Majorana fermions and the topological Kondo effect

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‡ Fact 1: conduction electrons + quantum spin with degenerate levels
Kondo effect
- key paradigm in strong correlations
- arising from the q. dynamics of a spin qubit

‡ Fact 2: Majorana fermions in condensed matter
Topological qubits ~ nonlocal spins
- level degeneracy ~ top. degeneracy

• Idea: coupling conduction electrons to topological qubits?
Topological Kondo effect
- Majorana induced strong correlations
- demonstrates q. dynamics of top. qubits via transport
Outline

- Intro to Majorana fermions
  - what are they?
  - how do they emerge?
  - key features & potential uses
  - some of the experimental signatures

- Topological Kondo effect
  - from Majoranas to Kondo – the topological Kondo idea
  - transport signatures, incl. NFL features
  - topological Kondo beyond the minimal setup
  - (exact) scaling functions for nonequilibrium transport

Further reading

Reviews on Majorana fermions:

Background on the Kondo effect:
A. C. Hewson, The Kondo Problem to Heavy Fermions (CUP 1997)

Background on field theory/CFT approaches:
Majorana fermions

Consider an arbitrary fermion problem with operators

\[ c_j : \{c_j, c_k^\dagger\} = \delta_{jk}, \{c_j, c_k\} = 0 \]

We can always take the Hermitian & anti-Hermitian parts:

\[ \gamma_{j1} = c_j + c_j^\dagger, \quad \gamma_{j2} = -i(c_j - c_j^\dagger) \]

\[ \gamma_\alpha = \gamma_\alpha^\dagger, \quad \{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \]

Always works as a maths trick... But can also emerge as a form of fractionalisation:

[A. Kitaev, Phys.-Usp. 2001]

Delft experiment: InSb nanowire

[V. Mourik et al., Science, 2012]

Superconductors & E-H symmetry

BCS mean field description

\[ H = \sum_{\alpha\beta} \frac{\Delta_{\alpha\beta}}{2} c_{\alpha}^\dagger c_{\beta} + \frac{1}{2} \Delta_{\alpha\beta} c_{\alpha}^\dagger c_{\beta}^\dagger + \frac{1}{2} \Delta_{\alpha\beta}^* c_{\beta} c_{\alpha} \quad \Delta_{\alpha\beta} = -\Delta_{\beta\alpha} \]

\[ H = \frac{1}{2}(c^\dagger \cdot c) \left( \begin{array}{cc} h & \Delta \\ -\Delta^* & -h^* \end{array} \right) \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) + \text{const.} \]

\[ \mathcal{C} H_{\text{BdG}} C^{-1} = -H_{\text{BdG}} \quad \mathcal{C} = K\Sigma_1 \quad : \text{e-h symmetry} \]

Spectral symmetry:

\[ H_{\text{BdG}} \psi = E \psi \]

\[ \mathcal{C} H_{\text{BdG}} C^{-1} C \psi = E \mathcal{C} \psi \]

\[ -H_{\text{BdG}} \]

\[ H_{\text{BdG}} (C \psi) = -E (C \psi) \]
E-H symmetry & negative energy “modes”

\[ H = \frac{1}{2} (c^\dagger c) H_{\text{BdG}} \begin{pmatrix} c \\ c^\dagger \end{pmatrix} \]

\[ CH_{\text{BdG}} C^{-1} = -H_{\text{BdG}} \implies H_{\text{BdG}} \psi = E \psi, \quad H_{\text{BdG}}(C \psi) = -E(C \psi), \quad C = K \Sigma_1 \]

\[ H_{\text{BdG}} = U \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & -E_1 \end{pmatrix} U^\dagger, \quad U = \begin{pmatrix} \psi_{E_1} & & \vert C \psi_{E_1} \rangle & \cdots \\ \vdots & \ddots & \vdots \\ \psi_{-E_1} & \cdots & \psi_{E_1} \end{pmatrix} \]

\[ H = \frac{1}{2} (c^\dagger c) U \text{diag}(E_1 \ldots, -E_1 \ldots) U^\dagger \begin{pmatrix} c \\ c^\dagger \end{pmatrix} \]

\[ (\alpha_{E>0}^\dagger \alpha_{E<0}^\dagger) = (c^\dagger c) U \]

Leads to an apparently unusual form (note the 1/2, negative energies):

\[ H = \frac{1}{2} (\alpha_{E>0}^\dagger \alpha_{E<0}^\dagger) \begin{pmatrix} E_1 \\ \vdots \\ \alpha_{E>0}^\dagger \\ \alpha_{E<0}^\dagger \end{pmatrix} = \frac{1}{2} \sum_j E_j \alpha_j \alpha_j^\dagger \]

E-H symmetry & negative energy “modes”

\[ H = \frac{1}{2} (c^\dagger c) H_{\text{BdG}} \begin{pmatrix} c \\ c^\dagger \end{pmatrix} \]

\[ CH_{\text{BdG}} C^{-1} = -H_{\text{BdG}} \implies H_{\text{BdG}} \psi = E \psi, \quad H_{\text{BdG}}(C \psi) = -E(C \psi), \quad C = K \Sigma_1 \]

\[ H_{\text{BdG}} = U \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & -E_1 \end{pmatrix} U^\dagger, \quad U = \begin{pmatrix} \psi_{E_1} & & \vert C \psi_{E_1} \rangle & \cdots \\ \vdots & \ddots & \vdots \\ \psi_{-E_1} & \cdots & \psi_{E_1} \end{pmatrix} \]

\[ (\alpha_{E>0}^\dagger \alpha_{E<0}^\dagger) = (c^\dagger c) U \]

\[ \alpha_{E_1}^\dagger = (c^\dagger c) \cdot \psi_{E_1}, \quad \alpha_{-E_1}^\dagger = (c^\dagger c) \cdot \Sigma_1 \psi_{E_1}^* = (c^\dagger c) \cdot \psi_{E_1}^* \]

Redundancy relation:

\[ \alpha_{-E_1}^\dagger = \alpha_{E_1} \]

Hamiltonian diagonalises to the usual form:

\[ H = \frac{1}{2} \sum_j E_j \alpha_{E_j}^\dagger \alpha_{E_j} = \frac{1}{2} \sum_{E_j>0} E_j \left( \alpha_{E_j}^\dagger \alpha_{E_j} - \alpha_{-E_j}^\dagger \alpha_{-E_j} \right) \]

\[ = \frac{1}{2} \sum_{E_j>0} E_j \left( \alpha_{E_j}^\dagger \alpha_{E_j} - \alpha_{E_j}^\dagger \alpha_{E_j}^\dagger \right) = \sum_{E_j>0} E_j \alpha_{E_j}^\dagger \alpha_{E_j} + \text{const} \]
E-H symmetry & negative energy “modes”

\[ H = \frac{1}{2} (c^\dagger \ c) H_{\text{BdG}} \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \]

\[ C H_{\text{BdG}} C^{-1} = -H_{\text{BdG}} \Rightarrow H_{\text{BdG}} \psi = E \psi, \ H_{\text{BdG}}(C \psi) = -E(C \psi), \ C = K \Sigma_1 \]

\[ H_{\text{BdG}} = U \left( \begin{array}{ccc} E_1 & & \\ & \ddots & \\ & & -E_1 \end{array} \right) U^\dagger, \quad U = \left( \begin{array}{c} \psi_{E_1} \\ \vdots \\ |C \psi_{E_1}| \\ \vdots \end{array} \right) \]

\[ (\alpha_{E>0}^\dagger \ \alpha_{E<0}^\dagger) = (c^\dagger \ c) U \]

\[ \alpha_{E_1}^\dagger = (c^\dagger \ c) \cdot \psi_{E_1}, \quad \alpha_{-E_1}^\dagger = (c^\dagger \ c) \cdot \Sigma_1 \psi_{E_1}^*, \quad \psi_{E_1}^* = (c \ c^\dagger) \cdot \psi_{E_1}^* \]

Redundancy relation:

\[ \alpha_{-E_1}^\dagger = \alpha_{E_1}^\dagger \]

For zero modes this suggests:

\[ \alpha_{E=0}^\dagger = \alpha_{E=0} \]

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E-H symmetry & zero modes

\[ H = \frac{1}{2} (c^\dagger \ c) H_{\text{BdG}} \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \]

\[ C H_{\text{BdG}} C^{-1} = -H_{\text{BdG}} \Rightarrow H_{\text{BdG}} \psi = E \psi, \ H_{\text{BdG}}(C \psi) = -E(C \psi), \ C = K \Sigma_1 \]

(Locally) nondegenerate zero mode:

\[ C \psi_0 = e^{i \chi} \psi_0 \Rightarrow e^{-i \chi/2} C \psi_0 = C e^{i \chi/2} \psi_0 = e^{i \chi/2} \psi_0 \]

Can choose:

\[ C \psi_0^* = \psi_0 \]

\[ \gamma = (c^\dagger \ c) \cdot \psi_0 = (c^\dagger \ c) \cdot \Sigma_1 \psi_0^* = (c \ c^\dagger) \cdot \psi_0^* = \gamma^\dagger \]

\[ \gamma = \gamma^\dagger \]
E-H symmetry & zero modes

\[ H = \frac{1}{2}(c^\dagger \ c)H_{BdG} \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \]

\[ C H_{BdG} C^{-1} = -H_{BdG} \Rightarrow H_{BdG}\psi = E\psi, \ H_{BdG}(C\psi) = -E(C\psi), \ C = K\Sigma_1 \]

(Locally) nondegenerate zero mode:
\[ C\psi_0 = e^{ix}\psi_0 \Rightarrow e^{-ix/2}C\psi_0 = Ce^{ix/2}\psi_0 = e^{ix/2}\psi_0 \]

Can choose: \[ C\psi_0 = \psi_0 \]

\[ \gamma = (c^\dagger \ c) \cdot \psi_0 = (c^\dagger \ c) \cdot \Sigma_1\psi_0^* = (c^\dagger \ c^\dagger) \cdot \psi_0^* = \gamma^\dagger \]

\[ \gamma = \gamma^\dagger \]

With more spatially separated zero modes:
\[ \psi_0^{(j)} \rightarrow \gamma_j, \ \langle \psi_0^{(j)} , \psi_0^{(k)} \rangle = \delta_{jk} \Rightarrow \{ \gamma_j, \gamma_k \} = 2\delta_{jk} \]

(Locally) nondegenerate zero mode in superconductor: guaranteed to be Majorana mode

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E-H symmetry & zero modes

\[ H = \frac{1}{2}(c^\dagger \ c)H_{BdG} \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \]

\[ C H_{BdG} C^{-1} = -H_{BdG} \Rightarrow H_{BdG}\psi = E\psi, \ H_{BdG}(C\psi) = -E(C\psi), \ C = K\Sigma_1 \]

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Can choose: \[ C\psi_0 = \psi_0 \]

\[ \gamma = (c^\dagger \ c) \cdot \psi_0 = (c^\dagger \ c) \cdot \Sigma_1\psi_0^* = (c^\dagger \ c^\dagger) \cdot \psi_0^* = \gamma^\dagger \]

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(Locally) nondegenerate zero mode in superconductor: guaranteed to be Majorana mode
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Nanowire realisation

\[ H_{\text{BdG}} = \left( \frac{p^2}{2m} - \mu \right) \Sigma_3 + \alpha \sigma_3 \Sigma_3 + E_Z \sigma_1 + \Delta \Sigma_1 \]

\[ H_0 = \frac{p^2}{2m} - \mu + \alpha \sigma_3 + E_Z \sigma_1 \]

Jackiw-Rebbi-type picture

Linear (Dirac/Majorana) gap closing described by

$$H_{\text{eff}} = -i v \partial_x \tau_1 + m \tau_3 \quad E = \pm \sqrt{(vk)^2 + m^2} \quad \tau_1 (H_{\text{eff}})^* \tau_1 = -H_{\text{eff}}$$

Consider an interface across which the gap parameter changes sign:

$$H_{\text{eff}} = -i v \partial_x \tau_1 + m(x) \tau_3$$

Jackiw-Rebbi: interface binds a zero mode.

$$H_{\text{eff}} \Psi = 0$$
$$i \partial_x \tau_1 \Psi = \mu(x) \tau_3 \Psi \quad \mu(x) = m/v$$
$$\Rightarrow \quad \partial_x \Psi = -i \tau_1 \tau_3 \mu(x) \Psi = -\tau_2 \mu(x) \Psi$$

$$\Psi = e^{-\int_0^x \mu(x')dx'} \tau_2 \Psi(0) \quad \Rightarrow \quad \Psi = e^{\mp \int_0^x \mu(x')dx'} \psi_{\pm}$$

The convergent one for the profile above:

$$\Psi = e^{\int_0^x \mu(x')dx'} \psi_-$$

NB: exponentially localised to interface

Nanowire realisation

Nanowire realisation

\[ H_{\text{BdG}} = \left( \frac{p^2}{2m} - \mu \right) \Sigma_3 + \alpha p \sigma_3 \Sigma_3 + E_Z \sigma_1 + \Delta \Sigma_1 \]

Gapped but top. trivial:
• TR inv levels degenerate

Zeeman breaks TR invariance.
Top. regime? Look for linear gap closing.

\[ H(p = 0) = -\mu \Sigma_3 + E_Z \sigma_1 + \Delta \Sigma_1 \]
\[ E = \pm \left( \sqrt{\Delta^2 + \mu^2} \pm |E_Z| \right) \]

\[ H_{\text{BdG}} = \left( \frac{p^2}{2m} - \mu \right) \Sigma_3 + \alpha \sigma_3 \Sigma_3 + E_Z \sigma_1 + \Delta \Sigma_1 \]

\[ \mu = 0 \]

\[ E = \pm (\sqrt{\Delta^2 + \mu^2} \pm |E_Z|) \]

Gap closing @ \[ |E_Z| = \sqrt{\Delta^2 + \mu^2} \]

\[ H(p = 0) = -\mu \Sigma_3 + E_Z \sigma_1 + \Delta \Sigma_1 \]

\[ E = \pm (\sqrt{\Delta^2 + \mu^2} \pm |E_Z|) \]

\[ |E_Z| > \sqrt{\Delta^2 + \mu^2} \text{ : topological phase} \]
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Majorana fermions: key features

\[ \gamma_j = \gamma_j^\dagger, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij} \]
\[ c_0 = \frac{1}{2}(\gamma_1 + i\gamma_2) \]

(recall: Majoranas as Hermitean and anti-Hermitean parts of fermions)

More generally:
1 fermion per 2 Majorana; system of ordinary fermions

→ Majoranas must come in pairs!
Majorana fermions in nanodevices: envisioned applications

\[ \gamma_j = \gamma_j^\dagger, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij} \]
\[ c_0 = \frac{1}{2} (\gamma_1 + i\gamma_2) \]
- \( c_0^\dagger \) costs no energy \textbf{topological GS degeneracy}
- \( |0\rangle, |1\rangle \) topological qubit
- more Majoranas \( \Rightarrow \) qubit operations

[N. Majoranas; 1 fermion per pair \( \Rightarrow \) \( N/2 \) zero energy fermions
\( \Rightarrow 2^{N/2} \)- fold GS degeneracy

However, overall parity is conserved (in a closed system)
\( \Rightarrow 2^{N/2-1} \)- fold degenerate space to operate on

Preliminary considerations:

- Groundstate degeneracy for \( N \) Majoranas:
  \( N \) Majoranas; 1 fermion per pair \( \Rightarrow \) \( N/2 \) zero energy fermions
  \( \Rightarrow 2^{N/2} \)- fold GS degeneracy

Envisioned applications: some underlying principles

- Fermion parity in terms of Majoranas: parity of the pair \( i, j \):
  \[ c_{ij} = \frac{1}{2} (\gamma_i + i\gamma_j) \]
  \( \Rightarrow \)
  \[ \pi_{ij} = 1 - 2c_{ij}^\dagger c_{ij} = i\gamma_j\gamma_i \]

  \( \Rightarrow \)
  \[ \| \text{overall fermion parity:} \]
  \[ \Pi_{\text{tot}} = \pi_1\pi_3\cdots\pi_{N-1,N} \]

  NB: even (odd) products of Majoranas preserve (flip) overall parity
Envisioned applications: some underlying principles

Majorana advantages include:

1) **Topologically protected information storage:**
   - low energy (subgap), fermion parity conserving operators
     \[ O = \sum_{ijkl} a_{ij} \gamma_i \gamma_j + \sum_{ijkl} b_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l + \ldots \text{(even powers)} \]
   - low energy, local, fermion parity conserving operators
     \[ O_{\text{loc}} = \sum_i a_i \gamma_i^2 + b_i \gamma_i^4 \propto \mathbb{I} \]
     resilience against local, parity conserving, perturbations

2) **Topologically protected gates** (though not universal set), e.g., via non-Abelian statistics: exchanging Majorana i and j implements

   \[ |GS\rangle \rightarrow \frac{1}{\sqrt{2}} (1 \pm \gamma_i \gamma_j) |GS\rangle \]

Non-Abelian statistics

Exchanging Majorana i and j implements

\[ |GS\rangle \rightarrow \frac{1}{\sqrt{2}} (1 \pm \gamma_i \gamma_j) |GS\rangle \]

How does this come about and what is non-Abelian about it?

- Exchanging \( \gamma_i \) and \( \gamma_j \):
  \[ U_{i_j}^\dagger \gamma_i U_{i_j} = \eta_i \gamma_j, \quad U_{i_j}^\dagger \gamma_j U_{i_j} = \eta_i \gamma_i \]  

Most general unitary involving only \( \gamma_i \) and \( \gamma_j \):

\[ U_{i_j} = a + b \gamma_i \gamma_j, \quad |a|^2 + |b|^2 = 1, \quad a^* b = b^* a \]  

(1) & (2) \[ U_{i_j}^\pm = \frac{e^{i\eta_i \gamma_j}}{\sqrt{2}} (1 \pm \gamma_i \gamma_j), \quad \eta_j = -\eta_i = \pm 1 \]

- Non-Abelian because successive exchanges do not commute:
  \[ U_{i_j}^\pm U_{i_j}^\mp \neq U_{i_j}^\mp U_{i_j}^\pm \]
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Majorana fermions in nanodevices:
first signatures: zero energy nature

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Majorana mediated resonant transport
(resonant Andreev reflection)

Theory
[K. T. Law & P. A. Lee, PRL 2009; Wimmer et al. NJP 2011;
Fig.: A. Zazunov et al. PRB 2016]

Experiment (2012)
[V. Mourik et al. Science 2012]
Majorana fermions in nanodevices: zero energy nature – recent demonstration

Majorana mediated resonant transport (resonant Andreev reflection)

Theory [K. T. Law & P. A. Lee, PRL 2009; Wimmer et al. NJP 2011; Fig.: A. Zazunov et al. PRB 2016]

Experiment (2017) [H. Zhang et al., Nature 2018]

$2e^2/h$ peak finally seen:

Majorana fermions in nanodevices: some of the confirmed features

Zero energy nature via $2e^2/h$ conductance peak in hard gap [H. Zhang et al., Nature 2018 (Kouwenhoven group); F. Nichele et al. PRL 2018 (Marcus group)]
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Localised end-mode nature of state

Exponential protection against level splitting
[S. M. Albrecht et al. Nature 2016]

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... all via various forms of conductance measurements

But yet untested: nonlocal topological qubit
... can one see this via conductance?

‡ Fact 1: conduction electrons + quantum spin with degenerate levels
Kondo effect
- key paradigm in strong correlations
- arising from the q. dynamics of a spin qubit

‡ Fact 2: Majorana fermions in condensed matter
topological qubits ~ nonlocal spins
- level degeneracy ~ top. degeneracy

‡ Idea: coupling conduction electrons to topological qubits?
topological Kondo effect
- Majorana induced strong correlations
- demonstrates q. dynamics of top. qubits via transport

Topological Kondo: idea in 1 slide
[BB, N. R. Cooper, PRL 109, 156803 (2012)]
Kondo reminder

Scattering of conduction electrons on a spin with degenerate levels
- spin often “effective” spin (e.g., quantum dot/island)
- strong correlation paradigm: “asymptotic freedom”
- non-Fermi liquid behaviour possible (e.g., multichannel Kondo)

\[ H_K = J S_{\text{imp}} \cdot S_{\text{cond}}(0) \]

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From Majoranas to Kondo: effective spin

4 Majoranas:

\[ |n_{12}, n_{34} \rangle = |0, 0\rangle, |1, 1\rangle \equiv |\uparrow\rangle, |\downarrow\rangle \]

Considerations also apply to setups with Tis; Fe adatom chains, etc.

Topological Kondo effect

\[ H = H_{\text{lead}} + H_c(N) \]
\[ + \sum_j t_j \gamma_j e^{i\phi/2} \psi_j + \text{H.c.} \]
\( (e^{\pm i\phi/2}|N\rangle = |N \pm 1\rangle ) \) \[ \text{[L. Fu PRL 2010]} \]

Tunneling at \( j \)-th lead:

\[ H^{(j)}_T = \tau_j \psi^\dagger_S(x_j) \psi_j + \text{H.c.} \]

\( \psi^\dagger_S(x_j) \) in terms of excitations?
Recall: BdG vs electron operators

\[ H = \frac{1}{2} (c^\dagger c) H_{\text{BdG}} \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \]

\[ H_{\text{BdG}} = U \left( \begin{array}{ccc} E_1 & \cdots & -E_i \\ \cdots & \ddots & \cdots \\ \cdots & \cdots & \ddots \end{array} \right) U^\dagger, \quad U = \left( \begin{array}{c|c|c|c} \psi_{E_1} & \cdots & c \psi_{E_1} & \cdots \end{array} \right) \]

\[
\left( \begin{array}{c} \alpha_{E \geq 0} \\ \alpha_{E < 0} \end{array} \right) = U^\dagger \left( \begin{array}{c} \text{spinless} \\ \text{Majoranas exponentially localized:} \\ \text{Part with operators above the gap in terms of excitations?} \end{array} \right) U \left( \begin{array}{c} \alpha_{E > 0} \\ \gamma \\ \alpha_{E > 0}^\dagger \end{array} \right) 
\]

\[ \psi_{E_i > 0} = \left( \begin{array}{c} u_{E_i} \\ v_{E_i} \end{array} \right) \quad \psi_{E_i = 0} = \left( \begin{array}{c} \xi_i \\ \xi_i^* \end{array} \right) \]

\[
c_j = \sum_l U_{ji} \left( \begin{array}{c} \alpha_{E > 0}^\dagger \\ \gamma \\ \alpha_{E > 0} \end{array} \right) l = \sum_{E_i = 0} (\xi_l)_{j} \gamma_l + \sum_{E_i > 0} \left[ (u_{E_i})_{j} \alpha_{E_i} + (v_{E_i}^*)_{j} \alpha_{E_i}^\dagger \right] \]

\[ \rightarrow \sum_l \xi_l(x_j) \gamma_l + \sum_{E_i > 0} \left[ u_{E_i} (x_j) \alpha_{E_i} + v_{E_i}^* (x_j) \alpha_{E_i}^\dagger \right] = \sum_l \xi_l(x_j) \gamma_l + c_{E_i}(x_j) \]

Topological Kondo effect: lead-island term

\[ H = H_{\text{lead}} + H_c(N) + \sum_j t_j \gamma_j e^{i\phi/2} \psi_j + \text{H.c.} \]

(\[ e^{i\phi/2} [N] = [N \pm 1] \]) \[ [L. Fu PRL 2010] \]

Tunneling at \( j \)-th lead:

\[ H_T^{(j)} = \tau_j \psi_{S_j}^\dagger (x_j) \psi_j + \text{H.c.} \]

\[ \psi_{S_j}^\dagger (x_j) \text{ in terms of excitations?} \]

\[ \psi_{S_j}^\dagger (x_j) = e^{i\phi/2} \left[ \sum_l \xi_l^* (x_j) \gamma_l + \psi_{S_j,>}(x_j) \right] \]

Majorana wavefn. Part with operators above the gap

Majoranas exponentially localized:

\[ \xi_l(x_j) \approx \xi_j(x_j) \delta_{l,j} \]

working much below the gap

\[ \psi_{S_j}^\dagger (x_j) \approx e^{i\phi/2} \xi_j^* (x_j) \gamma_j \]
Topological Kondo effect

\[
H = H_{\text{lead}} + H_c(N)
+ \sum_j t_j \gamma_j e^{i\phi/2} \psi_j + \text{H.c.}
\]

where \( e^{\pm i\phi/2} |N\rangle = |N \pm 1\rangle \) [L. Fu PRL 2010]

**Kondo regime** \( T, V, t_j \ll E_c \): virtual transitions \( N_0 \leftrightarrow N_0 \pm 1 \)

\[ \rightarrow \] effective Hamiltonian in the space spanned by \( |m\rangle = |\sigma\rangle |\Psi\rangle \)

**Note:** Above is shorthand for state with \( N_0 \); more generally:

\[ |j\rangle = |n_{12}, n_{34}; N; \Psi\rangle \quad n_{12} + n_{34} = N \mod 2 \]

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Topological Kondo effect

\[
H = H_{\text{lead}} + H_c(N)
+ \sum_j t_j \gamma_j e^{i\phi/2} \psi_j + \text{H.c.}
\]

where \( e^{\pm i\phi/2} |N\rangle = |N \pm 1\rangle \) [L. Fu PRL 2010]

**Kondo regime** \( T, V, t_j \ll E_c \): virtual transitions \( N_0 \leftrightarrow N_0 \pm 1 \)

\[ \rightarrow \] effective Hamiltonian in the space spanned by \( |m\rangle = |\sigma\rangle |\Psi\rangle \)

from 2nd order p.t. (Schrieffer-Wolff):

\[
H = H_0 + V, \quad H_0 |j\rangle = E_j |j\rangle, \quad \langle j|V|j\rangle = 0
\]

\[ \Rightarrow \] \( H_{\text{eff}} = H_0 + \delta H \)

\[
\langle m|\delta H|n\rangle = -\frac{1}{2} \left[ \sum_{j \neq m} \frac{\langle m|V|j\rangle \langle j|V|n\rangle}{E_j - E_m} + \sum_{j \neq n} \frac{\langle m|V|j\rangle \langle j|V|n\rangle}{E_j - E_n} \right]
\]

simplifies if \( E_j - E_{m,n} \gg |E_m - E_n| \)

**here:** \( \sim E_c \gg \sim \delta \varepsilon_{\text{leads}} \sim T, V \)
Topological Kondo effect

\[ H = H_{\text{lead}} + H_c(N) \]
\[ + \sum_j t_j \gamma_j e^{i\phi/2} \psi_j + \text{H.c.} \]
\[ \left( e^{\pm i\phi/2} |N\rangle = |N \pm 1\rangle \right) \]

If from 2nd order p.t. (Schrieffer-Wolff):

\[ H = H_0 + V, \quad H_0 |j\rangle = E_j |j\rangle, \quad \langle j | V | j \rangle = 0 \]

\[ H_{\text{eff}} = H_0 - \sum_{E_j \neq E_m} \frac{V|j\rangle \langle j | V}{E_j - E_m} \]

**Kondo regime** \( T, V, t_j \ll E_c \): virtual transitions \( N_0 \leftrightarrow N_0 \pm 1 \)

\[ \Rightarrow \text{effective Hamiltonian} \] in the space spanned by \( |m\rangle = |\sigma\rangle |\Psi\rangle \)

from 2nd order p.t. (Schrieffer-Wolff):

\[ H_{\text{eff}} = H_0 + \delta H \]

\[ N_0 - 2 \]
\[ N_0 - 1 \]
\[ N_0 \]
\[ N_0 + 1 \]
\[ N_0 + 2 \]

\[ |0, 0\rangle, |1, 1\rangle \]
\[ |0, 1\rangle, |1, 0\rangle \]

\[ \gamma_{ij} \psi_i^\dagger \psi_j e^{-i\phi/2} \]

\[ \sim E_c \]
Topological Kondo effect

\[ H_{\text{eff}} = H_{\text{lead}} + \sum_{i \neq j} \lambda_{ij}^+ \gamma_j \gamma_i \psi_i^\dagger \psi_j - \sum_i \lambda_i^\pm \psi_i^\dagger \psi_i, \quad (\lambda_i^\pm = \frac{t_i^\dagger t_i}{E_c}). \]

3 Majoranas \( \sigma_\alpha : \quad \sigma_1 = -i \gamma_2 \gamma_3, \quad \sigma_2 = -i \gamma_3 \gamma_1, \quad \sigma_3 = -i \gamma_1 \gamma_2 \)

3 leads \( \lambda_{12}^+ = \lambda_{21}^+ \)

\[ \gamma_1 \gamma_2 = -\gamma_2 \gamma_1 \]

\[ \lambda_{12}^+ \gamma_1 \gamma_2 (\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2) = \lambda_{12}^+ \sigma_3 i(\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2) \]

\[ i(\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2) = \psi_3^\dagger J_3 \psi = \hat{J}_3 \]

\[ J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \]

spin-1 generator

\[ \hat{J}_3 : \text{spin-1 density} \]

\[ \lambda_{12}^+ \gamma_1 \gamma_2 (\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2) = \lambda_{12}^+ \sigma_3 \hat{J}_3 \]

Topological Kondo effect

\[ H_{\text{eff}} = H_{\text{lead}} + \frac{1}{2} \sum_{\alpha} \lambda_\alpha \sigma_\alpha \hat{J}_\alpha - \sum_i \lambda_i^\pm \psi_i^\dagger \psi_i, \quad (\lambda_\alpha = \sum_{a,b} |e_{a,b}| \lambda_{ab}^+). \]

Antiferromagnetic Kondo

- \( S = \frac{1}{2} \) impurity coupled to
- \( S = 1 \) conduction electrons.

[Fabrizio&Gogolin PRB 1994, Sengupta&Kim, PRB 1996]

Overscreened Kondo

- “Spin” distributed nonlocally
- curious transport features
  (e.g., \( G_{ij} \) Kondo anisotropy)
- 3 leads is minimal for Kondo:
  “smoking gun” for non-locality
Outline

- Intro to Majorana fermions
  - what are they?
  - how do they emerge?
  - key features & potential uses
  - some of the experimental signatures

- Topological Kondo effect
  - from Majoranas to Kondo – the topological Kondo idea
  - transport signatures, incl. NFL features
  - topological Kondo beyond the minimal setup
  - (exact) scaling functions for nonequilibrium transport

Transport signatures

Focus on T-dep. of linear conductance

\[ H_{\text{eff}} = H_{\text{lead}} + \sum_{i \neq j}^{\lambda_{ij}^+ \gamma_j \gamma_i \psi_i^\dagger \psi_j - \sum_i^{\lambda_{ii}^- \psi_i^\dagger \psi_i} \]

\[ G_{21}(T) \sim (\lambda_{12}^+)^2 + \ldots \]

Log-singularities \( \sim [\ln \left( \frac{D}{T} \right)]^n \)

Sum up by RG

Standard Kondo flow:

\[ \frac{d\lambda_1}{dt} = \rho \lambda_2 \lambda_3, \text{ cycl. perm.,} \]

\[ \frac{d\lambda_{kk}}{dt} = 0 \quad (\lambda_a = \sum_{ab} |\epsilon_{ab}| \lambda_{ab}^\pm) \]
Transport signatures

**Focus on T-dep. of linear conductance**

$$H_{\text{eff}} = H_{\text{lead}} + \sum_{i \neq j} \lambda_{ij}^+ \gamma_j \psi_i^\dagger \psi_j - \sum_i \lambda_i^+ \psi_i^\dagger \psi_i$$

$$G_{21}(T) \sim (\lambda_{12}^+)^2 + \ldots$$

Log-singularities \( \sim [\ln (\frac{T}{T_K})]^n \)

Conductance from RG:

$$\lambda^+(D) \sim \frac{1}{\ln(D/T_K)}$$

$$\lambda_\alpha^+/\lambda_\beta^+ \to 1$$

$$G_{21}(T) \sim \frac{1}{\ln^2(T/T_K)} \quad (T \gg T_K)$$

$$G_{21}/G_{31} \to 1 \quad (T_K \sim E_c e^{-1/\rho \lambda})$$

---

**Majorana-Klein hybridization**

[B. Béri, PRL 110, 216803 (2013)]

**Bosonization approach**

$$H^{(j)}_{\text{lead}} = H_0(\rho_j, \theta_j) \quad [\rho(x), \theta(y)] = i\delta(x - y)$$

$$\psi_j(0) \sim \Gamma_j e^{i\theta_j(0)}$$

**Klein factors** = auxiliary Majoranas

Coupling:

$$\gamma_j \psi_j \sim \gamma_j \Gamma_j e^{i\theta_j(0)}$$

Parity of hybrid fermion

$$d_j = \frac{1}{2} (\gamma_j + i \Gamma_j)$$

$$\pi_j = 1 - 2d_j d_j = -i \gamma_j \Gamma_j$$

/problem effectively in terms of \( \rho_j, \theta_j \) only, related to QBM.

Topological Kondo solved for

- arbitrary # of leads
- Luttinger liquid leads

[see also Altland&Egger PRL 2013, A. Zazunov et al. arXiv:1307.0210]
Transport signatures

Conductance for $T \ll T_K$

\[ G_{k\neq l}(T) = \frac{2e^2}{3h} + c_k T^{2/3} \]

- Noninteger power law: **NFL physics - w/o fine tuning!**

Fermi liquid transport

Resonant tunneling

\[ G(T) \sim \int dE \, t(E) \left( \frac{\partial f_\beta(E - \mu)}{\partial E} \right) \sim \begin{cases} G_0 + aT \quad (\mu \neq \varepsilon_0) \\ G_0 + aT^2 \quad (\mu = \varepsilon_0) \end{cases} \]

**Fermi liquid physics → integer exponents**

\[ G(T) \sim \int dE \left[ t(\mu) + e^{\alpha E} \right] \left( \frac{\partial f_\beta(E - \mu)}{\partial E} \right) = G_0 + e^{\alpha T} \int d(\beta E) \langle \beta E \rangle^{\alpha} \left( \frac{\partial f_\beta(E - \mu)}{\partial E} \right) \]
**NFL & Overscreened Kondo**

Multichannel Kondo


Topological Kondo

Stable NFL

Unstable NFL

Transport signatures

- Noninteger power law:
  - NFL physics - w/o fine tuning!
- \( T \to 0 \)
  - isotropic, universal
  - two terminal value beyond \( e^2/h \):
    - correlated Andreev reflection

\[ G_{21} = I_2/V_1 \]

Conductance for \( T \ll T_K \)

\[ G_{k\neq l}(T) = \left(\frac{2e^2}{3h}\right) + c_{kl}T^{2/3} \]

[Fig: A Toth et al. PRB 2007]
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Kondo with more Majoranas

3 Majoranas/leads:

Kondo “impurity”: \( \gamma_j \gamma_k \rightarrow \sigma_\alpha \)  \( S = 1/2 \)

Cond. electrons: \( J_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)  \( S = 1 \)

\( M \) Majoranas/leads

Kondo “impurity”: \( S = \)?

Cond. electrons: \( S = ? \)

assumes SU(2)
**Kondo with more Majoranas**

3 Majoranas/leads:

\[ \begin{array}{c}
\text{Kondo } \text{impurity}: \\
\gamma_j \gamma_k \rightarrow \sigma_\alpha \\
\text{Cond. electrons: } J_3 = i \begin{pmatrix} 0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \\
\end{array} \]

\[ \text{SO}(3) \text{ spinor} \]

Can also view as \( \text{SO}(3) \):

SO(3) spinor

\[ \text{SO}(3) \text{ def.} \]

\[ M = 5 \]

precisely how Clifford algebra gives spinors!

\[ \text{SO}(M) \text{ spinor} \]

\[ \text{SO}(M) \text{ def.} \]

\[ \begin{pmatrix}
0 & -1 & 0 & \cdots \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & \ddots
\end{pmatrix} \]

\[ H_K = \sum_{j<k} \lambda_{jk} \Gamma_{jk} (\psi^\dagger J_{jk} \psi) \]

\[ \text{SO}(M) \text{ Kondo problem} \]

**Majorana Klein & QBM connection**

SO(M) Kondo problem (\( H_{\text{eff}} = H_{\text{lead}} + H'_{\text{eff}} \))

\[ H'_{\text{eff}} = \sum_{j \neq k} \lambda^+_{jk} \gamma_j \gamma_k \psi_k^\dagger \psi_j - \sum_j \lambda^-_{jj} \psi_j^\dagger \psi_j \]

General features:

\[ \gamma_j \psi_j \sim \gamma_j \Gamma_j e^{i\theta_j(0)} \]

\[ H'_{\text{eff}} = -\sum_{j \neq k} \lambda^+_{jk} \frac{e^{i(\theta_j - \theta_k)}}{a} - \sum_j \lambda^-_{jj} \theta_j e^{\varphi_j} \]

Weak coupling RG gives:

\[ \lambda^+(D) \sim \frac{1}{\ln(D/T_K)} \]

\[ \lambda^+/\lambda^- \rightarrow 1 \]

\[ T \gg T_K : \]

\[ G_{kl}(T) \sim \frac{1}{\ln^2(T/T_K)} \]

\[ G_{kl}/G_{mn} \rightarrow 1 \]
Majorana Klein & QBM connection

SO(M) Kondo problem \( (H_{\text{eff}} = H_{\text{lead}} + H'_{\text{eff}}) \)

\[
H'_{\text{eff}} = \sum_{j \neq k} \lambda^+_j \gamma_j \gamma_k \psi_k^\dagger \psi_j - \sum_j \lambda^-_{jj} \psi_j^\dagger \psi_j
\]

General features:

\[
\gamma_j \psi_j \sim \gamma_j \Gamma_j \ e^{i \theta_j(0)}
\]

\[
H'_j = -\sum_{j \neq k} \lambda^+_j \frac{e^{i(\theta_j - \theta_k)}}{a} - \sum_j \lambda^-_{jj} \partial_x \varphi_j
\]

\[
\theta_j = w_j \cdot r + \frac{R_0}{\sqrt{M}}
\]

\[
\varphi_j = w_j \cdot k + \frac{R_0}{\sqrt{M}} \ w_j \cdot w_1 = \delta_{j1} - \frac{1}{M}
\]

\[
T \gg T_K : \quad G_{kl}(T) \sim \frac{1}{\ln^2(T/T_K)}
\]

\[
G_{kl} / G_{mn} \rightarrow 1
\]

\[
T \ll T_K : \quad G_{kl}(T) = \frac{2e^2}{M \hbar} + c_{kl} T^{2\Delta - 2}
\]

\[
G_{ll}(T) = \Delta \frac{e^2}{\hbar} + c_{ll} T^{2\Delta - 2}
\]

Tunneling between minima; dimension:

\[
\Delta = \frac{2(M-1)}{M} \quad \rightarrow \quad \text{robust NFL}
\]
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**Nanoscale Kondo experiments:** universal scaling functions

**Linear transport**

[D. Goldhaber-Gordon et al. PRL, (1998)]

**Non-equilibrium transport**


Theory for topological Kondo analogues?

Only for linear conductance, via numerics (NRG).


**Goal:** exact approach to the topological Kondo effect, able to access universal physics below $T_K$ – both in and out of equilibrium.
Setup & strategy

Setup:
- consider M Majoranas/leads;
- leads taken as Fermi liquids;
- focus on local (lead M) observables e.g., $G = \frac{\partial I}{\partial V}$

Strategy: **universality** for $k_B T, eV \lesssim T_K$

$\Rightarrow$ identify an “easily solvable” limit.

“Toulouse limit” for topological Kondo

Note: top. Kondo is exactly solvable w/o Toulouse limit – in *equilibrium*.  


Toulouse limit: expected performance

“Conventional” Toulouse limit vs Bethe ansatz for “conventional” Kondo:

[Desgranges & Schotte, Phys. Lett. A 1982]
Toulouse limit

**Main innovation:**
topological Kondo $\rightarrow$ backscattering in repulsive Luttinger liquid (massless BSG)

Powerful exact BSG transport results $\rightarrow$ exact top. Kondo transport in the Toulouse limit


**Framework:** Bosonisation + Majorana Klein

[B. Béri, PRL 110, 216803 (2013); Altland&Egger PRL 2013]

**Summary of the mapping:**

![Diagram](image)

Anisotropic limit w/ “RHS” @ Kondo FP $\rightarrow$ lead M tunneling to RHS charge mode $\rightarrow$ backscatt. in LL w/ quantized $g_b$

[Chamon&Fradkin, PRB 1997]

Key feature: transparent relation b/w physical & mapped charges

Nanoscale Kondo experiments: universal scaling functions

**Linear transport**

[D. Goldhaber-Gordon et al. PRL, (1998)]

![Linear transport graph](image)

**Non-equilibrium transport**


![Non-equilibrium transport graph](image)
From BSG to topological Kondo transport: Exact conductance

**Summary**
Conduction electrons + Majoranas → **“topological Kondo”**

- Demonstrates q-dynamics of top. qubits
  - Probes of the Majoranas’ quantum computing potential
- Robust realisation of **NFL Kondo physics**
- “Smoking gun” signature