 3, 1, 3, \ldots]$.

Alternatively, one can also compute x explicitly. Since $a_0(x) = 1$ by Part (a), $G(x) = \frac{1}{x} - a_0(x) = \frac{1}{x} - 1$. Similarly, since $a_1(x) = 3$ again by Part (a),

$$G^2(x) = \frac{1}{G(x)} - a_1(x) = \frac{1}{G(x)} - 3 = \frac{1}{\frac{1}{x} - 1} - 3.$$

Thus, if $G^2(x) = x$, this means that

$$x = \frac{1}{\frac{1}{x} - 1} - 3 \quad \Leftrightarrow \quad \frac{(4x - 4}{-x + 1}$$

This leads to a quadratic equation for x, namely $x^2 - 3x + 1$, which one can solve. Only one of the two solutions lies in the prescribed interval (one can also see that the any branch of G^2 which correspond to prescribing the two first entries intersect the diagonal in a unique point). Explicitely, x has the form

$$x = \frac{-3 + \sqrt{3^2 + 4 \cdot 3}}{2a} = -\frac{3}{2} + \frac{\sqrt{21}}{2}.$$

One can also compute x explicitly from the knowledge that $x = [1, 3, 1, 3, 1, 3, \ldots]$, by remarking that then $x = [1, 3 + x]$, i.e.

$$x = \frac{1}{1 + \frac{1}{3+x}} \quad \Leftrightarrow \quad \frac{1}{x} - 1 = \frac{1}{3 + x},$$

which leads to the same degree two equation.

Part (c) Let $x = [3, x_0, 3, x_1, 3, x_2, \ldots]$ where x_i is an increasing sequence of integers, that is satisfy $\lim_{i\to\infty} x_i = \infty$. Since the $2n^{th}$ entry is equal to 3 and the following digit is x_n and the Gauss map acts as a shift on entries of the continued fraction expansion, we have that $G^{2n}(x) = [3, x_n, \ldots]$ so that

$$G^{2n}(y) \in P_2 \cap G^{-1}(P_{x_n}) = P_2 \cap G^{-1}\left(\frac{1}{x_n + 1}, \frac{1}{x_n}\right].$$

Let us call this intersection I^n. The preimage $G^{-1}(P_{x_n})$ consists of countably many intervals, of the form

$$\left[\frac{1}{i + \frac{1}{x_n}}, \frac{1}{i + \frac{1}{x_n+1}}\right), \qquad i \in \mathbb{N}.$$

Since we are interesting it with $P_2 = (1/4, 1/3]$, we have that

$$G^{2n}(y) \in I_n = \left(\frac{1}{3 + \frac{1}{x_n}}, \frac{1}{3 + \frac{1}{x_n+1}}\right),$$

Since as n tends to infinity and hence $x_n \to +\infty$ both the endpoints of the interval I_n tend to $1/3$, this shows by the pinching or sandwitch theorem that $G^{2n}(y) \to 1/3$ as $n \to \infty$.

Solutions to Exercise

We will use Weyl Theorem to answer this question. We will show that the frequency of the digit k as leading digit in the sequence $(2^n)_{n\in\mathbb{N}}$ is given by

$$\lim_{N\to\infty} \frac{Card\{0 \le n < N \quad \text{s.t. the leading digit of } 2^n \text{ is } k\}}{N} = \log_{10}\left(1+\frac{1}{k}\right)$$

where $\log_{10}$ denotes the logarithm in base 10 (that is, $\log_{10}(a) = b$ if and only if $10^a = b$).

Notice that the leading digit of 2^n is k if and only if there exists an integer $r \ge 0$ such that

$$k10^r \le 2^n < (k+1)10^r.$$

For example, $2 \cdot 100 \le 256 < 3 \cdot 100$ shows that the leading digit of 256 is 2.

Taking logarithms in base 10 and using the properties of logarithms (as $\log_{10}(ab) = \log_{10}(a) + \log_{10}(b)$ and $\log_{10} 10^r = r$), this shows that

$$\begin{aligned} &\log_{10}(k10^r) \le \log_{10} 2^n < \log_{10}((k+1)10^r), \\ &\log_{10} k + r \le n\log_{10} 2 < \log_{10}(k+1) + r. \end{aligned}$$

Thus, equivalently,

$$(n\log_{10} 2 \mod 1) \in I_k = [\log_{10} k, \log_{10}(k+1)]\,.$$

Notice that if we call $\alpha = \log_{10} 2$, the sequence

$$\begin{aligned} (n\log_{10} 2 \mod 1)_{n\in\mathbb{N}} &= 0, \quad \log_{10} 2 \mod 1, \quad 2\log_{10} 2 \mod 1, \quad 3\log_{10} \mod 1, \dots \\ &= 0, \quad \log_{10} 2 \mod 1, \quad \log_{10} 2 + \log_{10} 2 \mod 1, \quad 2\log_{10} 2 + \log_{10} 2 \mod 1, \dots \end{aligned}$$

is the orbit $\mathcal{O}^+_{R_\alpha}(0)$ of 0 under the rotation by α. Thus,

$$\begin{aligned} &\frac{Card\{\,0 \le n < N \quad \text{such that the leading digit of } 2^n \text{ is } k\,\}}{N} = \\ &\frac{Card\{\,0 \le n < N \quad \text{such that } (n\log_{10} 2 \mod 1) \in I_k\,\}}{N} = \\ &\frac{Card\{\,0 \le n < N \quad \text{such that } R_\alpha^n(0) \in I_k\,\}}{N} = \frac{1}{N}\sum_{n=0}^{N-1} \chi_{I_k}(R_\alpha^n(0)). \end{aligned}$$

One can show that $\log_{10} 2$ is irrational, thus R_α is an irrational rotation and hence by Weyl theorem the orbit of any point, in particular 0, is equidistributed. This gives that

$$\begin{aligned} \lim_{N\to\infty} &\frac{Card\{0 \le n < N \quad \text{s.t. the leading digit of } 2^n \text{ is } k\}}{N} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \chi_{I_k}(R_\alpha^n(0)) \\ &= \lambda(I_k) = \log_{10}(k+1) - \log_{10} k = \log_{10}\left(1+\frac{1}{k}\right). \end{aligned}$$