$\mathcal{N}$=1 SQFT on curved backgrounds and rigid supersymmetry anomalies

Ioannis Papadimitriou

Korea Institute for Advanced Study

Workshop on Supersymmetric Localization and Holography:
Black Hole Entropy and Wilson Loops

ICTP, Trieste, 10 July 2018

JHEP07 (2017) 038, arXiv:1703.04299 [hep-th] and work in progress

Quantum and classical $\mathcal{Q}$-cohomology

- The expectation value of $\mathcal{Q}$-exact operators in a supersymmetric state vanishes:

$$\langle \delta_\zeta \mathcal{O}_F \rangle = \langle \{ \overline{\mathcal{Q}}_\zeta, \mathcal{O}_F \} \rangle = 0$$

- Supersymmetric localization relies on a *classical* version of this statement:

$$\langle \delta_\zeta^{cl} \mathcal{O}_F \rangle = \int [\mathcal{D}X] \{ \overline{\mathcal{Q}}_\zeta, \mathcal{O}_F \}_{cl} e^{-S[X]} = \int [\mathcal{D}X] \Big\{ \overline{\mathcal{Q}}_\zeta, \mathcal{O}_F e^{-S[X]} \Big\}_{cl} = 0$$

  where the last step *assumes* that the integration measure commutes with $\overline{\mathcal{Q}}_\zeta$, i.e.

$$\delta_\zeta = \delta_\zeta^{cl}$$

- In this talk I will provide evidence that the quantum and classical $\mathcal{Q}$-cohomologies do not always coincide and will discuss some of the consequences

A familiar example

- $\mathcal{N} = 1$ super Virasoro

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$
$$[L_m, G_r] = \frac{1}{2}(m-2r)G_{m+r}$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}$$

- We will see that a similar anomaly exists in 4 (and 6) dimensions, except that it only deforms the supersymmetry algebra on curved backgrounds

- In 4d flat space

$$(\delta_\zeta \mathcal{S}^i)_{\alpha\dot\beta} = \{\overline{Q}_{\dot\beta}, \mathcal{S}_{i\alpha}\} = \sigma^j_{\alpha\dot\beta}\Big(2\mathcal{T}_{ij} - i\eta_{ij}\partial^k \mathcal{J}_k + i\partial_j \mathcal{J}_i + \frac{1}{2}\epsilon_{ijkl}\partial^k \mathcal{J}^l\Big)$$

where $\mathcal{T}^{ij}$ is the stress tensor and $\mathcal{J}^i$ the R-current

- On curved background admitting a (conformal) Killing spinor ζ_+, i.e. $\mathcal{D}_i\zeta_+ = \Gamma_i\zeta_-$

$$\{\overline{\mathcal{Q}}[\zeta], \mathcal{S}^i\} = -\frac{1}{2}\mathcal{T}^{ij}\Gamma_j\zeta_+ + \frac{i}{8\sqrt{3}}\Gamma^{ijk}(\Gamma_{kl} - 2g_{kl})\zeta_+ D_j \mathcal{J}^l$$
$$+\frac{i}{2\sqrt{3}}\Big(\Gamma^i_l - 3\delta^i_l\Big)\zeta_- \mathcal{J}^l + \mathcal{A}^i_\zeta[g, A]$$

where $\mathcal{A}^i_\zeta[g, A]$ is a local functional of the background and represents an rigid supersymmetry anomaly

Outline

Outline

4d $\mathcal{N} = 1$ supercurrent multiplets

- $\mathcal{N} = 1$ supercurrent multiplets: $\mathcal{T}^{ij}$, $\mathcal{S}^i$, $(\mathcal{J}^i)$, auxiliary fields
- They are components of a real vector superfield

$$\mathbb{S}^i = \mathcal{J}^i + \bar{\theta}\mathcal{S}^i + \overline{\mathcal{S}}^i\theta + 2(\bar{\theta}\sigma_j\theta)\mathcal{T}^{ij} + \cdots$$

- Possible current multiplets differ in auxiliary field content and *improvement terms*:

$$\mathcal{T}_{ij} \to \mathcal{T}'_{ij} = \mathcal{T}_{ij} + (\eta_{ij}\partial^2 - \partial_i\partial_j)t, \qquad \mathcal{S}_{i\alpha} \to \mathcal{S}'_{i\alpha} = \mathcal{S}_{i\alpha} + (\sigma_{ij})_\alpha^{\ \beta}\partial^j s_\beta$$

- The **S-multiplet** [Komargodski, Seiberg '10] always exists and comprises 16+16 off-shell degrees of freedom in the real superfield $\mathbb{S}_{\alpha\dot{\alpha}}$, an auxiliary chiral superfield X, and an auxiliary spinor (chiral fieldstrength) superfield χ_α, satisfying

$$\overline{D}^{\dot{\alpha}}\mathbb{S}_{\alpha\dot{\alpha}} = D_\alpha X + \chi_\alpha, \qquad \overline{D}_{\dot{\alpha}} X = 0, \qquad \overline{D}_{\dot{\alpha}}\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} - D^\alpha \chi_\alpha = 0$$

- The **Ferrara-Zumino (FZ)-multiplet** is obtained by setting $\chi_\alpha = 0$ and comprises 12+12 off-shell degrees of freedom. It exists if there are no FI terms and the Kähler form of the target space is exact.

- The **R-multiplet** is obtained by setting $X = 0$ and contains also 12+12 off-shell degrees of freedom. It exists if there is a $U(1)_R$ symmetry.

- These defining relations correspond to classical Ward identities

Linear coupling to supergravity

- The background supergravity fields reside in a real vector superfield $\mathbb{H}_i$ that to linear order couples to the current superfield as

$$\int d^4\theta \; \mathbb{S}^i \mathbb{H}_i$$

- Gauging the global symmetries amounts to assigning a local gauge transformation to the background superfield

$$\mathbb{H}_{\alpha\dot{\alpha}} \to \mathbb{H}'_{\alpha\dot{\alpha}} = \mathbb{H}_{\alpha\dot{\alpha}} + D_\alpha \overline{L}_{\dot{\alpha}} - \overline{D}_{\dot{\alpha}} L_\alpha$$

  and demanding that the above linear coupling is gauge invariant.

- These local transformations include diffeomorphisms, local frame rotations, Weyl and $U(1)$ gauge transformations, as well as local Q- and S-supersymmetry transformations

- The defining relations of the supercurrent multiplet, i.e. the Ward identities, follow from the Noether's procedure.

Local transformation of the currents

- The linear coupling

$$W[\cdots, \mathbb{H}] = \cdots + \int d^4x \int d^4\theta\, \mathbb{S}^i \mathbb{H}_i$$

  in the effective action implies that the supercurrent multiplet operators can be defined in the Local Renormalization Group sense [Osborn '94] as

$$\mathbb{S}^i = \frac{\delta W}{\delta \mathbb{H}_i}$$

- This defines the **consistent** current multiplet, which couples supergravity. The **covariant** current multiplet differs by Bardeen-Zumino terms [Bardeen, Zumino '84]

- The transformation of the current superfield under the local symmetries is given by

$$\delta_L \mathbb{S}^i = \delta_L\left(\frac{\delta}{\delta \mathbb{H}_i}\right) W + \frac{\delta}{\delta \mathbb{H}_i} \delta_L W$$

  where the second term is non-zero only in the presence of 't Hooft anomalies

Ward identities as first class constraints

- An elegant way to compute the gauge transformation of local operators is utilizing an underlying symplectic structure
- The superfields $\mathbb{S}^i$ and $\mathbb{H}_i$ parameterize a symplectic manifold equipped with the Poisson bracket

$$\{,\}_{\text{PB}} = \int d^4x \int d^4\theta \Big(\frac{\delta}{\delta \mathbb{H}_i} \frac{\delta}{\delta \mathbb{S}^i} - \frac{\delta}{\delta \mathbb{S}^i} \frac{\delta}{\delta \mathbb{H}_i} \Big)$$

- The functional

$$\mathcal{C}[L] = \int d^4x \int d^4\theta \, L^\alpha \Big(\overline{D}^{\dot\alpha} \mathbb{S}_{\alpha\dot\alpha} - D_\alpha X - \chi_\alpha \Big) + \text{h.c.}$$

  is a first class constraint generating local gauge symmetries, i.e.

$$\{\mathcal{C}[L], \mathbb{H}_i\}_{\text{PB}} = -\frac{\delta \mathcal{C}[L]}{\delta \mathbb{S}^i} = D_\alpha \overline{L}_{\dot\alpha} - \overline{D}_{\dot\alpha} L_\alpha = \delta_L \mathbb{H}_i$$

- The gauge transformation of the current superfield is then given by

$$\boxed{\{\mathcal{C}[L], \mathbb{S}_i\}_{\text{PB}} = \delta_L \mathbb{S}^i = \frac{\delta \mathcal{C}[L]}{\delta \mathbb{H}_i}}$$

Killing symmetries and conserved charges

- So far the background fields in $\mathbb{H}_i$ and the gauge parameters in L^α are arbitrary
- For a given background $\mathbb{H}_i$, the gauge parameters L_o^α that satisfy

$$\boxed{\delta_{L_o}\mathbb{H}_i = D_\alpha \overline{L}_{o\dot\alpha} - \overline{D}_{\dot\alpha} L_{o\alpha} = 0}$$

  correspond to **Killing symmetries** of the background $\mathbb{H}_i$
- The Killing spinor of rigid supersymmetry corresponds to a specific component of the superfield L_o^α
- The conserved charges $\mathbb{Q}[L_o]$ associated with the Killing symmetries can be obtained through the Ward identities
- The **quantum** transformation of the currents under the Killing symmetries is

$$\boxed{\{\mathbb{Q}[L_o], \mathbb{S}^i] = \delta_{L_o}\mathbb{S}^i}$$

  which includes the anticommutators $\{\overline{Q}_{\dot\beta}, \mathcal{S}_{i\alpha}\}$ and $\{Q_\beta, \mathcal{S}_{i\alpha}\}$

- e.g. for the S-multiplet in flat space

$$\{\overline{Q}_{\dot\beta}, \mathcal{S}_{i\alpha}\} = \sigma^j_{\alpha\dot\beta}\Big(2\mathcal{T}_{ij} + \frac{1}{2}\epsilon_{ijkl}F^{kl} - i\eta_{ij}\partial^k\mathcal{J}_k + i\partial_j\mathcal{J}_i + \frac{1}{2}\epsilon_{ijkl}\partial^k\mathcal{J}^l\Big)$$

$$\{Q_\beta, \mathcal{S}_{i\alpha}\} = 2i\epsilon_{\lambda\beta}(\sigma_{ij})^\lambda_\beta\partial^j x^\dagger$$

where F_{ij} the closed two-form and the complex scalar x are auxiliary fields

Non linear coupling to supergravity

- To couple the theory non linear to supergravity one can use the **Festuccia-Seiberg argument** [Festuccia, Seiberg '11]
- The superconformal 't Hooft anomalies can be determined for arbitrary a and c anomaly coefficients by solving the Wess-Zumino consistency conditions
- For the Ferrara-Zumino multiplet this has been done in curved superspace by [Bonora, Pasti, Tonin '85]
- Extracting the fermionic components is still non trivial...

Quantum Ward identities from holography

- Minimal gauged supergravity in 5D holographically describes the current multiplet of $\mathcal{N}=1$ SCFTs in 4d, coupled to off-shell (conformal) supergravity
- It only describes theories with $a=c$
- The arbitrary sources of the bulk fields

$$e^a_{(0)i}, \qquad \Psi_{(0)+i}, \qquad A_{(0)i}$$

  specify an arbitrary (non-linear) field theory background
- The variation of the renormalized on-shell supergravity action defines the conjugate (consistent) current operators via

$$\delta W = \int d^dx\sqrt{-g_{(0)}}\Big(- \mathcal{T}^i_a \delta e^a_{i\,(0)} + \mathcal{J}^i\delta A_{(0)i} + \overline{\mathcal{S}}^i\delta\Psi_{(0)+i} + \delta\overline{\Psi}_{(0)+i}\mathcal{S}^i\Big)$$

Fermionic Ward identities

- Supersymmetric holographic renormalization determines the Ward identities

$$\mathcal{D}_i\mathcal{S}^i + \frac{1}{2}\mathcal{T}^i_a\Gamma^a\Psi_{(0)+i} - \frac{i}{8\sqrt{3}}\mathcal{J}^i(\Gamma_{ij} - 2g_{(0)ij})\Gamma^{jpq}\mathcal{D}_p\Psi_{(0)+q} = \mathcal{A}_S$$

$$\Gamma_i\mathcal{S}^i - \frac{i\sqrt{3}}{4}\mathcal{J}^i\Psi_{(0)+i} = \mathcal{A}_{sW}$$

including the **'t Hooft anomalies**

$$\mathcal{A}_S = \frac{ic}{18}\,\epsilon^{iskl}F_{(0)sk}A_{(0)l}(\Gamma_{ij} - 2g_{(0)ij})\Gamma^{jpq}\mathcal{D}_p\Psi_{(0)+q}$$

$$\mathcal{A}_{sW} = \frac{c}{2}\Big[\frac{\ell^2}{4}\Big(R_{ij} - \frac{1}{6}Rg_{(0)ij}\Big)\Gamma^i\Gamma^{jkl}\mathcal{D}_k\Psi_{(0)+l} + \frac{2i}{3}\,\epsilon^{ijkl}F_{(0)jk}A_{(0)l}\Psi_{(0)+i} + \frac{i}{4\sqrt{3}}F_{(0)jk}(2\Gamma^{jk}\Gamma^i - 3\Gamma^{jki})\Gamma_i{}^{pq}\mathcal{D}_p\Psi_{(0)+q}\Big]$$

- Moving the orange terms to the LHS of the fermionic Ward identities shifts the R-current from the **consistent** to the **covariant** (and gauge invariant) one:

$$\mathcal{J}^i \to \mathcal{J}^i{}_{\rm cov} = \mathcal{J}^i + \frac{4c}{3\sqrt{3}}\epsilon^{ijkl}F_{(0)jk}A_{(0)l}$$

Fermionic transformations of the sources

- The local supersymmetry and superWeyl transformations of the background fields, parameterized respectively by the spinors $\epsilon_{o+}(x)$ and $\epsilon_{o-}(x)$, are:

$$\delta_{\epsilon_{o+},\epsilon_{o-}} e^a_{i\,(0)} = \frac{1}{2}(\bar{\epsilon}_{o+}\Gamma^a\Psi_{(0)+i} - \overline{\Psi}_{(0)+i}\Gamma^a\epsilon_{o+}),$$

$$\delta_{\epsilon_{o+},\epsilon_{o-}} A_{(0)i} = \frac{i}{4\sqrt{3}}\Big(\overline{\Psi}_{(0)+i}\epsilon_{o-} + \overline{\Psi}_{(2)-i}\epsilon_{o+} - \bar{\epsilon}_{o+}\Psi_{(2)-i} - \bar{\epsilon}_{o-}\Psi_{(0)+i}\Big),$$

$$\delta_{\epsilon_{o+},\epsilon_{o-}} \Psi_{(0)+i} = \mathcal{D}_{(0)i}\epsilon_{o+} - \Gamma_{(0)i}\epsilon_{o-}$$

where

$$\Psi_{(2)-i} = -\frac{1}{6}(\Gamma_{(0)ij} - 2g_{(0)ij})\Gamma^{jkl}_{(0)}\mathcal{D}_{(0)k}\Psi_{(0)+l}.$$

- These are the transformations of **off-shell $\mathcal{N}=1$ conformal supergravity**

Fermionic transformations of the supercurrent

- The local supersymmetry and superWeyl transformations of the supercurrent are:

$$\delta_{\epsilon_{o+}}\mathcal{S}^i = -\frac{1}{2}\mathcal{T}^i_a\Gamma^a\epsilon_{o+} + \frac{i\ell}{8\sqrt{3}}\Gamma^{ijk}_{(0)}(\Gamma_{(0)kl} - 2g_{(0)kl})\mathcal{D}_{(0)j}\left[\left(\mathcal{J}^l + \frac{4c}{3\sqrt{3}}\epsilon^{lpqs}F_{(0)pq}A_{(0)s}\right)\epsilon_{o+}\right]$$

$$\delta_{\epsilon_{o-}}\mathcal{S}^i = -\frac{i\sqrt{3}}{4}\left(\mathcal{J}^i + \frac{4c}{3\sqrt{3}}\epsilon^{lpqs}F_{(0)pq}A_{(0)s}\right)\epsilon_{o-} - \frac{c}{8}\Gamma^{ijk}_{(0)}\Gamma^l_{(0)}\mathcal{D}_{(0)j}\left[\left(R_{kl}[g_{(0)}] - \frac{1}{6}R[g_{(0)}]g_{(0)kl}\right)\epsilon_{o-}\right] - \frac{ic}{8\sqrt{3}}\Gamma^{ij}_{(0)}{}_k\left(2\Gamma^k_{(0)}\Gamma^{pq}_{(0)} - 3\Gamma^{kpq}_{(0)}\right)\mathcal{D}_{(0)j}(F_{(0)pq}\epsilon_{o-})$$

- Notice contribution from ’t Hooft anomalies!

Outline

Notions of rigid supersymmetry

- Covariantly constant spinors (very restrictive):

$$\nabla_\mu \zeta = 0$$

- Twistor equation:

$$\nabla_\mu \zeta = \sigma_\mu \widetilde{\eta}, \qquad \widetilde{\eta} = -\frac{1}{4}\widetilde{\sigma}^\mu \nabla_\mu \zeta$$

- Twist by a line bundle (Kähler base):

$$(\nabla_\mu - iA_\mu)\zeta = 0$$

- Twistor equation twisted by line bundle (**conformal supergravity**):

$$(\nabla_\mu - iA_\mu)\zeta = \sigma_\mu \widetilde{\eta}$$

- **New minimal supergravity**:

$$(\nabla_\mu - iA_\mu)\zeta = -iV_\mu \zeta - iV^\nu \sigma_{\mu\nu}\zeta, \qquad \nabla_\mu V^\mu = 0$$

Classification of solutions

- Killing spinor equations have been studied extensively and the manifolds that support Killing spinors have been largely classified.
- Killing spinors of new minimal and conformal supergravity and the restrictions they impose on the manifold $\mathcal{M}$ were studied in [Klare, Tomansiello, Zaffaroni: 1205.1062; Dumitrescu, Festuccia, Seiberg: 1205.1115]
- $\mathcal{N} = 1$ theories in 4d can be coupled to different background supergravities [Festuccia, Seiberg: 1105.0689]. The Killing spinor equations arise from the gravitino variations of the corresponding supergravity.
- This talk concerns local properties of $\mathcal{M}$ and so the difference between the new minimal and conformal supergravity spinor equations is not important.
- Rigid supersymmetry is independent of the particular theory, since it only depends on the background supergravity fields!

Manifolds with two KSs of opposite R-charge

- Manifolds that admit two Killing spinors, ζ and $\widetilde{\zeta}$, of opposite R-charge are T^2 fibrations over a Riemann surface with metric

$$ds^2 = \Omega(z,\overline{z})^2 \Big((dw + h(z,\overline{z})dz)(d\overline{w} + \overline{h}(z,\overline{z})d\overline{z}) + c(z,\overline{z})^2 dz d\overline{z} \Big)$$

- Such manifolds possess a complex Killing vector $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$ that commutes with its conjugate.

- In Lorentzian signature ζ and $\widetilde{\zeta}$ are related by complex conjugation.

- I will focus on the special case when one cycle is trivially fibered:

$$\begin{aligned} ds^2 &= \Omega(z,\overline{z})^2 d\tau^2 + ds^2_{\mathcal{M}_3} \\ ds^2_{\mathcal{M}_3} &= \Omega(z,\overline{z})^2 \Big((d\psi + a(z,\overline{z}) + \overline{a}(z,\overline{z})d\overline{z})^2 + c(z,\overline{z})^2 dz d\overline{z} \Big) \end{aligned}$$

- By dimensional reduction, such backgrounds are related to Seifert manifolds in 3d and, if the second cycle is also trivially fibered, to the A-twist in 2d.

- Examples: $S^3 \times S^1$, $L(r,s) \times S^1$, where $L(r,s)$ is a Lens space.

Outline

Partition functions on Hermitian manifolds

- Closset, Dumitrescu, Festuccia and Komargodski studied the dependence of general supersymmetric partition functions on the geometric data of generic Hermitian manifolds $\mathcal{M}_4$ and the related line bundles.
- They first studied the linearized deformation problem around flat space [1309.5876] and later the non-linear problem by means of the holomorphic twist [1407.2598].
- For $\mathcal{M}_4$ that admit two Killing spinors of opposite R-charge they find that $Z_{\mathcal{M}_4}$:
  - does not depend on the Hermitian metric on $\mathcal{M}_4$
  - depends holomorphically on a subset of the complex structure and line bundle moduli
- In the case when one T^2 cycle is trivially fibered, the above conditions imply that the partition function is independent of the functions $a(z,\overline{z})$, $\overline{a}(z,\overline{z})$ and $c(z,\overline{z})$.

- The R-multiplet for $\mathcal{N}=1$ theories with a $U(1)_R$ symmetry in 4d contains the following operators:

$$j_\mu^{(R)}, \quad S_{\alpha\mu}, \quad \widetilde{S}_\mu^{\dot\alpha}, \quad T_{\mu\nu}, \quad \mathcal{F}^{\mu\nu}$$

- These operators couple to the background fields in new minimal supergravity:

$$A_\mu^{(R)}, \quad \Psi_{\alpha\mu}, \quad \widetilde{\Psi}^{\dot\alpha}{}_\mu, \quad g_{\mu\nu}, \quad B_{\mu\nu}$$

- The (flat space) supersymmetry algebra determines

$$\boxed{\left\{Q, \frac{1}{|\zeta|^2}\zeta^\dagger\sigma_\rho\widetilde{S}_\mu\right\} = -2i(\delta^\nu{}_\rho + iJ^\nu{}_\rho)\mathcal{T}_{\mu\nu}}$$

where

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\lambda} - \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho j^{(R)\lambda} - \frac{i}{2}\partial_\nu j_\mu^{(R)}$$

is conserved, $\partial^\mu\mathcal{T}_{\mu\nu}=0$, and the complex structure is given by

$$J^\nu{}_\rho = -\frac{2i}{|\zeta|^2}\zeta^\dagger\sigma^\nu{}_\rho\zeta$$

- The variation of the partition function with respect to the background fields is given by the linearized coupling of the R-multiplet operators to supergravity:

$$\Delta\mathcal{L} = -\frac{1}{2}\Delta g^{\mu\nu}T_{\mu\nu} + \Delta A^{(R)\mu} j_\mu^{(R)} + \frac{i}{4}\varepsilon^{\mu\nu\rho\lambda}\Delta B_{\mu\nu}\mathcal{F}_{\rho\lambda}$$

- An explicit calculation shows that the variation of the partition function with respect to the geometric data parameterizing the Hermitian manifold $\mathcal{M}_4$ around flat space, up to a total derivative, takes the form

$$\Delta\mathcal{L} = -\Delta g^{i\bar{j}}\mathcal{T}_{i\bar{j}} - i\sum_{j=\bar{j}}\Delta J^{\bar{j}}{}_j\mathcal{T}_{\bar{j}\bar{i}}$$

- Since $\mathcal{T}_{\mu\nu}$ is Q-exact, this completes the proof.

- **Caveat:** the argument relies on the classical supersymmetry algebra

Supersymmetric partition function from holography

- [Genolini, Cassani, Martelli, Sparks: 1612.06761] **holographically** computed the variation of the supersymmetric partition function with respect to the geometric data parameterizing the Lorentzian (conformal supergravity) backgrounds

$$
\begin{aligned}
ds^2_{(0)} &= -\mathrm{d}t^2 + \left(\mathrm{d}\psi + \frac{\mathrm{i}}{2}\partial_{\bar z}\mu \mathrm{d}\bar z - \frac{\mathrm{i}}{2}\partial_z \mu \mathrm{d}z\right)^2 + 4e^w \mathrm{d}z\mathrm{d}\bar z, \\
A^{\text{Conf.}}_{(0)} &= -\frac{1}{\sqrt{3}}\Big[-\frac{1}{8}e^{-w}\partial_z\partial_{\bar z}\mu \mathrm{d}t + \frac{1}{4}e^{-w}\partial_z\partial_{\bar z}\mu\left(\mathrm{d}\psi + \frac{\mathrm{i}}{2}\partial_{\bar z}\mu\mathrm{d}\bar z - \frac{\mathrm{i}}{2}\partial_z\mu\mathrm{d}z\right) \\
&\qquad + \frac{\mathrm{i}}{4}(\partial_{\bar z}w\mathrm{d}\bar z - \partial_z w \mathrm{d}z) + \gamma'\mathrm{d}t + \gamma\mathrm{d}\psi + \mathrm{d}\lambda\Big]
\end{aligned}
$$

  where $w(z,\bar z)$ and $\mu(z,\bar z)$ are arbitrary functions, and γ', γ and $\lambda(z,\bar z)$ are locally pure gauge but contain global information.

- These are analytically continued versions of the T^2-fibrations with one trivial fiber.

- Killing spinor equation (and complex conjugate)

$$
\mathcal{D}_{(0)i}\zeta_+ = \Gamma_{(0)i}\zeta_-, \qquad \zeta_- = \frac{1}{4}\Gamma^j_{(0)}\mathcal{D}_{(0)_j}\zeta_+ \neq 0
$$

- Under a local deformation of the function $w(z,\bar{z})$, keeping $\mu(z,\bar{z})$ fixed:

$$\delta_w W = \frac{1}{2^6 3\kappa^2}\int d^4x\sqrt{-g_{(0)}}\,\delta w\Big(-u^2 R_{2d} - \frac{1}{2}\Box_{2d}u^2 + \frac{19}{32}u^4 + \frac{8}{9}(\gamma+2\gamma')(2uR_{2d} + 2\Box_{2d}u - u^3)\Big)$$

where $u = e^{-w}\partial_z\partial_{\bar{z}}\mu$.

- Under a local deformation of the function $\mu(z,\bar{z})$, keeping $w(z,\bar{z})$ fixed:

$$\delta_\mu W = \frac{1}{2^9 3^2\kappa^2}\int d^4x\sqrt{-g_{(0)}}(e^{-w}\partial_z\partial_{\bar{z}}\delta\mu)\Big(24uR_{2d} - 19u^3 + \frac{32}{3}(\gamma+2\gamma')(3u^2 - 4R_{2d})\Big)$$

Outline

Rigid SUSY transformation of the supercurrent

- Restricting the local fermionic parameters $\epsilon_{o+}(x)$ and $\epsilon_{o-}(x)$ in the local transformation of the supercurrent to the conformal Killing spinor (ζ_+, ζ_-) gives the transformation of the supercurrent under **rigid supersymmetry**:

$$
\begin{aligned}
\delta_\zeta \mathcal{S}^i = & -\frac{1}{2}\mathcal{T}^{ij}\widehat{\Gamma}_{(0)j}\zeta_+ \\
& + \frac{i}{8\sqrt{3}}\Gamma^{ijk}_{(0)}(\Gamma_{(0)kl} - 2g_{(0)kl})\zeta_+ D_{(0)j}\left(\mathcal{J}^l + \frac{4c}{3\sqrt{3}}\widehat{\epsilon}^{lpqs}F_{(0)pq}A_{(0)s}\right) \\
& + \frac{i}{2\sqrt{3}}(\Gamma^i_{(0)l} - 3\delta^i_l)\zeta_-\left(\mathcal{J}^l + \frac{4c}{3\sqrt{3}}\epsilon^{lpqs}F_{(0)pq}A_{(0)s}\right) \\
& - \frac{c}{8}\Gamma^{ijk}_{(0)}\Gamma^l_{(0)}\mathcal{D}_{(0)j}\left[\left(R_{kl}[g_{(0)}] - \frac{1}{6}R[g_{(0)}]g_{(0)kl}\right)\zeta_-\right] \\
& - \frac{ic}{8\sqrt{3}}\Gamma^{ij}_{(0)k}\left(2\Gamma^k_{(0)}\Gamma^{pq}_{(0)} - 3\widehat{\Gamma}^{kpq}_{(0)}\right)\mathcal{D}_{(0)j}(F_{(0)pq}\zeta_-)
\end{aligned}
$$

- The anomalous terms in this transformation are non-vanishing for this class of supersymmetric backgrounds, even though all bosonic anomalies in the Ward identities are numerically zero!

- Under a local deformation of the function $w(z,\bar{z})$, keeping $\mu(z,\bar{z})$ fixed:

$$\delta_w W = \int d^4x\sqrt{-g_{(0)}}\;\delta w\;\mathrm{i}\sqrt{2}e^{w/2}\Big(\;\delta_\zeta^{\text{anom}}\mathcal{S}^z\Big|_1 + \delta_\zeta^{\text{anom}}\mathcal{S}^z\Big|_2\Big)$$
$$= \frac{1}{2^6 3\kappa^2}\int d^4x\sqrt{-g_{(0)}}\;\delta w\Big(-u^2R_{2d} - \frac{1}{2}\Box_{2d}u^2 + \frac{19}{32}u^4$$
$$+\frac{8}{9}(\gamma+2\gamma')(2uR_{2d} + 2\Box_{2d}u - u^3)\Big)$$

where $u = e^{-w}\partial_z\partial_{\bar{z}}\mu$ and we have used the fact that $\langle\delta_\zeta\mathcal{S}^i\rangle_{\text{susy}} = 0$.

- Under a local deformation of the function $\mu(z,\bar{z})$, keeping $w(z,\bar{z})$ fixed:

$$\delta_\mu W = \int d^4x\sqrt{-g_{(0)}}\Big\{\sqrt{2}\Big[\frac{\mathrm{i}}{2}\Big(\;\delta_\zeta^{\text{anom}}\mathcal{S}^{\bar{z}}\Big|_1 - \delta_\zeta^{\text{anom}}\mathcal{S}^{\bar{z}}\Big|_2\Big)$$
$$+\frac{1}{4}e^{-\frac{w}{2}}\Big(\;\delta_\zeta^{\text{anom}}\mathcal{S}^t\Big|_1 + \delta_\zeta^{\text{anom}}\mathcal{S}^t\Big|_2\Big)\Big]\partial_{\bar{z}}\delta\mu + \text{h.c.}\Big\}$$
$$= \frac{1}{2^9 3^2\kappa^2}\int d^4x\sqrt{-g_{(0)}}(e^{-w}\partial_z\partial_{\bar{z}}\delta\mu)\Big(24uR_{2d} - 19u^3$$
$$+\frac{32}{3}(\gamma+2\gamma')(3u^2 - 4R_{2d})\Big)$$

w and μ dependence of the partition function

- The final expressions in these variations agree with [Genolini, Cassani, Martelli, Sparks: 1612.06761], but the actual calculation and the explanation provided for the non-invariance of the partition function are different:
- We follow the argument of [Closset, Dumitrescu, Festuccia, Komargodski: 1309.5876, 1407.2598], except that:
  - we consider infinitesimal variations of the partition function around a generic Hermitian four-manifold (within the class specified above), instead of flat space
  - we have used the **quantum** transformation of the supercurrent, which is anomalous

  This leads to a specific dependence of the supersymmetric partition function on the complex structure moduli, i.e. $\mu(z, \bar{z})$, and the Hermitian metric, i.e. $w(z, \bar{z})$.
- Since the anomalous transformation of the supercurrent is derived for **local** supersymmetry transformations, it can be applied to any Hermitian manifold, beyond the class considered here.

Outline

Casimir charges and the BPS relation

- The conserved charges can be obtained from the Ward identities
- For the supersymmetric backgrounds specified above they take the form:

$$\mathcal{Q}_{\rm e}^{\omega} = \frac{1}{\sqrt{3}} \int d\sigma_i \left(\langle \mathcal{J}^i \rangle - \omega \frac{2\ell}{3\sqrt{3}\,\kappa^2} \epsilon^{ipqs} F_{(0)pq} A_{(0)s} \right)$$

$$\mathcal{Q}^{\omega}[\mathcal{K}] = -\int d\sigma_i \left[\langle \mathcal{T}^i_j \rangle - \left(\langle \mathcal{J}^i \rangle - \omega \frac{2\ell}{3\sqrt{3}\,\kappa^2} \epsilon^{ipqs} F_{(0)pq} A_{(0)s} \right) A_{(0)j} \right] \mathcal{K}^j$$

where the parameter ω is arbitrary. $\omega = -2$ corresponds to the **Maxwell charges** and $\omega = 1$ to the **Page charges**.

- Contracting the identity $\langle \delta_\zeta \mathcal{S}^i \rangle_{\rm susy} = 0$ with $\mathrm{i}\overline{\zeta}_+$ leads to the BPS relation

$$\boxed{M^{\omega} + J^{\omega} + (\gamma - \gamma')\mathcal{Q}_{\rm e}^{\omega} = \mathcal{Q}_{\rm anomaly}^{\omega}.}$$

where

$$M^{\omega} = \mathcal{Q}^{\omega}[-\partial_t], \qquad J^{\omega} = \mathcal{Q}^{\omega}[\partial_\psi],$$

and $\mathcal{Q}_{\rm anomaly}^{\omega}$ is a non-vanishing **anomaly charge** that is local in the background.

Outline

Conclusions and future directions

Conclusions:

- In generic 4d curved backgrounds admitting (conformal) Killing spinors the supercurrent transforms anomalously under rigid supersymmetry
- The supersymmetric partition function on such backgrounds is not invariant under deformations of the complex structure or of the Hermitian metric
- The BPS relation between the bosonic charges is **anomalous**

Future directions:

- Determine the supercurrent anomaly for arbitrary a and c and for different background supergravities
- Concrete examples where localization computations are affected?
- The supercurrent anomaly can be converted to a gravitational anomaly using a local **non-covariant** counterterm? (For $a = c$ local counterterm given in [Genolini, Cassani, Martelli, Sparks: 1612.06761])