Deformations of the circular Wilson loop, defect-CFT data and spectral independence

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Based on: `arXiv:1703.03812` - M. Cooke, A. Dekel and N.D.

`arXiv:18xx.xxxxx` - M Cooke, A. Dekel, N.D., D. Trancanelli and E. Vescovi

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$1/2$ BPS circular Wilson loop

- The simplest and most symmetric Wilson loop in a CFT is a circle.
- It preserves an $SL(2,\mathbb{R})$ subgroup of the full conformal group.
- In the case of $\mathcal{N}=4$ SYM it also preserves $1/2$ of the supercharges and an $OSp(4^*|4)$ supergroup which includes $SL(2,\mathbb{R})\times SO(3)\times SO(5)$.
- Its expectation value is well known [Erickson,Semenoff Zarembo] [drukker Gross] [Pestun]

$$\langle W\rangle_0=\frac{1}{N}L^1_{N-1}(\lambda/4N)e^{\lambda/8N}\sim\frac{2}{\sqrt{\lambda}}I_1\big(\sqrt{\lambda}\big)$$

Deformations of the circle

- We will view all closed loops as deformations of the circle.
- Consider a Wilson loop in $\mathcal{N} = 4$ SYM following a path in $\mathbb{R}^2$ given by

$$X(\theta) = x^1(\theta) + ix^2(\theta) = e^{i\theta + g(\theta)}.$$

- It is convenient to write $g(\theta)$ in a Fourier decomposition

$$g(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$$

  and without loss of generality $g(\theta)$ is real, so $b_{-n} = \bar{b}_n$.
- The expectation value of the Wilson loop can be written in an expansion in powers of b_n

$$\langle W_X \rangle = \langle W \rangle_0 + \langle W \rangle_2 + \langle W \rangle_4 + \cdots\,, \qquad\qquad \langle W \rangle_{2n} \sim \mathcal{O}(b^{2n})$$

- At order b^0 we have the circular Wilson loop whose VEV I quoted alread.
- As I will review, order b^2 is also known to all orders in the coupling. What I focus on is $\langle W \rangle_4$.

Insertions into the circle

[Drukker Kawamoto]

- One can insert any number of adjoint valued operators into the Wilson loop

$$\left\langle\!\left\langle \mathcal{O}^{(1)}(x(s_1))...\mathcal{O}^{(n)}(x(s_n)) \right\rangle\!\right\rangle$$
$$= \frac{1}{N}\mathrm{tr}\,\mathcal{P}\left[\mathcal{O}^{(1)}(x(s_1))...\mathcal{O}^{(n)}(x(s_n))\, e^{\int\left(i\dot{x}^\mu A_\mu(x(s)) - |\dot{x}|\Phi^1(x(s))\right)ds}\right].$$

- For example $\mathcal{O}$ can be a scalar field Φ^I or the field strength $F_{\mu\nu}$.
- This is true for any Wilson loop. In the case of the circle we have Ward identities for conformal symmetry so for two insertiona

$$\left\langle\!\left\langle \mathcal{O}^{(1)}(0)\mathcal{O}^{(2)}(\theta) \right\rangle\!\right\rangle = \frac{a_{\mathcal{O}^{(1)}\mathcal{O}^{(2)}}(\lambda)}{(2\sin\frac{\theta}{2})^{\Delta_{\mathcal{O}^{(1)}}+\Delta_{\mathcal{O}^{(2)}}}}\,,$$

- For three scalar primary insertions

$$\left\langle\!\left\langle \mathcal{O}^{(1)}(\theta_1)\mathcal{O}^{(2)}(\theta_2)\mathcal{O}^{(3)}(\theta_3) \right\rangle\!\right\rangle = \frac{c_{(123)}(\lambda)}{|d_{12}|^{\Delta_1+\Delta_2-\Delta_3}|d_{13}|^{\Delta_1-\Delta_2+\Delta_3}|d_{23}|^{-\Delta_1+\Delta_2+\Delta_3}}\,,$$

- In the case of the four point function we already have a single cross ratio.
- These insertions have normalizations, dimensions and structure constants and should satisfy the OPE.
- What can we say about those and how are they related to deformations of the circle?

Outline

- Introduction.
- Deformed circle and defect CFT.
- Integrability.
- Perturbation theory.
- dCFT data.
- Bremsstrahlung function.
- Results for $\langle W \rangle_4$.
- Construction of string solution for near circular Wilson loop.
- Spectral parameter (in)dependence.
- Conclusions.

Deformed circle and defect CFT

- Deformations away from the circle can be represented by insertions of adjoint valued fields into the Wilson loop. Normally the first insertion is $F_{\mu\nu}\dot{x}^\nu$.
- For a radial deformation of the circular Maldacena-Wilson loop this is replaced with

$$\mathbb{F}_{r\phi} = F_{r\phi} + iD_r\Phi^1$$

- All insertions can be classified by representations of $OSp(4^*|4)$, and this first insertion, the *displacement operator* is in fact a protected operator of dimension 2.
- The only insertions of classical dimension one are scalar fields Φ^I. They decompose to the singlet Φ^1 and the **5** of $SO(5)$.
- The **5** is also protected, it's a superpartner of the displacement operator.
- The singlet is not protected.

- Some operators of classical diemension two are

$$iF_{i\phi}, \quad ,iF_{ir}, \quad iF_{ij}, \quad D_\mu\Phi^1, \quad D_\mu\Phi^a, \quad \Phi^1\Phi^1, \quad \Phi^1\Phi^a, \quad \Phi^a\Phi^1, \quad \Phi^a\Phi^b.$$

- They can be arranged in representations of the symmetry group of the Wilson loop including into supermultiplets (notations slight simpler when considering insertions the line instead of the circle).
- There are also fermionic insertions, of course.
- Details can be found in my paper...
- Can calculate the dimensions and normalizations from
  - Perturbation theory.
  - AdS/CFT.
  - Integrability.
  - Localization.
  - Bootstrap.

Integrability

[Drukker][Correa, Maldacena, Sever]

- One can use integrability to calculate the anomalous dimension of a cusped Wilson loop.
- A Wilson loop is described by an open string in AdS, this translates to an open spin-chain (or other integrable model).
- It is non-trivial, but true, that the boundary conditions appropriate for a cusp satisfy the boundary Yang-Baxter equation.
- The same formalism allows to calculate a cusp with an operator insertion:
  - The insertion of Z^L is the length L ground state of the system.
  - All other insertions can be viewed as excitations of this state.
  - We can find the anomalous dimensions of insertions into the circle by taking the cusp angle to be zero.
- This procedure has not been applied in this case.

Perturbation theory

- One can consider the insertion of any operator and calculate Feynman diagrams.
- We chose instead to look at smooth Wilson loops, for which the one loop VEV is

$$\langle W[\mathcal{C}]\rangle_{\text{1-loop}} = -\frac{\lambda}{16\pi^2}\oint ds_1\, ds_2\, I(s_1,s_2)\,, \qquad I(s_1,s_2) = \frac{\dot{x}_1\cdot\dot{x}_2 + |\dot{x}_1||\dot{x}_2|}{x_{12}^2}\,.$$

- For curves in $\mathbb{R}^2$ there is also a compact formula for two loop graphs [Bassetto, Griguolo, Pucci, Seminara]

$$\langle W[\mathcal{C}]\rangle_{\text{2-loop}} = -\frac{\lambda^2}{128\pi^4}\oint ds_1\, ds_2\, ds_3\, \epsilon(s_1,s_2,s_3) I(s_1,s_3)\frac{x_{32}\cdot\dot{x}_2}{x_{32}^2}\log\frac{x_{21}^2}{x_{31}^2}$$

$$+\frac{\lambda^2}{2}\left(\frac{1}{16\pi^2}\oint ds_1\, ds_2\, I(s_1,s_2)\right)^2 - \frac{\lambda^2}{64\pi^4}\int_{s_1>s_2>s_3>s_4} ds_1\, ds_2\, ds_3\, ds_4\, I(s_1,s_3) I(s_2,s_4)\,.$$

- We have found efficient algorithms to calculate these integrals for arbitrary curves, and then using the relation between deformations and insertions, extracted some CFT data.

- Going back to a curve parametrized as

$$X(\theta) = x^1(\theta) + ix^2(\theta) = e^{i\theta + g(\theta)}.$$

- We can now expand the VEV of the WIlson loop in powers of $g(\theta)$, giving correlation functions in the defect-CFT, which are (schematically)

$$\begin{aligned}
\langle W \rangle = \langle W \rangle_0 &+ \oint g(\theta_1) g(\theta_2) \langle\!\langle \mathbb{F}(\theta_1) \mathbb{F}(\theta_2) \rangle\!\rangle d\theta_1\, d\theta_2 \\
&+ \oint g^2(\theta_1) g^2(\theta_2) \langle\!\langle D\mathbb{F}(\theta_1) D\mathbb{F}(\theta_2) \rangle\!\rangle d\theta_1\, d\theta_2 \\
&+ \oint g^2(\theta_1) g(\theta_2) g(\theta_3) \langle\!\langle D\mathbb{F}(\theta_1) \mathbb{F}(\theta_2) \mathbb{F}(\theta_3) \rangle\!\rangle d\theta_1\, d\theta_2\, d\theta_3 \\
&+ \oint g(\theta_1) g(\theta_2) g(\theta_3) g(\theta_4) \langle\!\langle \mathbb{F}(\theta_1) \mathbb{F}(\theta_2) \mathbb{F}(\theta_3) \mathbb{F}(\theta_4) \rangle\!\rangle d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4 + \cdots
\end{aligned}$$

- Expanding this further in powers of the coupling, we should find a match with the result of the Feynman diagram calculation.

- Note that the one-loop graph sees only 2-point functions.

The dCFT data

- Matching the two expressions we find for the displacement operator:

$$a_F = -\frac{3\lambda}{4\pi^2} + \frac{\lambda^2}{32\pi^2} + O(\lambda^3)\,, \qquad \gamma_F = 0\,.$$

- Some three point functions are

$$\langle\!\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2)\Phi^1(s_3)\rangle\!\rangle = \frac{c^0_\Phi\,\eta_{ij}}{|s_{12}|^3|s_{13}||s_{23}|} + \mathcal{O}(\lambda)\,,$$

$$\langle\!\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iF_{km}(s_3)\rangle\!\rangle = \frac{c^0_{\mathbf{3}}\,(\eta_{ik}\eta_{jm}-\eta_{im}\eta_{jk})}{|s_{12}|^2|s_{13}|^2|s_{23}|^2} + \mathcal{O}(\lambda)\,,$$

$$\langle\!\langle i\mathbb{F}_{i3}(s_1) i\mathbb{F}_{j3}(s_2) iD_{\{k}\mathbb{F}_{m\}3}(s_3)\rangle\!\rangle = \frac{c^0_{\mathbf{5}}\,\left(\eta_{ik}\eta_{jm}+\eta_{im}\eta_{jk}-\frac{2}{3}\eta_{ij}\eta_{km}\right)}{|s_{12}|^2|s_{13}|^2|s_{23}|^2} + \mathcal{O}(\lambda)\,,$$

  and we found

$$c^0_\Phi = -\frac{1}{32\pi^4}\,, \qquad c^0_{\mathbf{3}} = \frac{1}{16\pi^4}\,, \qquad c^0_{\mathbf{5}} = \frac{5}{16\pi^4}\,.$$

- For the unprotected singlet scalar we reproduced

$$a^0_\Phi = \frac{1}{8\pi^2}\,, \qquad \gamma_\Phi = \frac{1}{4\pi^2}\,.$$

- For the triplet and quintet states we calculated

$$a^0_{\mathbf{3}} = -\frac{1}{2\pi^2}\,, \quad \gamma_{\mathbf{3}} = \frac{1}{4\pi^2}\,, \qquad\qquad a^0_{\mathbf{5}} = \frac{5}{\pi^2}\,, \quad \gamma_{\mathbf{5}} = \frac{1}{4\pi^2}\,.$$

insertion dimension
function of all operators
The 2-pt function is
We find
loop.
dependant. For
structure constants:
npb/55 522
hep-th/010513

AdS/CFT

[Giombi,Roiban Tseytlin]

- The same story can be repeated using deformations propagating on a semiclassical string in AdS_5.
- One calculates Witten diagrams in the AdS_2 world-sheet.
- For the singlet they find

$$\gamma_\Phi = 2 - \frac{5}{\sqrt{\lambda}} + \cdots$$

- And some structure constant

$$C_{\Phi\Phi(\Phi^2)} = \frac{2}{5} - \frac{43}{30\sqrt{\lambda}} + \cdots$$

- They used the OPE decomposition of the 4-point function to extract this and other structure constants.

The bremsstrahlung function

- At order $g(\theta)^2$ the expectation value of the Wilson loop is given by

$$\begin{aligned}\langle W\rangle_2 &= \oint g(\theta_1)g(\theta_2)\langle\!\langle \mathbb{F}(\theta_1)\mathbb{F}(\theta_2)\rangle\!\rangle d\theta_1\, d\theta_2 \\ &= B(\lambda)\oint \frac{\bar{g}(\theta_1)g(\theta_2)}{16\sin^4\frac{\theta_1-\theta_2}{2}}d\theta_1\, d\theta_2 \\ &= 8\pi^2 B(\lambda)\sum_{n=2}^{\infty} n(n^2-1)|b_n|^2\,,\end{aligned}$$

  were the last expression replaces $g(\theta)$ by its Fourier coefficients b_n.

- The factor of $B(\lambda)$ is related to the normalization of the displacement operator $a_{\mathbb{F}}$, and by studying deformations that preserve supersymmetry can also be fixed from localization [Semenoff Young][Correa,Henn Maldacena,Sever] [Fiol,Garolera Lewkowycz]

$$B(\lambda) = \frac{1}{4\pi^2}\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}\,.$$

- This is of course consistent with the explicit calculations presented above.

Results to order b^4

- We expand the integrand of the one-loop diagram to order b^4 and integrate. The result is

$$
\begin{aligned}
\langle W\rangle_4 \sim \frac{\lambda}{4}\sum_{n=1}^{\infty}&\left[\frac{1}{6}(n^2-1)n(7n^2+3)|b_n|^4 - n^3(5n^2-1)b_n^3 b_{-3n}\right] \\
&- \frac{\lambda}{4}\sum_{0<n<m}\Big[n^2 m(7n^2+6mn+2m^2-3)b_n^2 b_m b_{-m-2n} \\
&\qquad + \frac{2}{3}n(n^4-3n^3m-6nm^3+n^2m^2-4n^2+9nm-m^2+3)|b_n b_m|^2\Big] \\
&- \frac{\lambda}{4}\sum_{0<n<m<l} 2nml(2n^2+2m^2+2l^2+3nm+3ml+3ln-3)b_n b_m b_l b_{-l-m-n}
\end{aligned}
$$

- We have the results also for two loops. They are more complicated.
- We are still studying the structure of these expressions.

Near circular holographic Wilson loop

- The holographic dual of a Wilson loop in $\mathbb{R}^2$ is a fundamental string in AdS_3.
- Using the Pohlmeyer reduction, one ends with the generalized cosh-Gordon equation

$$\partial\bar{\partial}\alpha = e^{2\alpha} + |f|^2 e^{-2\alpha}$$

  where f is an arbitrary holomorphic function.
- Given α and f, the regularized action is given by the integral over the disc

$$A_{\rm reg} = -2\pi - 2\int_\Sigma dz d\bar{z} |f(z)|^2 e^{-2\alpha(z,\bar{z})}\,,$$

- The shape of the Wilson loop is obscured in this description...
- For the circle $f = 0$ and $\alpha = -\ln(1 - z\bar{z})$, so for nearly circular Wilson loops we can take small f

$$f(z) = \epsilon \sum_{p=0}^{\infty} a_p z^p\,.$$

  and solve for α perturbatively.

- Using this one can evaluate the string action as a power series in ϵ

$$
\begin{aligned}
A = & -2\pi + \epsilon^2 \sum_{p=2}^{\infty} \frac{4}{p(p^2-1)} |a_{p-2}|^2 \\
& + 16\pi\epsilon^4 \sum_{p>q>r\geq 2} S_{p,q,r} \frac{a_{p-2}\bar{a}_{q-2}\bar{a}_{r-2}a_{q+r-p-2} + \bar{a}_{p-2}a_{q-2}a_{r-2}\bar{a}_{q+r-p-2}}{p(p^2-1)q(q^2-1)r(r^2-1)(q+r)((q+r)^2-1)} \\
& \qquad \big(p^2(q+r)((q+r)^2-1) + pqr(q^2+3qr+r^2+1) \\
& \qquad - (q+r)(q^4+q^3r-q^2r^2+qr^3+r^4-q^2-qr-r^2)\big)
\end{aligned}
$$

With the symmetry factor

$$
S_{p,q,r} = \begin{cases} \frac{1}{8}\,, & p=q=r\,, \\ \frac{1}{2}\,, & p=q \quad \text{or} \quad q=r\,, \\ 1\,, & \text{otherwise.} \end{cases}
$$

- There is also a prescription to derive the shape of the Wilson loop perturbatively in ϵ.
- Expressing the result in terms of b_n, at linear order in ϵ we find

$$b_n = \begin{cases} 0\,, & n = -1, 0, 1 \\ \dfrac{2\epsilon a_{n-2} e^{i\varphi}}{n(n^2-1)}\,, & n \geq 2 \\ -\dfrac{2\epsilon \bar{a}_{-n-2} e^{-i\varphi}}{n(n^2-1)}\,, & n \leq -2 \end{cases}$$

- Note that there is an extra *spectral parameter* φ in the mapping.
- Given f and α there is a one-parameter family of different string solutions, (and Wilson loops). All have the same area, so the same VEV for the Wilson loop at strong coupling.
- Dekel studied many examples of curves and found that at weak coupling there is a dependence on φ, but only at order ϵ^8 or higher.
- One of the reasons for our examination was to verify whether this is correct and to try to understand why. What is the analog of the spectral parameter at weak coupling?

- To the next order in ϵ

$$
\begin{aligned}
g(\theta) = \epsilon \sum_{p=2}^{\infty} &\left(\frac{2a_{p-2}e^{i(p\theta+\varphi)}}{p(p^2-1)} + \frac{2\bar{a}_{p-2}e^{-i(p\theta+\varphi)}}{p(p^2-1)} \right) \\
&+ \epsilon^2 \sum_{p=0}^{\infty} \left[\frac{a_{p-2}^2(5p^2+1)e^{2i(p\theta+\varphi)}}{p(p^2-1)^2(4p^2-1)} - \frac{4|a_{p-2}|^2}{p^2(p^2-1)^2} - \frac{\bar{a}_{p-2}^2(5p^2+1)e^{-2i(p\theta+\varphi)}}{p(p^2-1)^2(4p^2-1)} \right] \\
&+ \sum_{p>q} \left[\frac{4a_{p-2}a_{q-2}(p^2+3pq+q^2+1)e^{i((p+q)\theta+2\varphi)}}{(p-1)(p+1)(q-1)(q+1)(p+q-1)(p+q)(p+q+1)} \right. \\
&\qquad - \frac{4\bar{a}_{p-2}\bar{a}_{q-2}(p^2+3pq+q^2+1)e^{-i((p+q)\theta+2\varphi)}}{(p-1)(p+1)(q-1)(q+1)(p+q-1)(p+q)(p+q+1)} \\
&\qquad \left. - \frac{4a_{p-2}\bar{a}_{q-2}e^{i(p-q)\theta}}{p(p^2-1)(q-1)(q+1)} + \frac{4\bar{a}_{p-2}a_{q-2}e^{-i(p-q)\theta}}{p(p^2-1)(q-1)(q+1)} \right] \\
&+ \mathcal{O}(\epsilon^3)\,.
\end{aligned}
$$

- We calculated also the $\mathcal{O}(\epsilon^3)$ term.

- We can plug the $b_n(\epsilon, a_p)$ into the expression we found for the one loop VEV of the Wilson loop.
- The result is

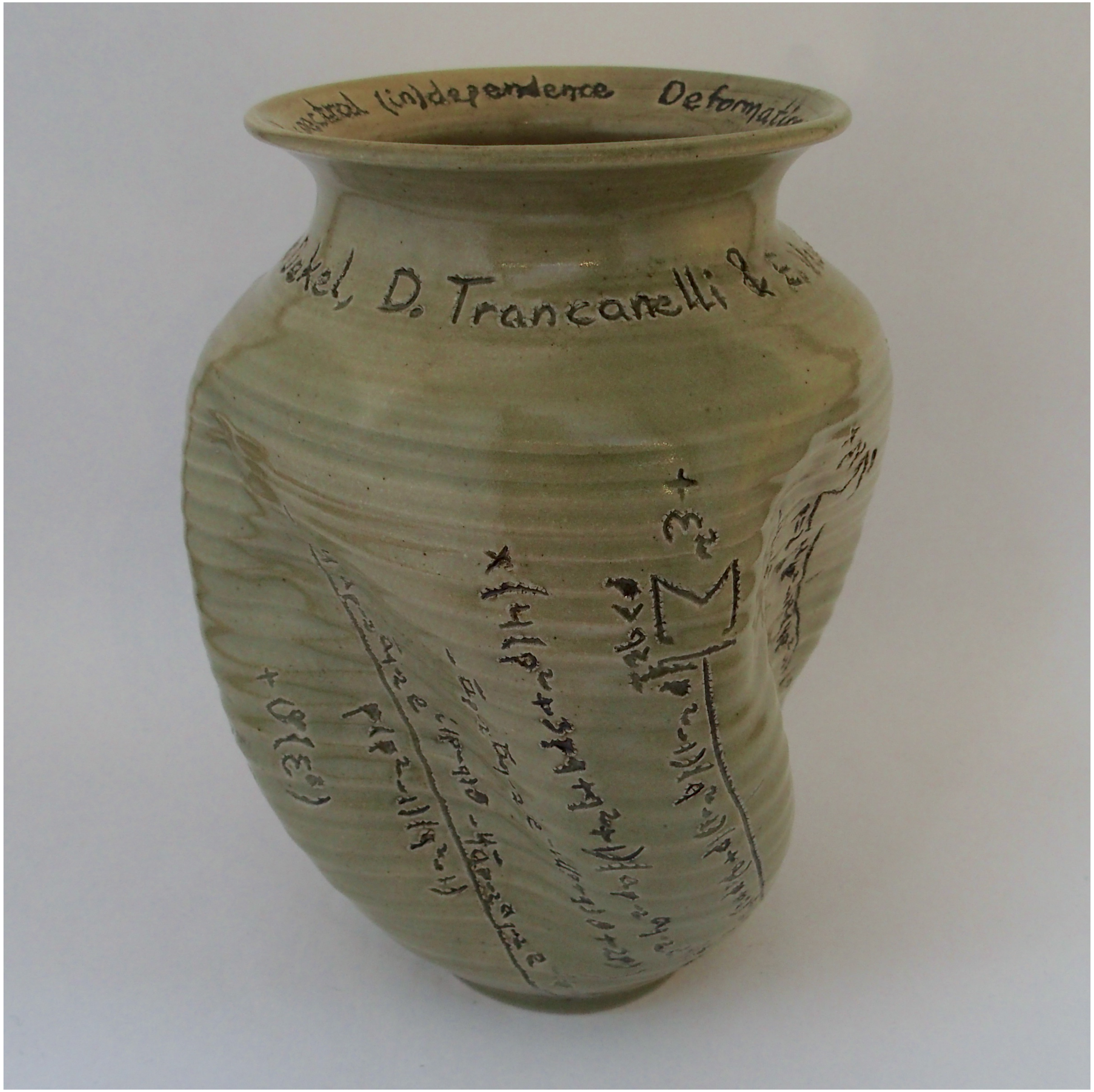
$$
\frac{\lambda}{8} + \lambda\epsilon^2 \sum_{p=2}^{\infty} \frac{4}{p(p^2-1)} |a_{p-2}|^2
$$

$$
- \frac{16}{3}\lambda\epsilon^4 \sum_{p\geq q\geq r\geq 2} S_{p,q,r} \frac{a_{p-2}\bar{a}_{q-2}\bar{a}_{r-2}a_{q+r-p-2} + \bar{a}_{p-2}a_{q-2}a_{r-2}\bar{a}_{q+r-p-2}}{p(p^2-1)q(q^2-1)r(r^2-1)(q+r)((q+r)^2-1)}
$$

$$
\big(4p^2(q+r)((q+r)^2-1) + p(-q^4+2q^3r+12q^2r^2+2qr^3-r^4+q^2+8qr+r^2)
$$

$$
-(q+r)(2q^4+3q^3r-4q^2r^2+3qr^3+2r^4+q^2-5qr+r^2-3)\big)
$$

- Note:
  - Very similar to strong coupling expression. This is obvious for order ϵ^2, because of the universality of the displacement operator, but very nontrivial for ϵ^4.
  - There is no dependence on φ at this order in λ and ϵ.
  - The same is true at order λ^2.
  - We found some examples with φ dependence at $\mathcal{O}(\epsilon^6)$.
- We don't understand the structures appearing here.


(in)dependence
D. Trancanelli &

Summary

- The circular Wilson loop in $\mathcal{N}=4$ SYM is an ideal lab to study dCFT. One can use perturbation theory, localization, integrability, OPE and AdS/CFT.
- We are calculating the expectation value of deformed circular Wilson loops or alternatively insertions into the Wilson loop.
- Explicit results for one-loop, two-loop and strong coupling.
- Explicit results for the dimensions and structure constants of insertions into the Wilson loop.
- A surprising indpendance on the spectral parameter φ at order ϵ^4.
- Are there any other hidden structures in our results?

- This calculation could have been done almost 19 years ago.
- We still find surprises in classical AdS/CFT calculations.

The end