Black Hole Entropy from 5D Twisted Indices

based on:

S. M. Hosseini, I.Y. and A. Zaffaroni - in preparation

Itamar Yaakov
University of Tokyo - Kavli IPMU

Workshop on Supersymmetric Localization and Holography: Black Hole Entropy and Wilson Loops
International Center for Theoretical Physics (ICTP), Trieste

July 13, 2018

Black hole entropy

Identification of the microstates contributing to the entropy of black holes is a long standing problem since the work of Bekenstein and Hawking.

- String theory has had some success at computing the entropy of supersymmetric black holes [Strominger and Vafa (1996)].
- For black holes in AdS, the AdS/CFT correspondence gives, in principle, a way of counting the microstates using the dual conformal field theory.
- Attempts have been made to do this by counting operators preserving some fraction of supersymmetry in $\mathcal{N} = 4$ super-Yang-Mills, but with only partial success [Kinney, Maldacena, Minwalla, and Raju (2005), Grant, Grassi, Kim, and Minwalla (2008)].

Twisted partition functions vs black hole entropy

Benini, Hristov, and Zaffaroni matched the partition function on twisted $S^2 \times S^1$ to the entropy of a 4d black hole [Benini, Hristov, and Zaffaroni (2015)]

$$\mathcal{I}(\mathfrak{s}, \hat{\Delta}) \equiv \log Z(\mathfrak{s}, \hat{\Delta}) - i \sum_I q_I \hat{\Delta}_I = S_{\text{BH}}(\mathfrak{s}, q)$$

- The field theory model is ABJM [Aharony, Bergman, Jafferis, Maldacena (2008)]: A Lagrangian, maximally supersymmetric SCFT in 3d.
- The entropy is represented by the **finite** part of the partition function, not an anomaly.
- The partition function is computable at large N using an effective twisted superpotential and its Bethe Ansatz Equations.
- The comparison is to an AdS$_4$ supersymmetric black hole [Cacciatori and Klemm (2010)].

Aspects of the BHZ calculation

Several technical aspects of the BHZ calculation seem crucial to its success relative to previous endeavors

- The index is **topological** and all of the states contributing are regarded as ground states.
- The one loop contributions to the effective action for the matrix model are simple.
- The flavor symmetries are manifest.
- The complex scalar modulus is integrated over a contour given by the Jeffrey-Kirwan prescription.
- There are supersymmetric **fluxes** for dynamical and background gauge fields.
- There is an **equivariant deformation**, which represents a black hole with angular momentum, but it can be turned off.

A five dimensional analogue

We are using the same approach in 5d, by considering an appropriate theory on a manifold of the type $\mathcal{M}_4 \times S^1$ where $\mathcal{M}_4$ is toric Kähler, which shares many of the features of BHZ

- There is a twisted topological partition function amenable to localization.
- There are gravity solutions with which to compare.
- The theory lives in an odd dimension and the finite part of the partition function is expected to be universal.
- There is an equivariant deformation which can be turned off [Nekrasov (2002)].
- There are fluxes and a contour prescription for the evaluation of the matrix model [Nekrasov (2006), Bawani,Bonelli,Ronzani,Tanzini (2014), Bershtein, Bonelli, Ronzani, Tanzini (2015)].

Many of the necessary calculations have already been done [citations on this slide + Källén and Zabzine (2012), Jafferis and Pufu (2012)].

Toward an entropy formula in five dimensions

There are a number of differences from the three dimensional case

- We can consider more complicated topologies.
- There is a Lagrangian theory with $\mathcal{N} = 2$ supersymmetry, but it is not conformal. The strong coupling limit is believed to represent a 6d $(2, 0)$ SCFT.
- Instanton contributions are present, but presumably go away at leading order at large N.

There are also some technical challenges

- The integration contour and sum over fluxes is not well understood.
- The correct analogue of the Bethe Ansatz Equations is not clear.

Donaldson - Witten theory

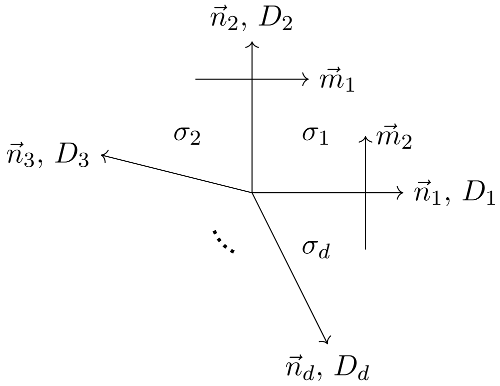
A 4d $\mathcal{N} = 2$ theory can be twisted: coupled to curvature on $\mathcal{M}_4$ using a diagonal combination of the spin group $SU(2)_l \times SU(2)_r$ and the R-symmetry group $SU(2)_\mathrm{R}$ [Witten (1988)]

- A scalar supercharge Q is preserved on an arbitrary manifold.
- The energy-momentum tensor is Q exact.

For $G = SU(2)$, the theory of Q closed observables is the cohomological Donaldson-Witten TQFT

- Computes the intersection theory on the moduli space of G-instantons on $\mathcal{M}_4$: Donaldson invariants.
- The Seiberg-Witten solution is an effective computational tool.
- The low energy effective field theory approach includes a sum over SW monopoles and an integral over a moduli space: the u-plane [Moore and Witten (1997)].

The toric Kähler manifold $\mathcal{M}_4$



Canonical construction for a metric [Guillemin (1994) and Abreu (2003)]

$$ds^2 = G_{ij}dx^i dx^j + \left(G^{-1}\right)_{ij} dy^i dy^j, \quad i,j \in \{1,2\}$$

- x^i, y^i coordinates on the Delzant polytope and torus.

When $\mathcal{M}_4$ admits a metric with an isometry, there is a refinement due to Nekrasov, using the Ω-deformation

- Introduce a supercharge which squares to the isometry with vector v.
- On $\mathbb{R}^4$ this is the setting for the Nekrasov partition function. Recall the relationship to the effective prepotential [Nekrasov (2002)]

$$\log Z_{\text{inst}}\left(\vec{a},\epsilon_1,\epsilon_2;q\right) \approx \frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_0\left(\vec{a},\Lambda\right),$$
$$q \to \Lambda^{2h^\vee(G)-k(R)}.$$

On a toric Kahler manifold we use the torus isometry to localize to the vertices of the polytope. We will have to use the 5d version of the Nekrasov partition function.

Nekrasov's equivariant extension II

The Nekrasov partition function contains much more information than just the effective prepotential. We can expand in ϵ_1, ϵ_2

$$\log Z_{\text{inst}}\left(\vec{a}, \epsilon_1, \epsilon_2; q\right) = \frac{1}{\epsilon_1 \epsilon_2}\mathcal{F}_0 + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2}\mathcal{H}_{\frac{1}{2}} + \mathcal{F}_1 + \frac{\left(\epsilon_1 + \epsilon_2\right)^2}{\epsilon_1 \epsilon_2}\mathcal{G}_1 + \dots$$

- The extra terms show up in calculations on curved manifolds.
- The expansion has been worked out for the 5d version of the Nekrasov partition function [Göttsche, Nakajima, Yoshioka (2006)].

Nekrasov's equivariant extension III

On a compact toric Kähler manifold $\mathcal{M}_4$ [Nekrasov (2006)]

$$Z_{\mathcal{M}_4} = \sum_{\{\vec{k}_a \in \mathbb{Z}^N\}} \oint da \prod_{i \in \text{vertices}} Z_{\text{inst}}\left(\vec{a} + \epsilon_i^a \vec{k}^a\,;\, q\right)$$

$$\xrightarrow[\epsilon_1,\epsilon_2 \to 0]{} \sum_{\{\vec{k}_a \in \mathbb{Z}^N\}} \oint da \exp \int_{\mathcal{M}_4} \mathcal{F}_0\left(\vec{a} + \sum_a \vec{k}_a c_1\left(L_a\right)\right)$$

$$+ c_1\left(\mathcal{M}_4\right) H_{\frac{1}{2}}\left(\vec{a} + \sum_a \vec{k}_a c_1\left(L_a\right)\right)$$

$$+ \chi\left(\mathcal{M}_4\right) \mathcal{F}_1\left(\vec{a}\right) + \sigma\left(\mathcal{M}_4\right) \mathcal{F}_1\left(\vec{a}\right)$$

- I have omitted the insertion of observables. The contour for a and the exact sum are unknown.
- $\vec{k}_a$ is an integer flux vector and ϵ_i is the action on the fixed point i.

Geometry of $\mathcal{M}_4^\Omega \times S^1$

We let e be a vielbein for the canonical metric on $\mathcal{M}_4$, parameterize the Euclidean time direction as

$$x^5 \in [0, 2\pi\beta)\,,$$

and define

$$\tilde{v} = \sqrt{\beta}\epsilon_i \partial_{y^i}\,,$$
$$x^3 \equiv y^1, \quad x^4 = y^2.$$

We define the metric on X by augmenting ${e_a}^\mu$ with

$${e_5}^\mu = \tilde{v}^\mu, \quad {e_5}^5 = 1.$$

- This metric is the one which implements the 5d Ω background.
- Preserving supersymmetry requires additional work.

Rigid supergravity in 5D $\mathcal{N}=1$

The modern approach is to couple to "rigid" supergravity [Festuccia and Seiberg (2011)]

- An appropriate choice of supergravity needs to be made.
- An alternative is Superconformal Tensor Calculus which seems to capture all flavors.

We start with the 5d Weyl multiplet [Fujita and Ohashi (2001)]

$$\begin{cases} e_\mu{}^a, \quad v_{ab}, \quad A_\mu^{(\mathrm{R})}, \quad b_\mu, \quad D & \text{bosons} \\ \psi_\mu, \quad \chi & \text{fermions} \end{cases}$$

and find a bosonic fixed point of

$$\delta\psi_\mu = \mathcal{D}_\mu\zeta + \frac{1}{2}v^{ab}\Gamma_{ab\mu}\zeta - \Gamma_\mu\eta,$$

$$\delta\chi = D\,\zeta + \Gamma^{\mu\nu}F\,(V)_{\mu\nu}\,\zeta + v_{ab}\ \text{dependent terms}$$

Twisted supersymmetry on $\mathcal{M}_4 \times S^1$

Twisted supersymmetry, including the Ω-background, is a special case

$$T_{\mu\nu} = b_\mu = 0.$$

The variation is simply

$$\delta\psi_\mu = \mathcal{D}_\mu\zeta - \gamma_\mu\eta,$$

$$\mathcal{D}_\mu\zeta \equiv \partial_\mu\zeta - \frac{1}{4}{\omega_\mu}^{ab}\gamma_{ab}\zeta + \zeta\left(A_\mu^{(\mathsf{R})}\right)^{\mathsf{T}}.$$

We can now choose

$$A_\mu^{(\mathsf{R})} \propto \omega_\mu^{ab}\sigma_{ab},$$

and preserve a spinor

$$\zeta = \begin{pmatrix} 0 \\ \mathbb{i}\sigma_2 \end{pmatrix}, \quad \eta = 0.$$

Supersymmetry algebra

The supersymmetry algebra is similar to the usual one for the Ω-background in 4d, and is the same as in the 5d contact case [Källén and Zabzine (2012)]

$$
\begin{aligned}
\delta A &= \Psi \\
\delta \Psi &= i_v F + i d_A \sigma \\
\delta \sigma &= -\mathbb{i} i_v \Psi \\
\delta \chi &= H \\
\delta H &= \mathcal{L}_v^A \chi - \mathbb{i}\,[\sigma, \chi]
\end{aligned}
$$

Similar expressions exist for the hypermultiplet

- Scalars in the hypermultiplet become spinors after twisting.
- When $\mathcal{M}_4$ is not spin, one has to work with sections of a Spin^C bundle.

The basics of localization

Deformation

- Identify an appropriate conserved fermionic charge: Q.
- Choose V such that $\{Q, V\}$ is a positive semi-definite functional (Q should square to 0 on V).
- Deform the action by a total Q variation $S \to S + t\{Q, V\}$. The resulting path integral is independent of t!
- Add some Q closed operators (Wilson loops, defect operators).

Localization

- Take the limit $t \to \infty$.
- The measure $exp(-S)$ is very small for $\{Q, V\} \neq 0$.
- The semi-classical approximation becomes exact, but there may be many saddle points to sum over: the moduli space.

The moduli

Non-trivial saddle points in this setup come only from vector multipets: fixed points of the equation

$$\delta\Psi = i_v F + i d_A \sigma$$

These come in three classes [Bawani,Bonelli,Ronzani,Tanzini (2014), Bershtein, Bonelli, Ronzani, Tanzini (2015)]

1. Bulk modulus: basically a flat connection in the original 6d theory - here it is a constant profile for σ and a (commuting) holonomy around S^1 which combine into a complex modulus a.
2. Instanton contributions: along the circles where the equivariant action on $\mathcal{M}_4$ degenerates.
3. Fluxes: one flux for every equivariant divisor subject to topological and stability conditions.

The partition function

The final result for the partition function is a simple lift of Nekrasov's calculation and those of Bershtein, Bonelli, Ronzani, and Tanzini. Denote the 5d Nekrasov partition function as

$$Z^{\mathbb{C}^2\times S^1}_{\text{full}}(a, \Delta_{\mathfrak{R}}; \epsilon_1, \epsilon_2, \beta) = Z^{\mathbb{C}^2\times S^1}_{\text{cl}} Z^{\mathbb{C}^2\times S^1}_{\text{1-loop}} Z^{\mathbb{C}^2\times S^1}_{\text{inst}}$$

Then

$$Z_{\mathcal{M}_4\times S^1} = \sum_{\{k^{(\ell)}\}|\text{semi-stable}} \oint_{\text{JK}} \mathrm{d}a \prod_{\ell=1}^{\chi(\mathcal{M}_4)} Z^{\mathbb{C}^2\times S^1}_{\text{full}}(a^{(\ell)}; \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, \beta)$$

where I shortened

$$a^{(\ell)} = \vec{a} + \epsilon_1^{(\ell)}\vec{k}_1 + \epsilon_2^{(\ell)}\vec{k}_2.$$

We can write the sum down explicitly only is some cases. A direct large N computation is hard.

The Nekrasov-Shatashvilli approach

The vacua of a massive 2d gauge theory correspond to solutions of the equation

$$\exp\left(\frac{\partial\tilde{\mathcal{W}}\left(a\right)}{\partial a^{i}}\right)=1,$$

where $\tilde{\mathcal{W}}$ is the effective twisted superpotential of the theory and a is the vev of the scalar in the vector multiplet. Nekrasov and Shatashvili identified this equation with the Bethe Ansatz Equations arising in integrable systems [Nekrasov and Shatashvili (2009)], and showed how to produce interesting systems from higher dimensions.

- Partition functions of the theory on a twisted compact 2-manifold can be calculated by solving the equations.
- The large N limit becomes tractable for specific cases.

The Nekrasov-Shatashvili approach

It seems reasonable to think that the effective prepotential $\mathcal{F}(a,\tau)$ plays a role analogous to that of $\tilde{\mathcal{W}}$ on a compact twisted four manifold. Indeed [Nekrasov and Shatashvili (2009)]

- In the Nekrasov-Shatashvili limit on $\mathbb{R}^4$ $(\epsilon_1 \to 0, \epsilon_2 = \hbar)$:

$$\tilde{\mathcal{W}}_\hbar(a,\tau) = \frac{1}{\hbar}\mathcal{F}(a,\tau) + \ldots$$

  leading to an equation for the vacua of the form

$$\exp\left(\frac{1}{\hbar}\frac{\partial \mathcal{F}(a,\tau)}{\partial a}\right) = 1.$$

- In a twisted compactification of $U(N)$, $\mathcal{N} = 2^*$ $(i \in \{1 \ldots N\})$

$$\tilde{\mathcal{W}}_{\text{eff}}(a,m,\tau) = 2\frac{\partial \mathcal{F}(a,m,\tau)}{\partial m} + \mathfrak{m}^i \frac{\partial \mathcal{F}(a,m,\tau)}{\partial a^i} + \mathfrak{n}_i a^i$$

  leading to an equation with a second derivative.

A black string solution

The gravity dual of the 6d $(2,0)$ A_N theory compactified on $\Sigma_{g_1} \times \Sigma_{g_2} \times T^2$ is known [Benini and Bobev 2013]

- A truncation of $SO(5)$ maximal gauged supergravity in 7d ($\mathcal{N} = 4$)
- Contains two $U(1)$ gauge fields and 2 real scalars.
- The solution interpolates between AdS$_7$ and AdS$_3 \times \Sigma_{g_1} \times \Sigma_{g_2}$.
- We identify fluxes for the $U(1)^2$ gauge fields with flavor fluxes in the SCFT.

There is a compactification yielding an AdS$_6$ black hole [Hristov (2014)]

- Supersymmetric compactification with momentum.
- Black hole entropy related by the Cardy formula to the central charge.

Paritition function for the $U(N)$ $\mathcal{N}=2$ theory

The perturbative part of the partition function for the 5d $U(N)$ $\mathcal{N}=2$ theory in the non-equivariant limit is

$$Z^{\text{pert}}(y,\mathfrak{s},\mathfrak{t}) = \frac{1}{N!}\sum_{\{\mathfrak{m},\mathfrak{n}\}\in\mathbb{Z}^N}\oint_{\text{JK}}\prod_{i=1}^{N}\frac{\mathrm{d}x_i}{2\pi i x_i}e^{\frac{8\pi^2\beta}{g^2_{\text{YM}}}(\mathfrak{m}_i-\mathfrak{m}_j)(\mathfrak{n}_i-\mathfrak{n}_j)}$$
$$\times\prod_{i\neq j}^{N}\left(\frac{1-x_i/x_j}{\sqrt{x_i/x_j}}\right)^{(\mathfrak{m}_i-\mathfrak{m}_j+1)(\mathfrak{n}_i-\mathfrak{n}_j+1)}$$
$$\times\prod_{i,j=1}^{N}\left(\frac{\sqrt{x_i y/x_j}}{1-x_i y/x_j}\right)^{(\mathfrak{m}_i-\mathfrak{m}_j+\mathfrak{s}-1)(\mathfrak{n}_i-\mathfrak{n}_j+\mathfrak{t}-1)}$$

- $x=\exp(\mathbb{i}\beta a)$ and $y=\exp(\mathbb{i}\beta\Delta)$. Δ is the flavor fugacity.
- $\mathfrak{m},\mathfrak{n}$ and $\mathfrak{s},\mathfrak{t}$ are gauge and flavor fluxes.

The effective prepotential

Concentrate on the first equation $\frac{\partial \mathcal{F}(a)}{\partial a_i} = 0$

$$\mathcal{F}(a,\Delta) = \frac{2\pi i\beta}{g^2_{\text{YM}}} \sum_{i=1}^{N} a_i^2 + \frac{i}{2\pi\beta^2} \sum_{i\neq j}^{N} \mathrm{Li}_3(e^{i\beta(a_i-a_j)}) - \frac{i}{2\pi\beta^2} \sum_{i,j=1}^{N} \mathrm{Li}_3(e^{i\beta(a_i-a_j+\Delta)}) + \text{polynomial}$$

At strong 't Hooft coupling eigenvalues are pushed apart [Minahan, Nedelin, and Zabzine (2013)]

$$N \gg 1, \qquad \lambda = \frac{g^2_{\text{YM}} N}{\beta} \gg 1,$$

$$a_k = \frac{\mathbb{i}\lambda}{16\pi^2 N}\left[\Delta(2\pi - \beta\Delta)(2k - N - 1)\right]$$

If we plug this back into the prepotential we recover the S^5 free energy!

Summing up one flux

We sum over one of the fluxes to produce the partition function in the form where solutions of the BAEs can be plugged in

$$Z_{S^2\times(S^2\times S^1)} = \frac{(-1)^{\mathsf{rk}(G)}}{|\mathfrak{W}|}\sum_{\mathfrak{n}}\sum_{a=a_{(i)}} Z_{\mathsf{full}}\Big|_{\mathfrak{m}=0}(a,\mathfrak{n}) \left(\det_{ij}\frac{\partial^2\widetilde{\mathcal{W}}(a,\mathfrak{n})}{\partial a_i\partial a_j}\right)^{-1}$$

- There are still an infinite number of solutions! (one for each flux sector)
- Thankfully, the second equation, resulting from

$$\tilde{\mathcal{W}} \propto \partial^2\mathcal{F},$$

yields a constraint on the **flux**!

The result

We can now plug both solutions back in to the partition function and recover

$$\log Z^{\text{pert}} = \frac{\lambda\beta^2(N^2-1)}{96\pi^2}\Big[\Delta_1\Delta_2(\mathfrak{t}_1\mathfrak{s}_2+\mathfrak{t}_2\mathfrak{s}_1) + (\Delta_1\mathfrak{s}_2+\Delta_2\mathfrak{s}_1)(\Delta_1\mathfrak{t}_2+\Delta_2\mathfrak{t}_1)\Big]$$

This matches the trial right moving central charge computed by Benini and Bobev

$$\log Z^{\text{pert}}(\mathfrak{s},\mathfrak{t},\Delta) = \frac{g_{\text{YM}}^2}{48\beta}c_r(\mathfrak{s},\mathfrak{t},\Delta),$$

hence the calculation matches the black string. The relationship to the modular parameter is

$$\tau = \frac{4\pi i\beta}{g_{\text{YM}}^2} = \frac{\beta}{2\pi R_6}.$$

Microstates

To recover the microstate counting of the black hole, we need to move to the micro-canonical ensemble

$$d_{\text{micro}}(\mathfrak{s},\mathfrak{t},n) = \text{const}\int \mathrm{d}\tilde{\beta}\mathrm{d}\Delta\, Z(\mathfrak{s},\mathfrak{t},\Delta)\, e^{\tilde{\beta}n}$$

with $\tilde{\beta} = -2\pi\mathbb{i}\tau$. Define

$$\mathcal{I}_{\text{SCFT}}(\tilde{\beta},\Delta) \equiv \log Z(\mathfrak{s},\mathfrak{t},\Delta) + \tilde{\beta}n,$$
$$\log d_{\text{micro}}(\mathfrak{s},\mathfrak{t},n) = \mathcal{I}\big|_{\text{crit}}(\mathfrak{s},\mathfrak{t},n)$$

which should be evaluated using a saddle point approximation

$$\frac{\partial\mathcal{I}(\tilde{\beta},\Delta)}{\partial\Delta} = 0\,, \qquad \frac{\partial\mathcal{I}(\tilde{\beta},\Delta)}{\partial\tilde{\beta}} = 0.$$

then the approximation yields the expected Cardy formula result for the number of d.o.f.

$$\mathcal{I}\big|_{\text{crit}}(\mathfrak{s},\mathfrak{t},n) = 2\pi\sqrt{\frac{n\, c_{\text{CFT}}\,(\mathfrak{s},\mathfrak{t})}{6}}.$$

Summary

To summarize

- 5d twisted indices are a direct analogue of the BHZ computation.
- We can compute these indices using localization on a toric Kähler manifold.
- A matching to black hold entropy can be shown in a simple case.

There remain a few points to sort out before this can be done on a general $\mathcal{M}_4$

- Stability conditions for the fluxes are known, in principle, for this class of manifolds [Kool (2009)], but are hard to decode.
- The identification of the contour of integration for a with the JK contour needs to be understood in a more rigorous fashion.
- The appropriate BAEs in terms of the effective prepotential $\mathcal{F}$ are a conjecture at this point.

Thank you

Thank you!