Ergodic and Non-Ergodic Dynamics -II

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Unitary Quantum Dynamics

$$\rho(t+\delta t) = U\rho(t)U^\dagger$$

Dynamics of isolated, MB systems undergoing unitary time evolution:

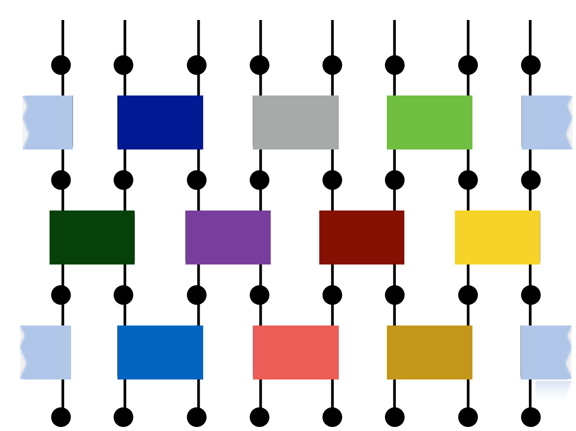
spins/cold atom molecules/ black holes/…

strongly interacting, excited (no quasiparticles)

Time-independent Hamiltonian: $$U(t) = e^{-iHt}$$

Floquet: $$U(nT) = [U(T)]^n$$

Random unitary circuit: $$U(t) =$$



Can *reversible* unitary time evolution bring a system to thermal equilibrium at late times?

If so, *how* does the system reach thermal equilibrium? For local operators A, how does the system "hide" $\langle A \rangle_{t=0}$?

What are the dynamics of quantum entanglement?

How does hydrodynamics emerge from reversible reversible unitary dynamics?

Many-Body "Quantum Chaos" vs. Thermalization

What is a precise formulation for many-body quantum chaos?

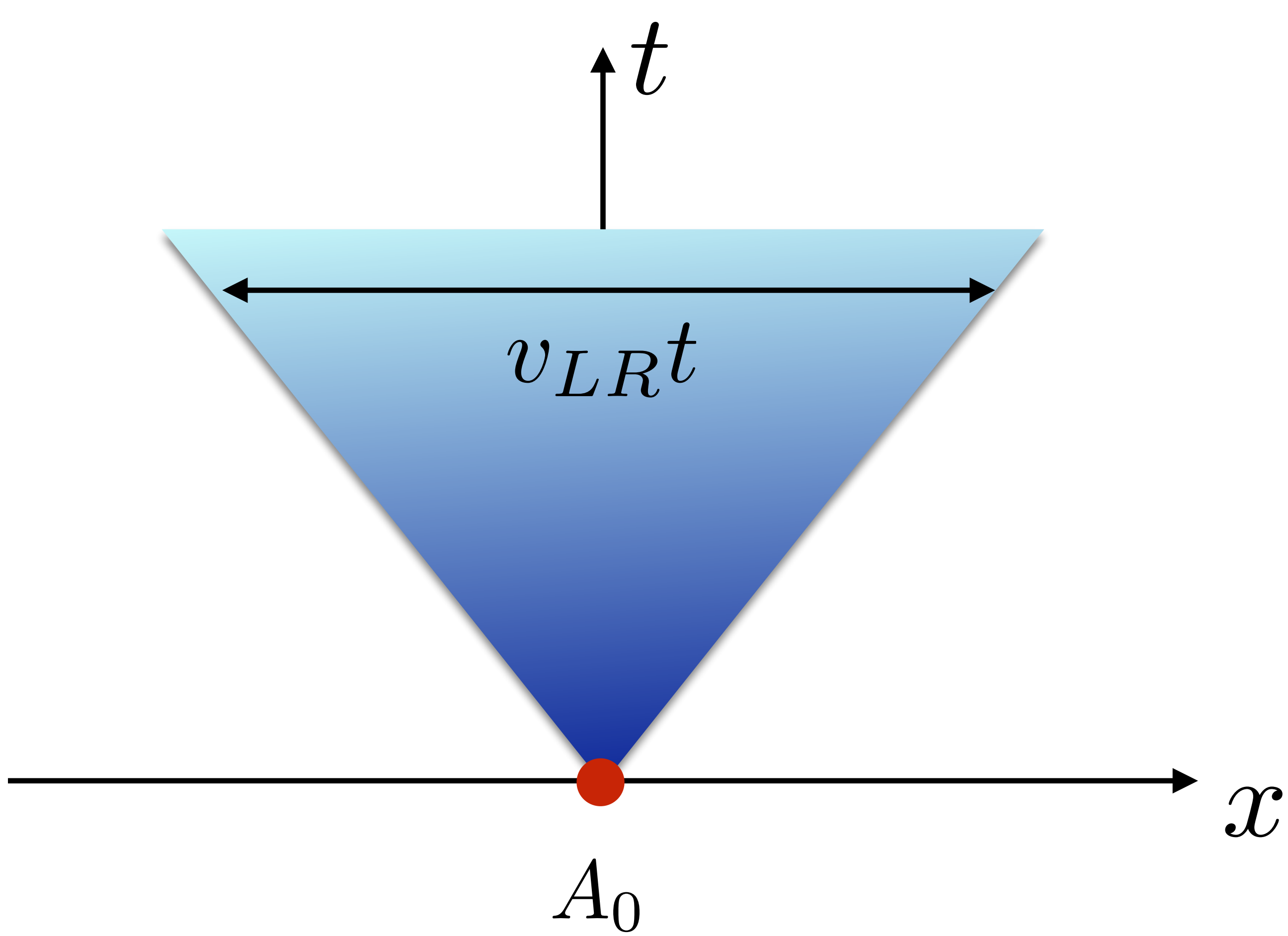
Is there a useful definition for chaos that is distinct from thermalization?

Are there distinct (universal) signatures of chaos at early/intermediate/late times? What are the most appropriate observables for probing these regimes?

For local operators A, how does the system "hide" $\langle A \rangle_{t=0}$?

Look at the dynamics of "operator spreading" *i.e.* time evolution of operators in the Heisenberg picture

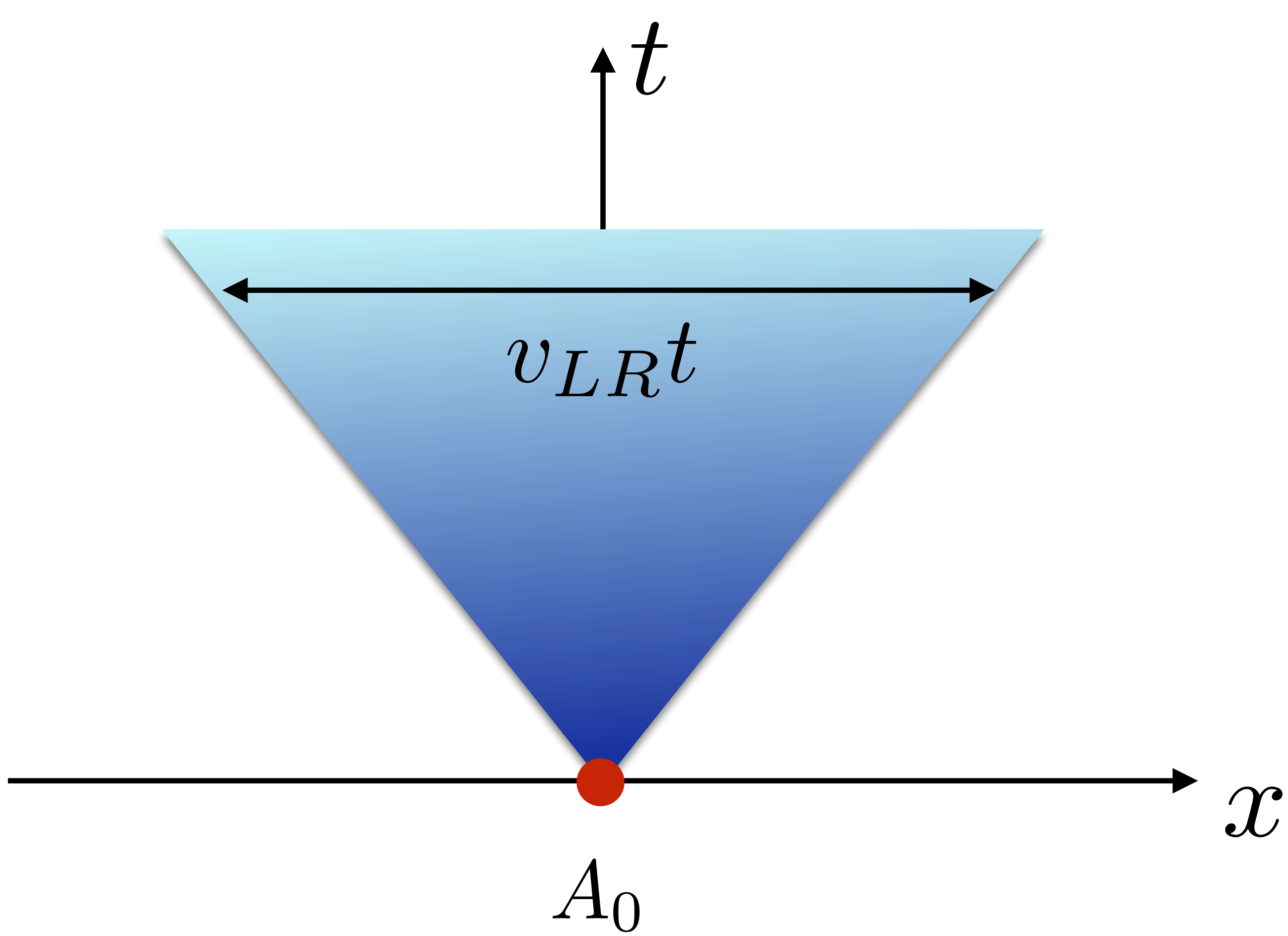
$$A_0(t) = U^\dagger(t) A_0 U(t)$$



Operator generically spreads ballistically within a "Lieb-Robinson" cone — getting highly entangled within the cone — for clean, thermalizing local quantum systems.

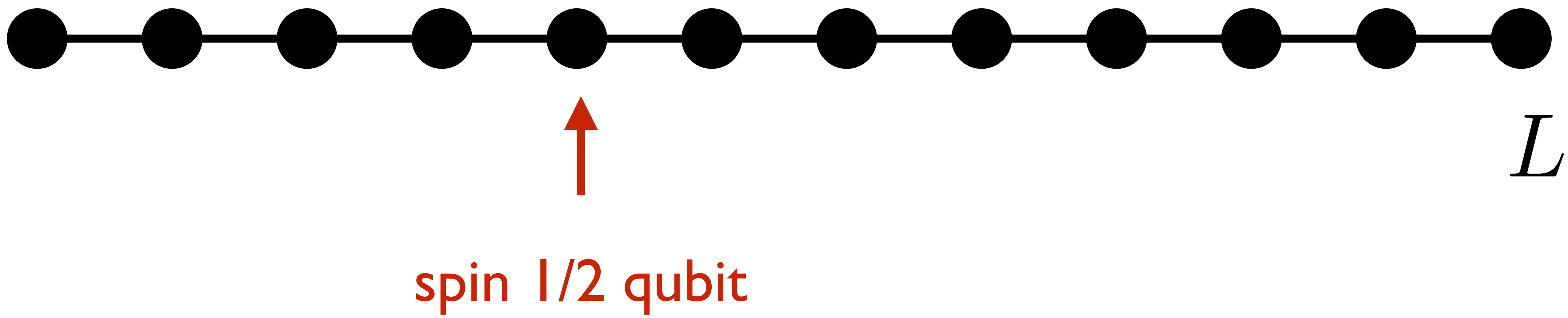
For local operators A, how does the system "hide" $\langle A \rangle_{t=0}$?

$$A_0(t) = U^\dagger(t) A_0 U(t)$$



- Spreading can be sub-ballistic $\sim t^a$, $a<1$ for disordered thermalizing systems due to Griffiths effects
- Spreading is logarithmic for MBL systems.
- Spreading is also ballistic for integrable systems with quasiparticles

Setup



Local Hilbert space dimension: 2 (can also consider qudits with q)

4 operators per site: σ_i^μ $\mu \in \{0, 1, 2, 3\}$

Orthonormal basis of operators:

$(4)^L$ "Pauli strings"

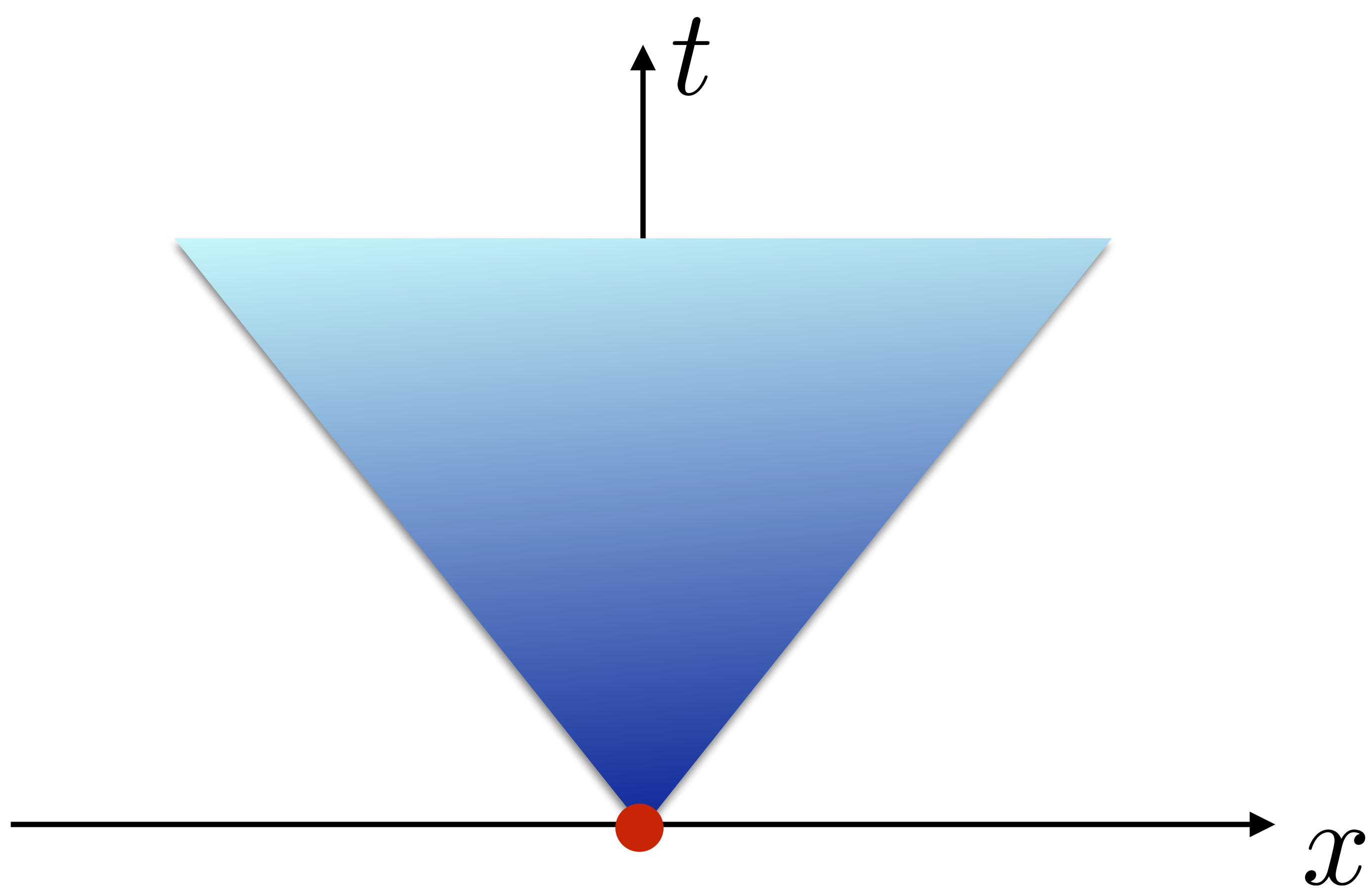
$$S = \prod_i \otimes \sigma_i^{\mu_i}$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

$$xIyz, IzII, xxxx \cdots$$

VK Vishwanath Huse (2017)

Operator Spreading



$$O(t) = U^{\dagger}(t) O_0 U(t)$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t) \mathcal{S}$$

sum over $(4)^L$
Pauli strings

Operator Spreading: unitarity

Unitarity preserves operator norm

$$\mathrm{Tr}[O_0^\dagger(t)O_0(t)] = \mathrm{Tr}[O_0^\dagger O_0] = 2^L$$

$$\Downarrow$$

$$\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

Operator shape: Right weight

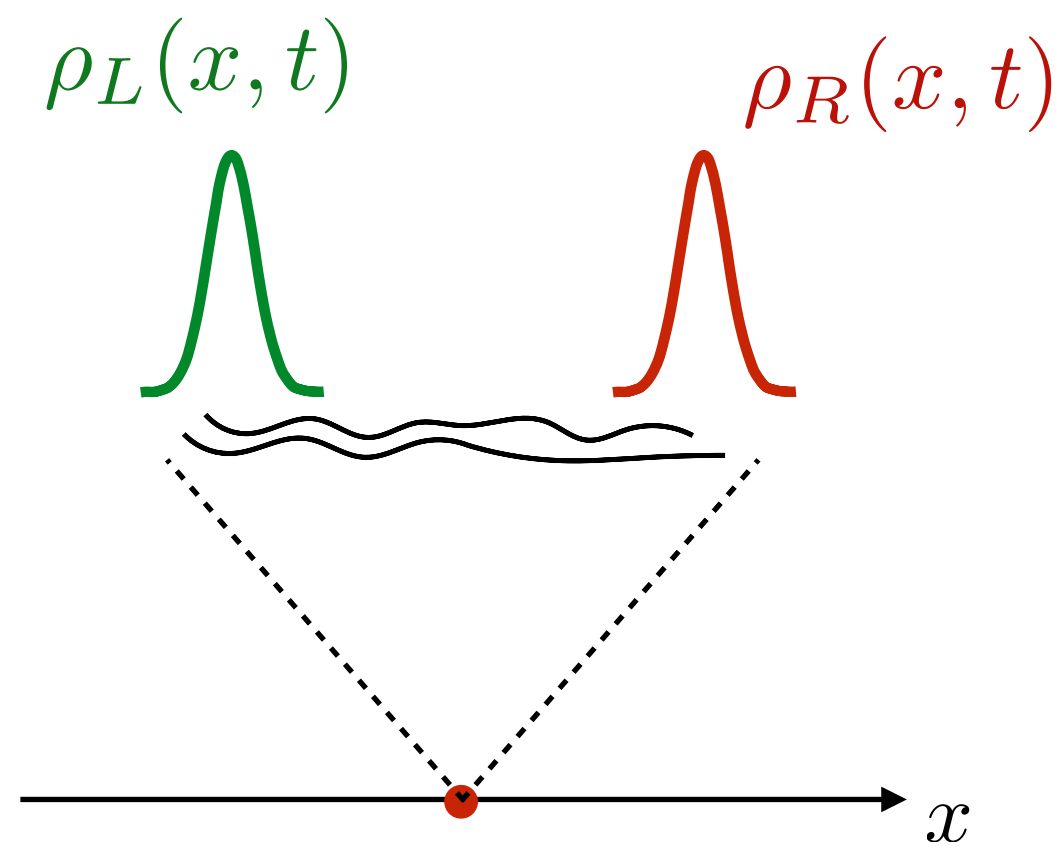
Right-Weight: "emergent" density following from unitarity

$$\rho_R(i,t) = \sum_{\substack{\text{strings } \mathcal{S} \text{ with} \\ \text{rightmost non-} \\ \text{identity on site } i}} |a_{\mathcal{S}}|^2, \qquad \sum_i \rho_R(i,t) = 1.$$

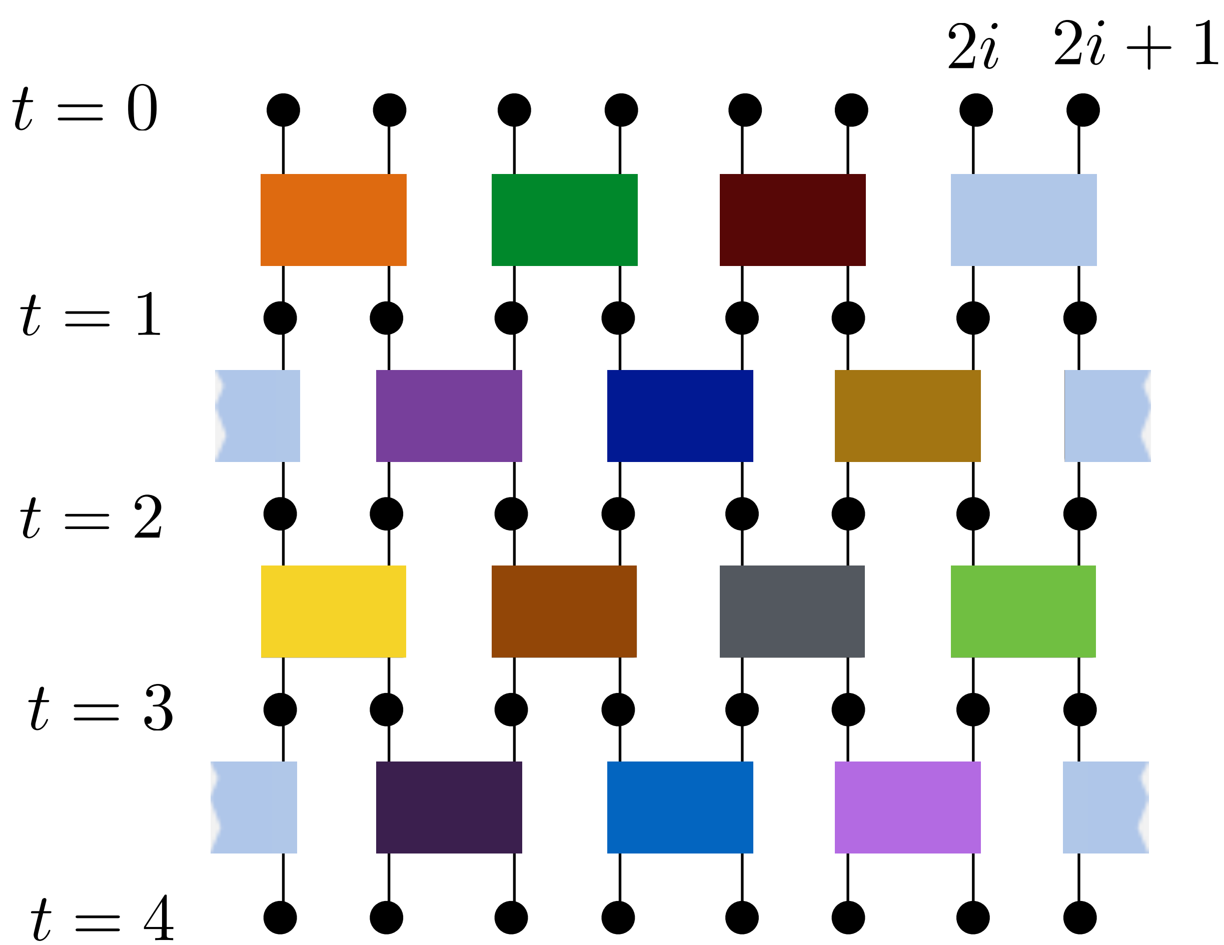
Each string has right/left edges beyond which it is purely identity.

ρ looks at the density distribution of the "right front" of the operator.

As operator spreads, weight moves to longer Pauli strings.



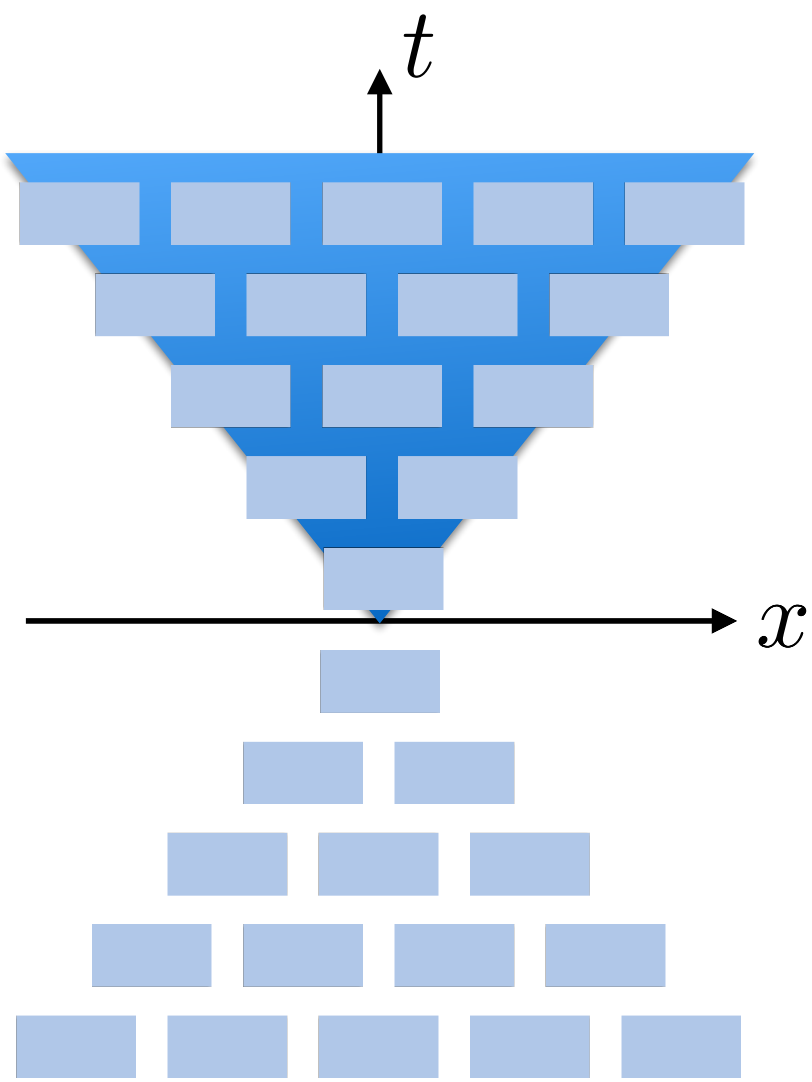
Dynamics with Random circuits



- Unitary gates independent and random in space and time.
- Allows us to derive exact results about operator spreading, building in only the requirements of unitarity and locality.
- Hope (and numerically verify) that results generalize to more realistic setting like time-independent Hamiltonians

Nahum et. al., (2016, 2017),
von Keyserlingk et. al (2017).

Operator shape: random circuit



Front dynamics: biased diffusion

$$U^\dagger(\delta t)\mathcal{S}U(\delta t)$$

has amplitudes for

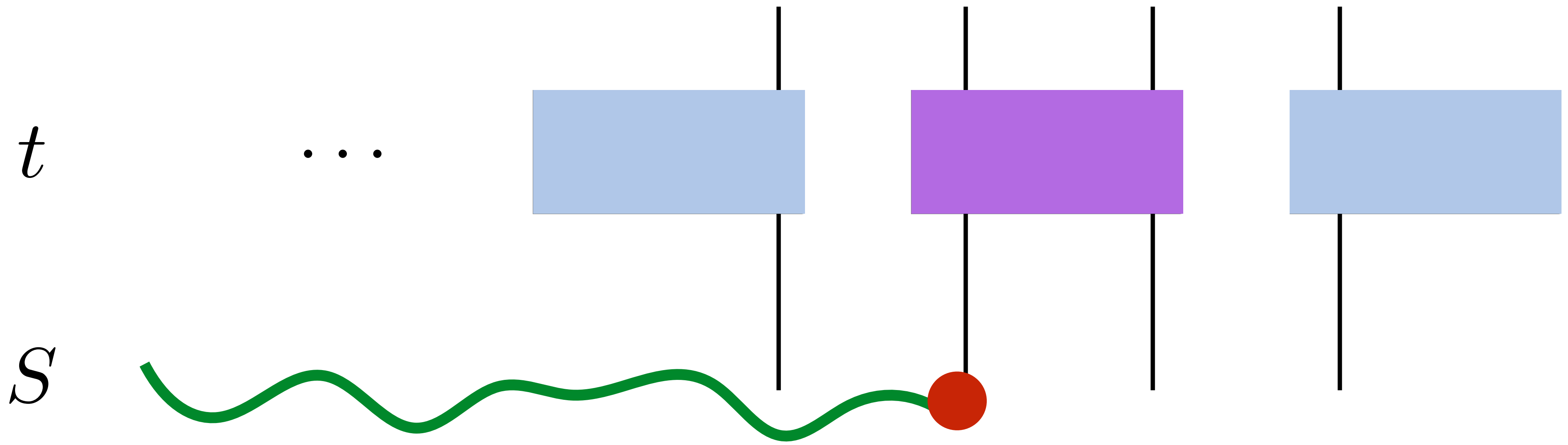
- making S shorter
- leaving it same length
- making S longer

But, biased towards making S longer.

Example, only 3/15 non-identity two-site spin 1/2 operators have identity on the right site.

Nahum et. al., (2017)
von Keyserlingk et. al (2017)

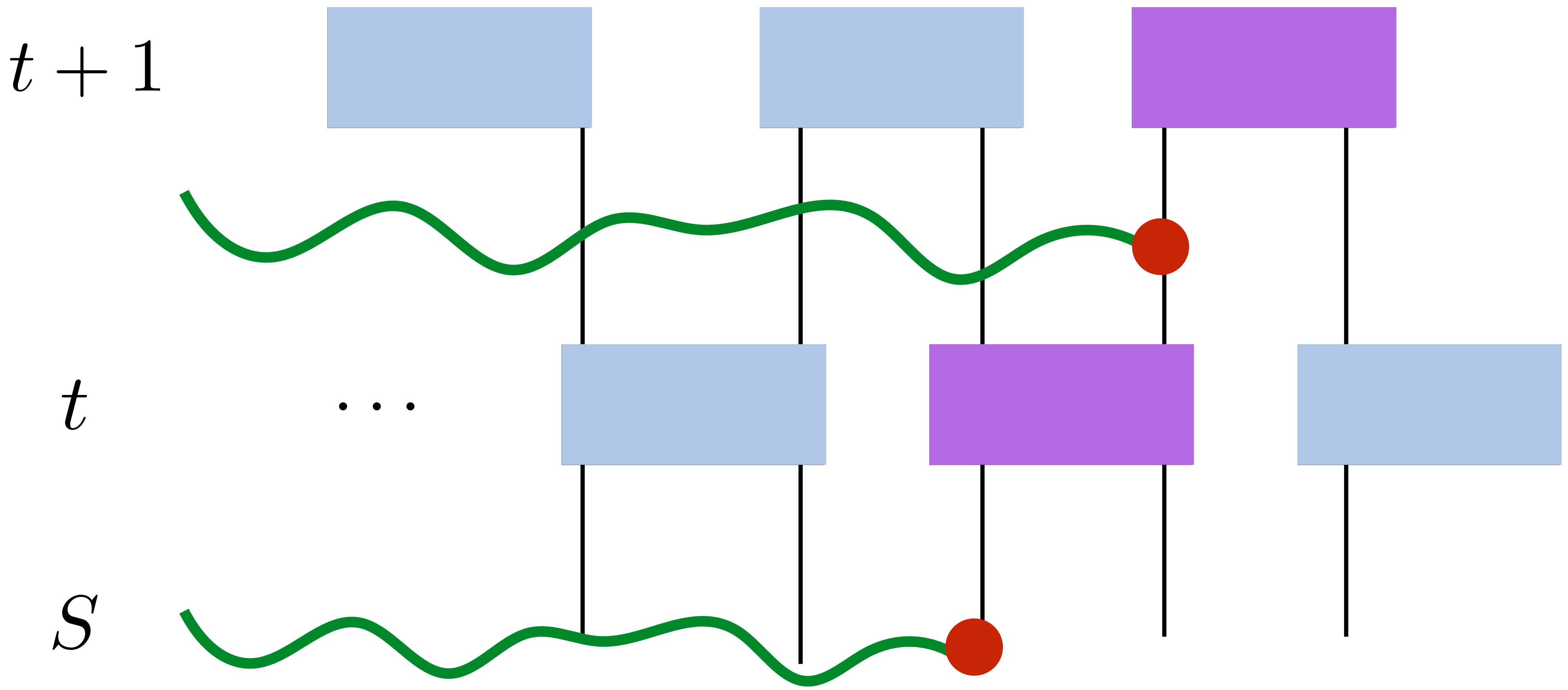
Operator shape: unconstrained circuit



Nahum et. al., (2017)
von Keyserlingk et. al (2017)

Operator shape: unconstrained circuit

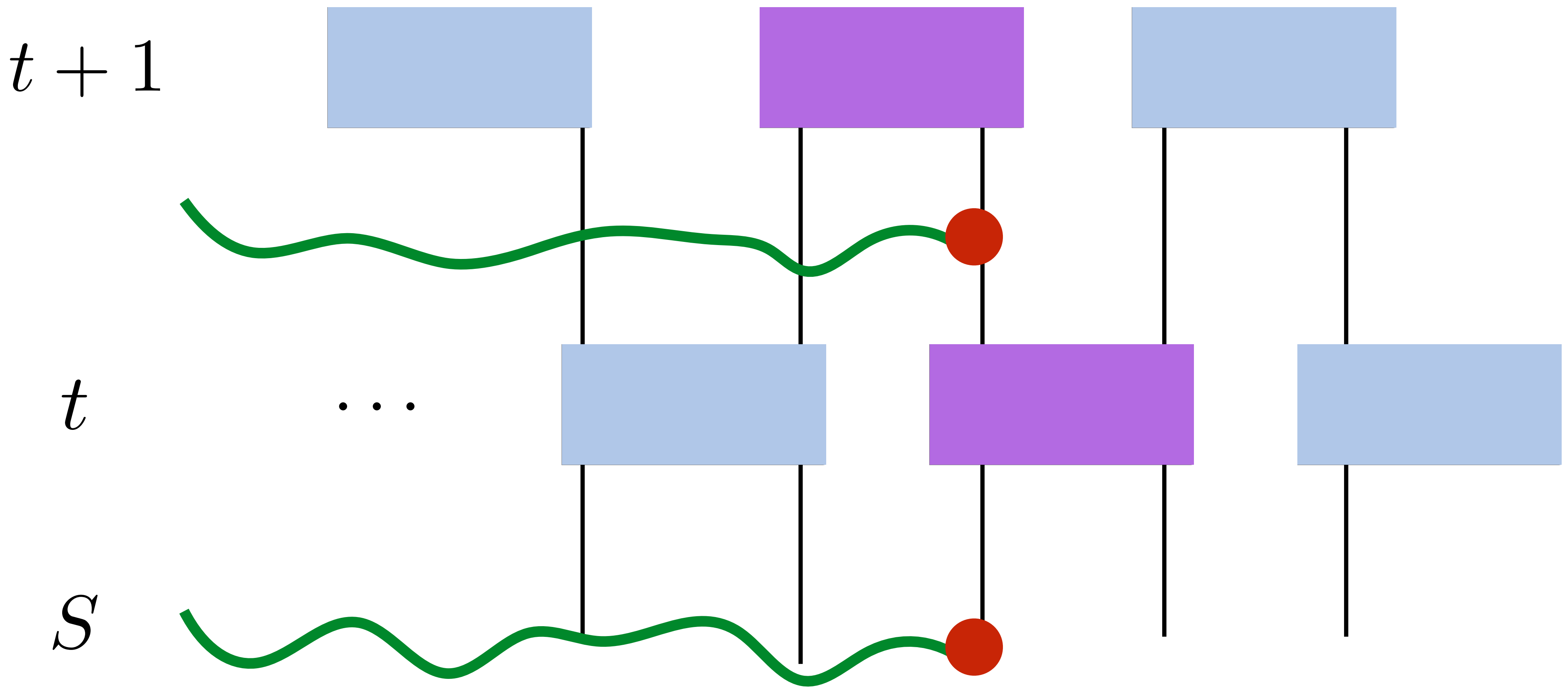
Probability: 12/15



Nahum et. al., (2017)
von Keyserlingk et. al (2017)

Operator shape: unconstrained circuit

Probability: 3/15



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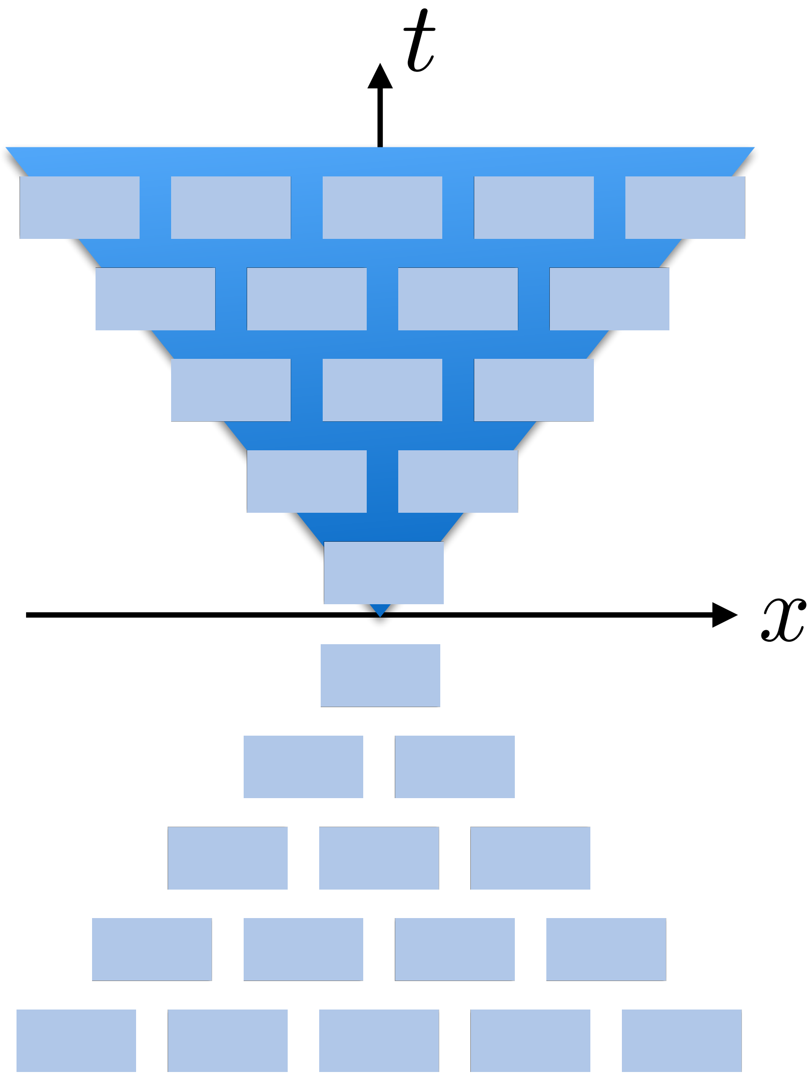
Operator shape: unconstrained circuit



Front dynamics: biased random-walk

Emergent hydrodynamics:

$$\partial_t \rho_R(x,t) = v_B \partial_x \rho_R(x,t) + D_\rho \partial_x^2 \rho_R(x,t)$$

$$\rho_R(x,t) \approx \frac{1}{\sqrt{4\pi D_\rho t}} e^{-\frac{(x - v_B t)^2}{4 D_\rho t}}$$

$$v_B \sim 1 - \frac{2}{q^2};\ D_\rho \sim \frac{2}{q^2}$$

Nahum et. al., (2017)
von Keyserlingk et. al (2017)

Operator shape: unconstrained circuit

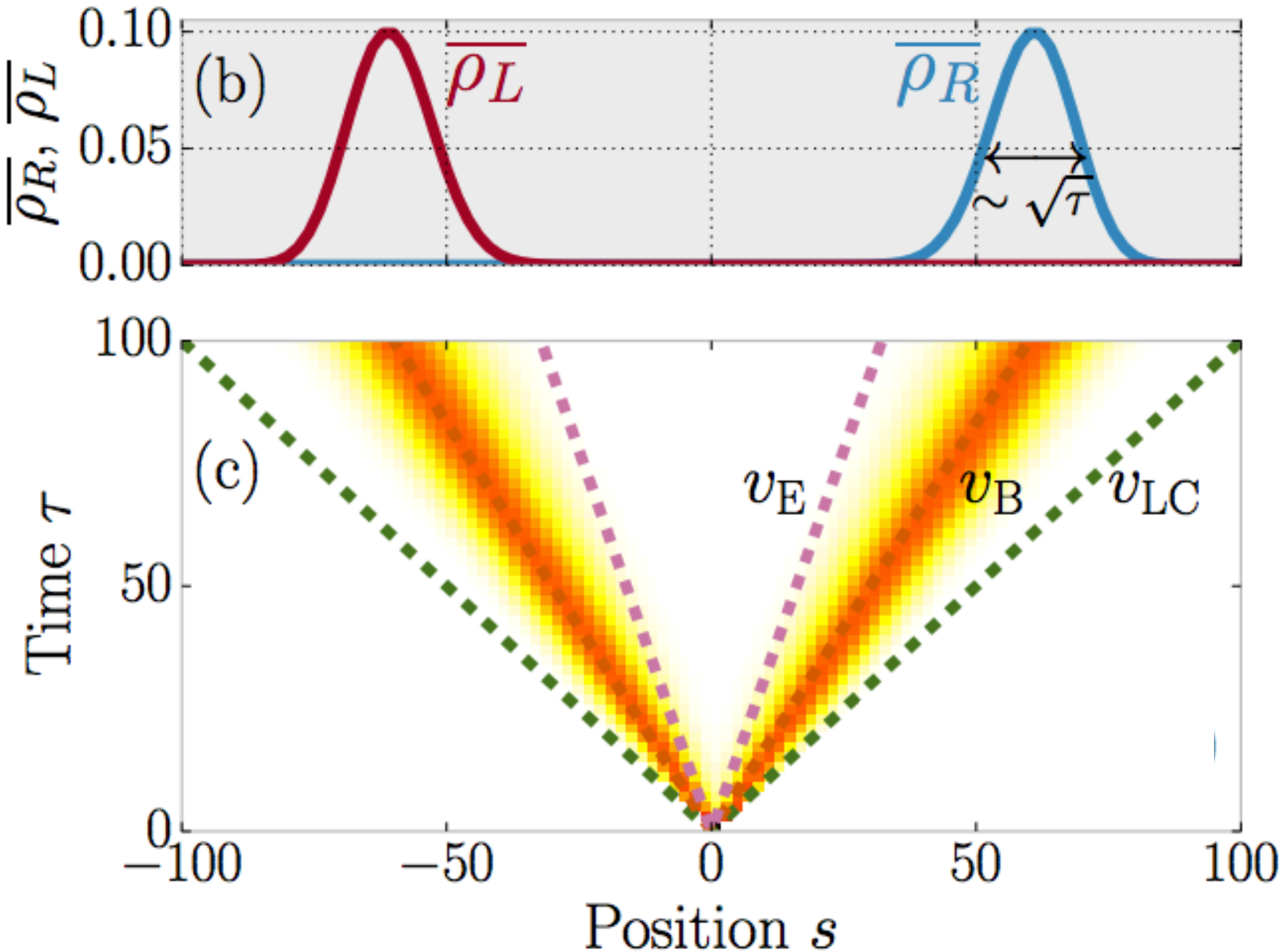


Figure from:
von Keyserlingk et. al (2017)

Thermalization + Conservation Law

Chaotic many-body system (ballistic information spreading)

+

locally conserved diffusive densities (energy/charge/..)

VK Vishwanath Huse (2017)

Unitarity vs. Dissipation

Chaotic many-body system (ballistic information spreading)
+
locally conserved diffusive densities (energy/charge/..)

Q: How does unitary quantum dynamics, which is reversible, give rise to diffusive hydrodynamics, which is dissipative (increases entropy)?

Unitary Dynamics: Reversible

Diffusion: Irreversible/Dissipation

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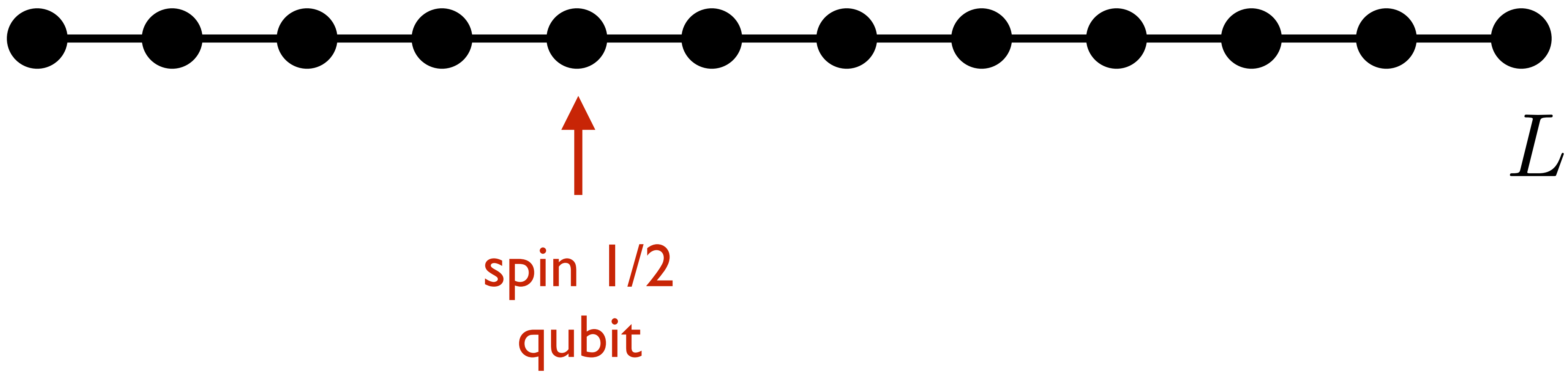
Setup



z component of spin 1/2 qubits conserved

$$S_z^{\mathrm{tot}} = \sum_i^L z_i$$

$$[U(t), S_z^{\mathrm{tot}}] = 0$$

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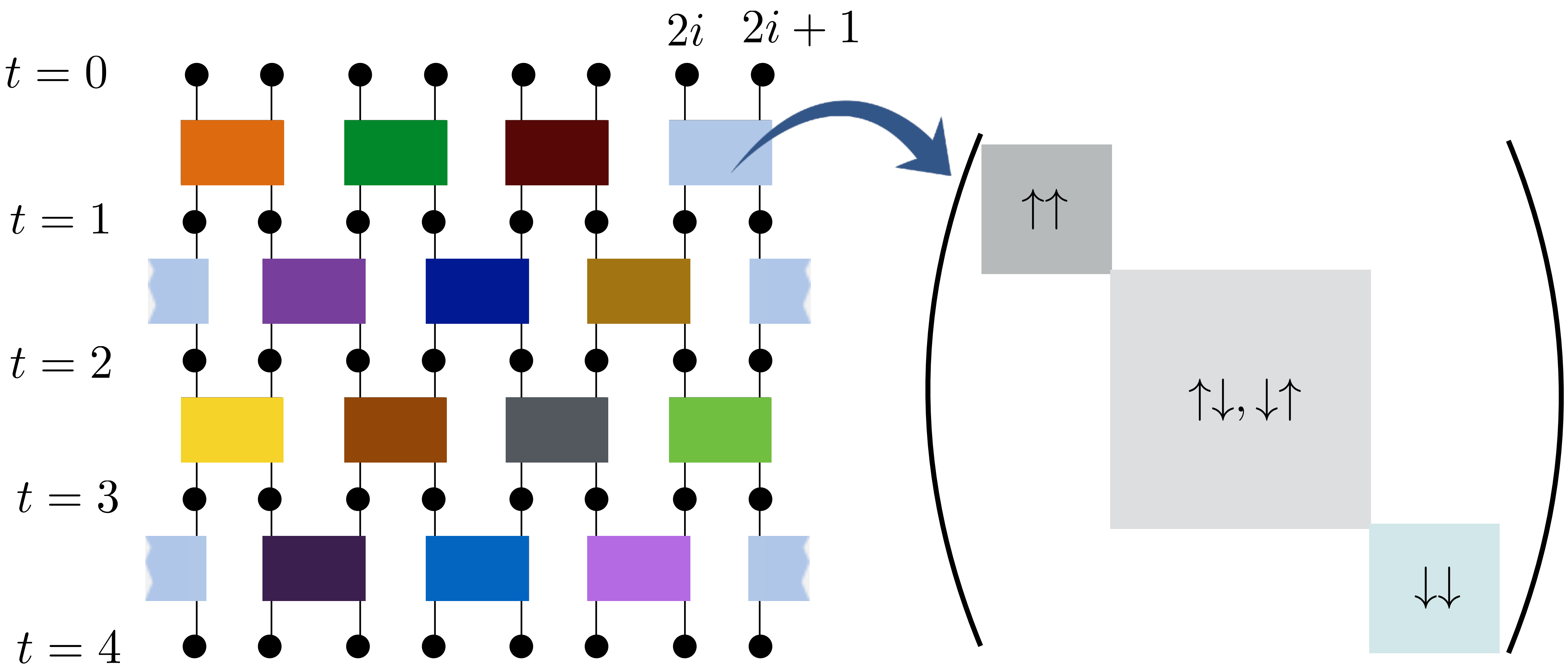
Setup: Random Conserving Circuit Model



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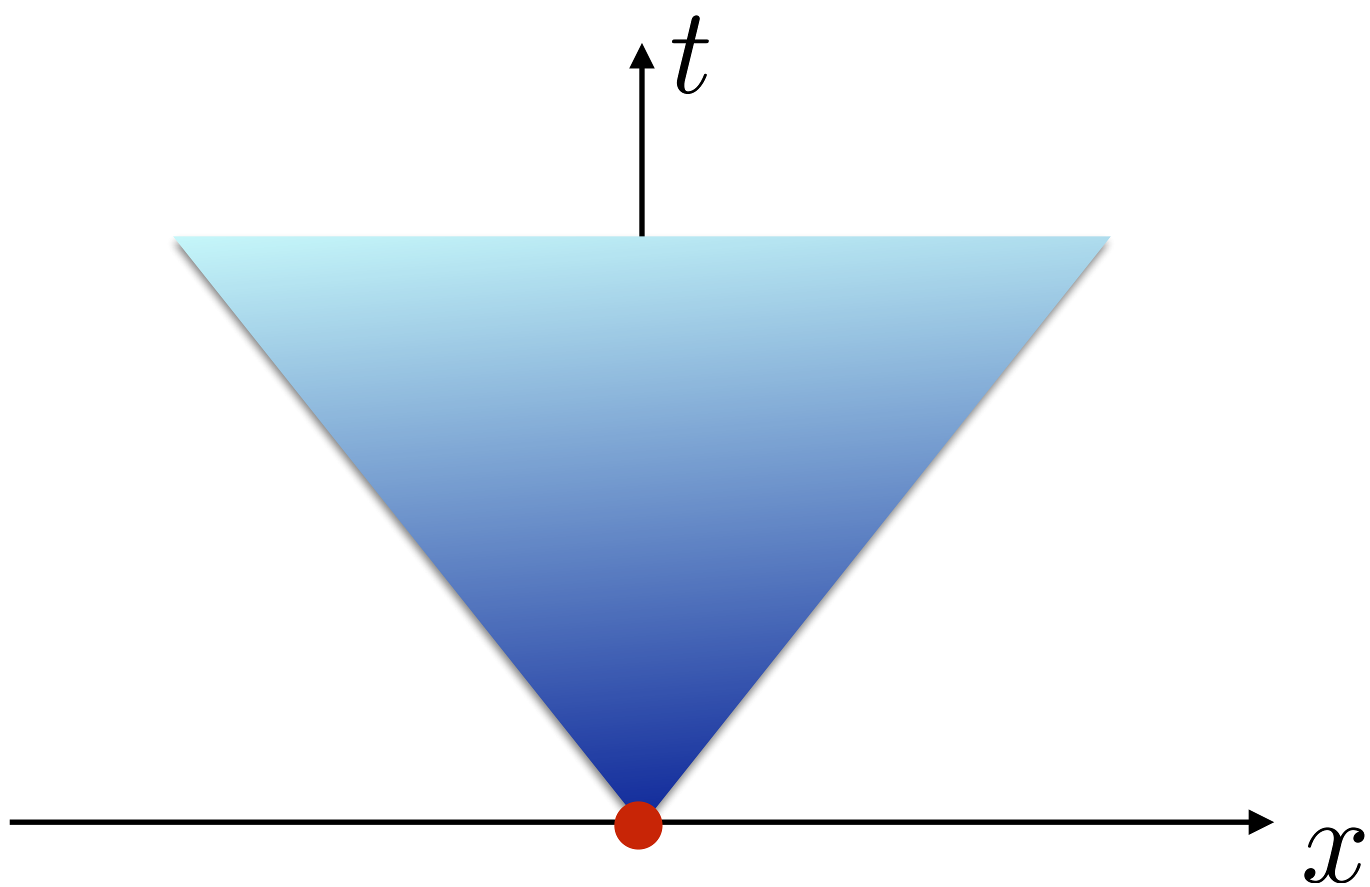
Builds on: Nahum et. al., (2016, 2017), von Keyserlingk et. al (2017).

Operator Spreading



Spreading constrained by:

- Unitarity
- Conservation Law(s)

$$O(t) = U^\dagger(t) O_0 U(t)$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t) \mathcal{S}$$

sum over $(4)^L$ strings

Operator Spreading: conservation law

Separate operator into conserved and non-conserved pieces

$$O_0(t) = O_0^c(t) + O_0^{\mathrm{nc}}(t)$$

$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

$$\mathrm{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

Operator Spreading: conservation law

Separate operator into conserved and non-conserved pieces

$$O_0(t) = O_0^c(t) + O_0^{\mathrm{nc}}(t)$$

exp(L) mostly non-local strings, thus "hidden"

$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

L local operator "strings", conserved densities

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\mathcal{S}$$

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Operator Spreading: conservation law

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L local operator "strings", conserved densities

$$\text{Tr}[O_0(t) S_z^{\text{tot}}] = \text{constant} \implies$$

$$\sum_{i=1}^{L} a_i^c(t) = \text{constant}$$

$$O(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t) \mathcal{S}$$

$$\text{Tr}[S^\dagger S']/(2^L) = \delta_{SS'}$$

Operator Spreading

Operator dynamics governed by the interplay between:

Unitarity: $$\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1$$

Conservation law: $$\sum_{i=1}^{L} a_i^c(t) = \text{constant}$$

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Spreading of conserved charges

First, consider spreading of conserved density

$$O_0 = z_0$$

$$a_i^c(t=0) = \delta_{i0}$$

$$\sum_i a_i^c(t) = 1$$

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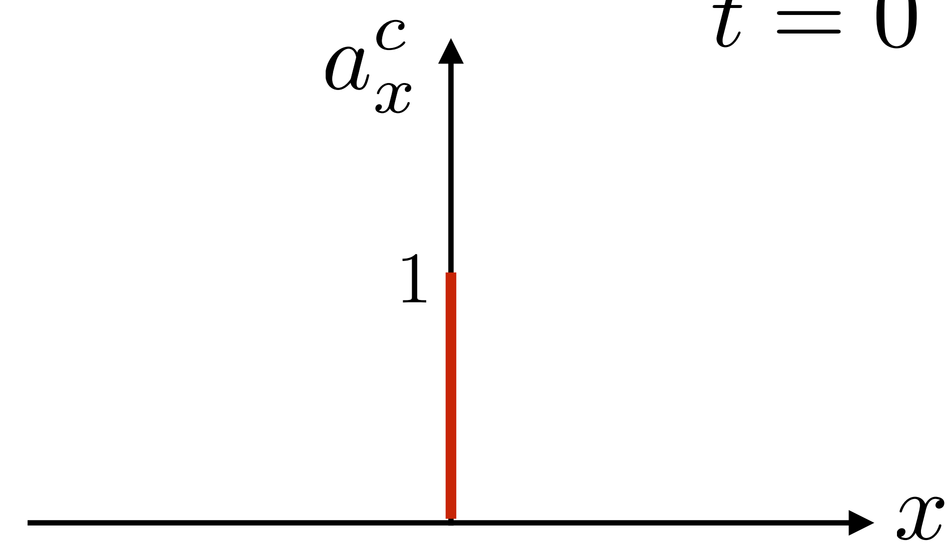
Diffusion & conserved amplitudes: intuition

Initial state: Infinite temperature equilibrium
+ local charge perturbation

$$\rho_0 = \frac{1}{2^L}\left[\mathbb{I} + \epsilon O_0\right]$$

$$O_0 = z_0$$

$t = 0$



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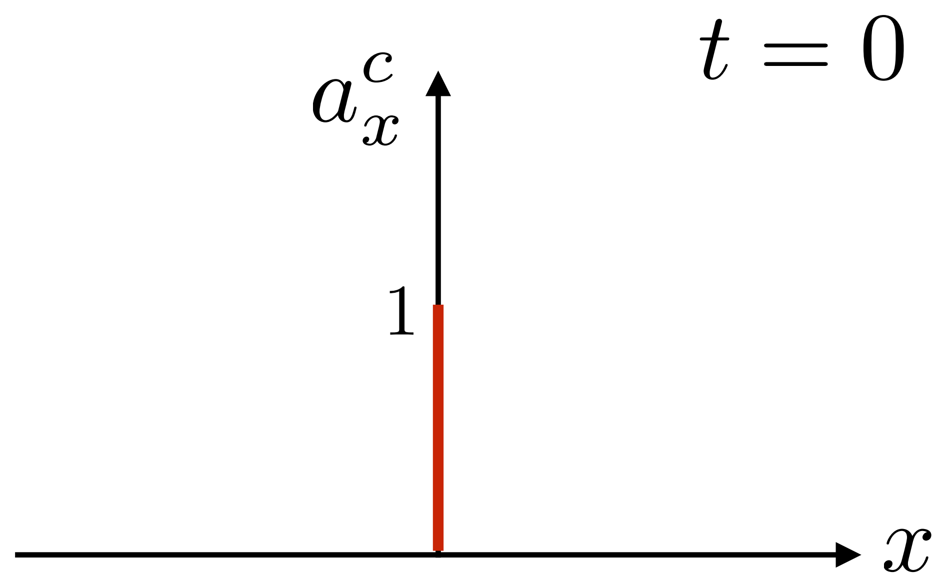
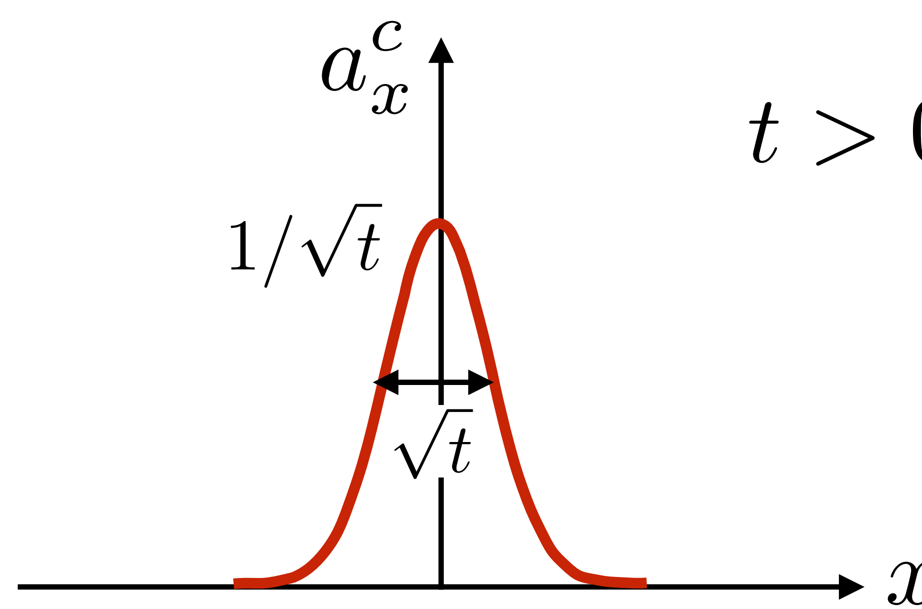


Diffusive charge spreading (coarse grained):

$$\begin{aligned}\langle z\rangle(x,t) &= \mathrm{Tr}[\rho(t) z_x] \\ &= \frac{\epsilon}{2^L}\mathrm{Tr}[\rho(t) z_x] \\ &= \epsilon\, a_x^c(t) \sim \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4D_c t}}\end{aligned}$$



Diffusion & conserved amplitudes

$$O_0 = z_0$$

Random conserving circuit model

$$\overline{a_i^c(t)} = \frac{1}{2^t}\binom{t-1}{\lfloor \frac{i+t-1}{2} \rfloor} \qquad \sum_i a_i^c(t) = 1$$

$$\overline{a^c(x,t)} \approx \sqrt{\frac{1}{2\pi t}} e^{-\frac{x^2}{2t}}$$

coarse grain+
scaling limit

$$D_c = \frac{1}{2}$$

independent of q

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Diffusive Lump

$$\sum_i a_i^c(t) = 1$$

Total operator **weight** in the diffusive lump of conserved charges decreases as a power-law in time.

$$\rho_{\text{tot}}^c \equiv \sum_i |a_i^c(t)|^2$$

$$\overline{\rho_{\text{tot}}^c(t)} \approx \int dx |\overline{a_x^c(t)}|^2 = \int dx \, \frac{1}{2\pi t} e^{-\frac{x^2}{t}} = \frac{1}{2\sqrt{\pi t}}$$

Significant weight in a "diffusive cone" near the origin, even at late times.

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Slow emission of non-conserved operators

- No net loss in operator weight (unitarity).
- Conserved parts emit a steady flux of "non-conserved" operators.
- The local production of non-conserved operators is proportional to the square of the diffusion current, as in Ohm's law:

$$\delta\rho_i^{\mathrm{nc}}(t) \sim (a_i^c(t) - a_{i+1}^c(t))^2 \;\; \sim (\partial_x a^c(x,t))^2$$

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Emergence of dissipation

The dissipative process is the **conversion** of operator weight from locally observable conserved parts to non-conserved, non-local (non-observable) parts at a *slow* hydrodynamic rate.

Observable entropy increases, while total von Neumann entropy of the full system is conserved.

Increase in *observable* entropy

$$\rho(t) = \frac{1}{2^L}\left[\mathbb{I} + \epsilon O_0(t)\right] \qquad S_{\mathrm{vn}}(t) = \mathrm{const}$$

$$O_0^c(t) = \sum_i a_i^c(t) z_i$$

$$S_{\mathrm{vn}}^c(t) = -\mathrm{Tr}[\rho^c(t)\log\rho^c(t)]$$
$$= L\log(2) - \frac{1}{2}\sum_i |a_i^c(t)|^2 + \cdots$$

$$\frac{d}{dt} S_{\mathrm{vn}}^c(t) \sim \frac{1}{2D_c}\int dx |j_c(x)|^2$$

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Putting it all together

- **Diffusion of conserved densities:** Local conserved densities spread diffusively. The *weight* of O(t) on the conserved parts (which live in a diffusive cone near the origin) slowly decreases as a power-law in time. Thus significant weight near the origin even at late times.

Putting it all together

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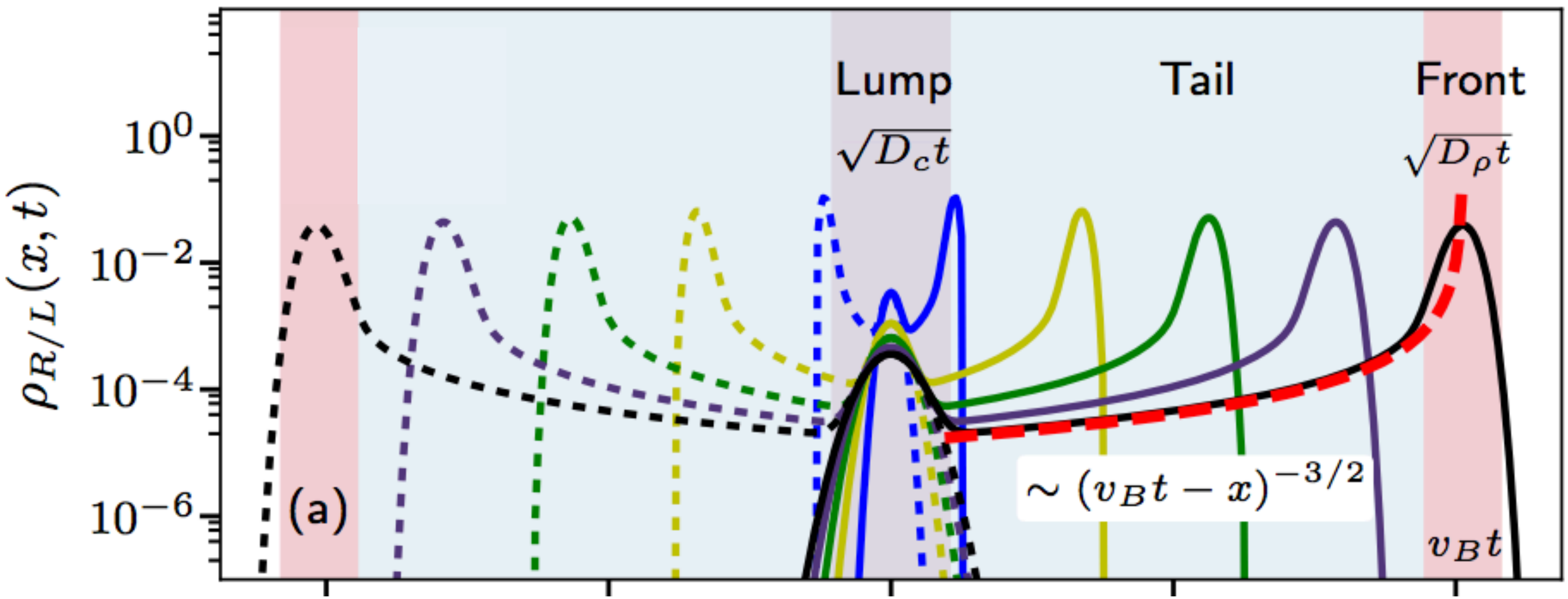
- **Ballistic spreading of non-conserved operators:** Once emitted, the non-conserved parts spread ballistically, quickly becoming non-local and hence non-observable.

Putting it all together

- **Diffusion of conserved densities:** Local conserved densities spread diffusively. The *weight* of O(t) on the conserved parts (which live in a diffusive cone near the origin) slowly decreases as a power-law in time. Thus significant weight near the origin even at late times.
- **Slow Emission of non-conserved operators:** No net loss in operator weight (unitarity). Conserved parts emit a steady flux of "non-conserved" operators. The emission happens at a slow hydrodynamic rate set by the local diffusive currents of the conserved densities.
- **Ballistic spreading of non-conserved operators:** Once emitted, the non-conserved parts spread ballistically, quickly becoming non-local and hence non-observable.
- **Diffusive tails behind ballistic front:** Slow diffusive modes lead to power-law "tails" behind the leading ballistic front, coming from ``lagging" fronts emitted at later times. Show up in the OTOC.

Operator shape: conserving circuit



Coupled hydrodynamic description

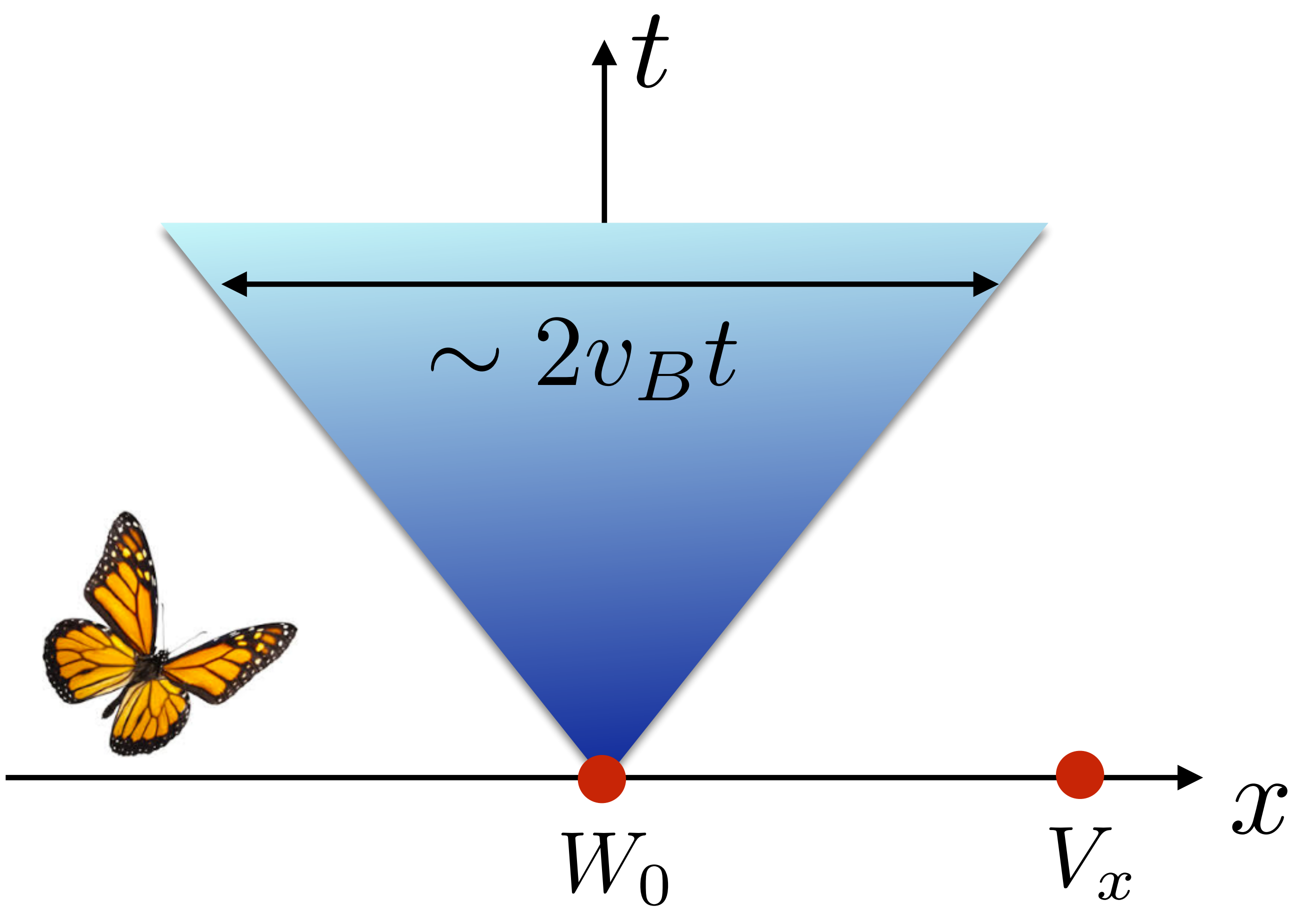
Diffusion of conserved charges

$$\partial_t a^c(\mathbf{x},t) = D_c \nabla^2 a^c(\mathbf{x},t)$$

Biased diffusion of non-conserved fronts emitted from local gradients in the conserved charges

$$\partial_t \rho_R^{nc}(x,t) = v_B \partial_x \rho_R^{\mathrm{nc}}(x,t) + D_\rho \partial_x^2 \rho_R^{\mathrm{nc}}(x,t) + 2D_c |\partial_x a^c(x,t)|^2$$

Operator Spreading & OTOC



$$W(t) = U^\dagger(t) W_0 U(t)$$

$$\mathcal{C}(x,t) = \frac{1}{2}\langle |[W(t), V_x]|^2 \rangle$$

"Out-of-time-ordered-commutator"

semi-classical analog:

$$|i\hbar\{q(t), p\}|^2 = \hbar^2 \left(\frac{\partial q(t)}{\partial q(0)}\right)^2$$

$$\sim \hbar^2 e^{\lambda t}$$

for classically chaotic systems with exponential sensitivity to initial conditions

Three aspects of dynamics

- Butterfly effect: ballistic operator growth with butterfly velocity v_B
- Diffusive hydrodynamics of conserved charges
- Lyapunov regime: exponential early-time sensitivity to perturbations

OTOC in the quantum setting

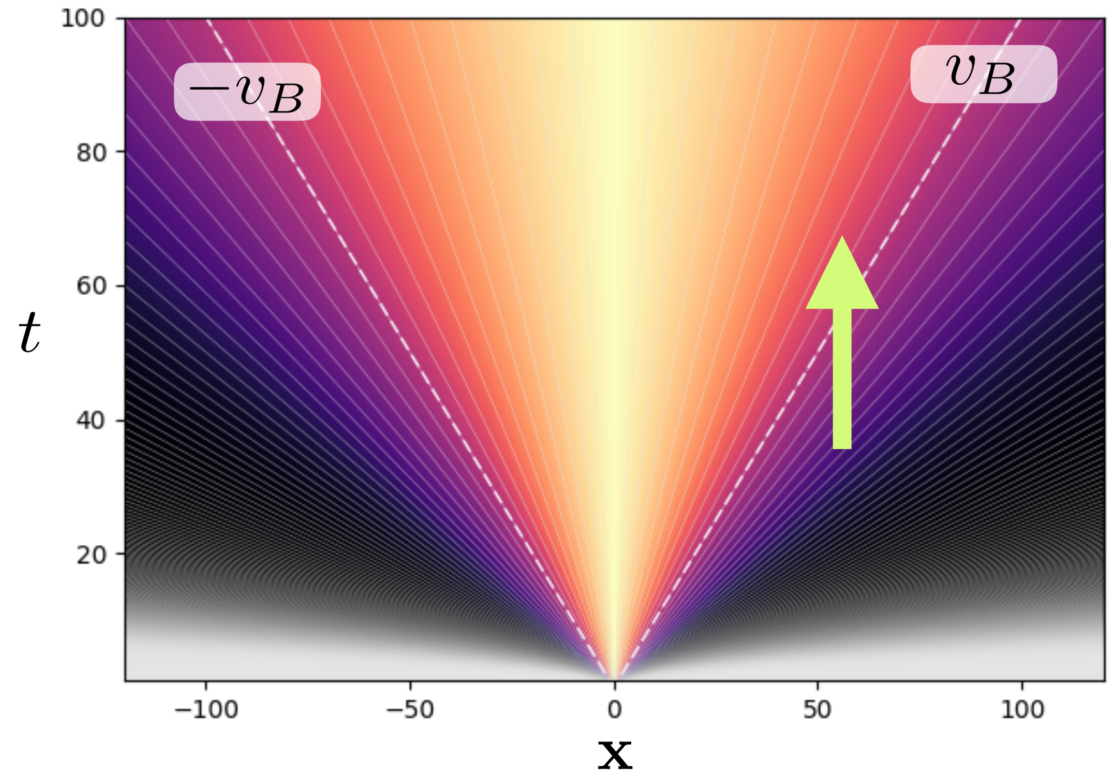
- Displays exponential growth in many large N/ holographic/ semiclassical models
- Saturates to O(1) value at late times due to unitarity; No unbounded growth possible
- Defining a quantum Lyapunov exponent requires a small parameter epsilon such that $C \sim \epsilon\, e^{\lambda t}$ at early times. Defines a long time for observing exponential growth ~
$$t_* = \frac{1}{\lambda} \log \frac{1}{\epsilon}$$
- OTOC is an "intermediate" time diagnostic of chaos.
    - *t** can be parametrically smaller than other thermalization time scales associated (e.g. the Thouless time, or inverse level spacing)

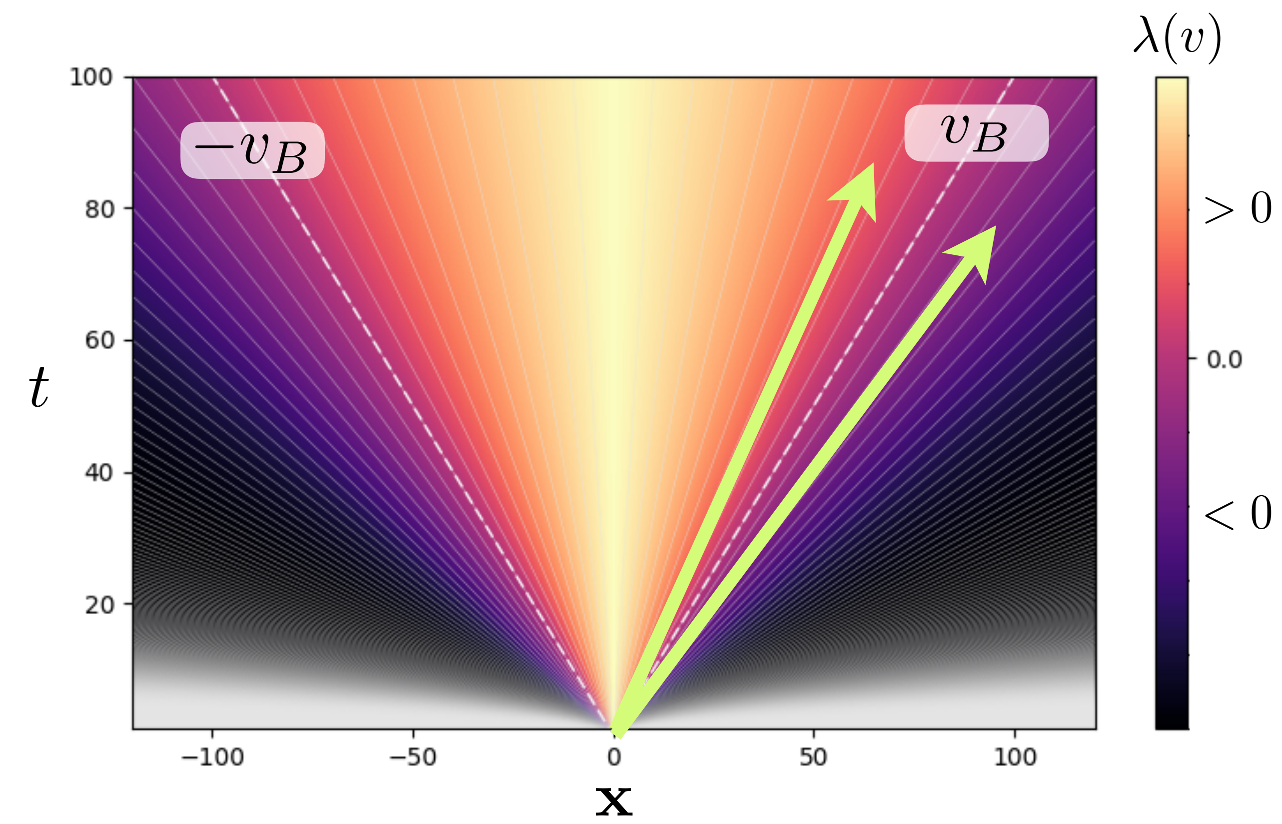
Existence of a Lyapunov Regime

- What about "strongly quantum" systems away from large N/ weak coupling limits (like a thermalizing spin 1/2 chain)?

- Spatially local systems potentially have a small parameter because it takes a large time $t_* \sim |x|/v_B$ for a large commutator to build up. Simple exponential regime may <u>*still*</u> not exist due to front broadening. But velocity dependent Lyapov exponents can still be defined.

$$C(x_0, t) \sim \exp\left[-\frac{(x_0 - v_B t)^2}{2Dt}\right]$$





$$C(\mathbf{x_0}, t) = \langle |[V(0,t), W(\mathbf{x_0})]|^2 \rangle$$

$$C(\mathbf{x}, t) \sim e^{\lambda(\mathbf{v})t} \quad \text{for} \quad \mathbf{x} = \mathbf{v}t$$

OTOC at fixed x_0 → OTOC at fixed *v*

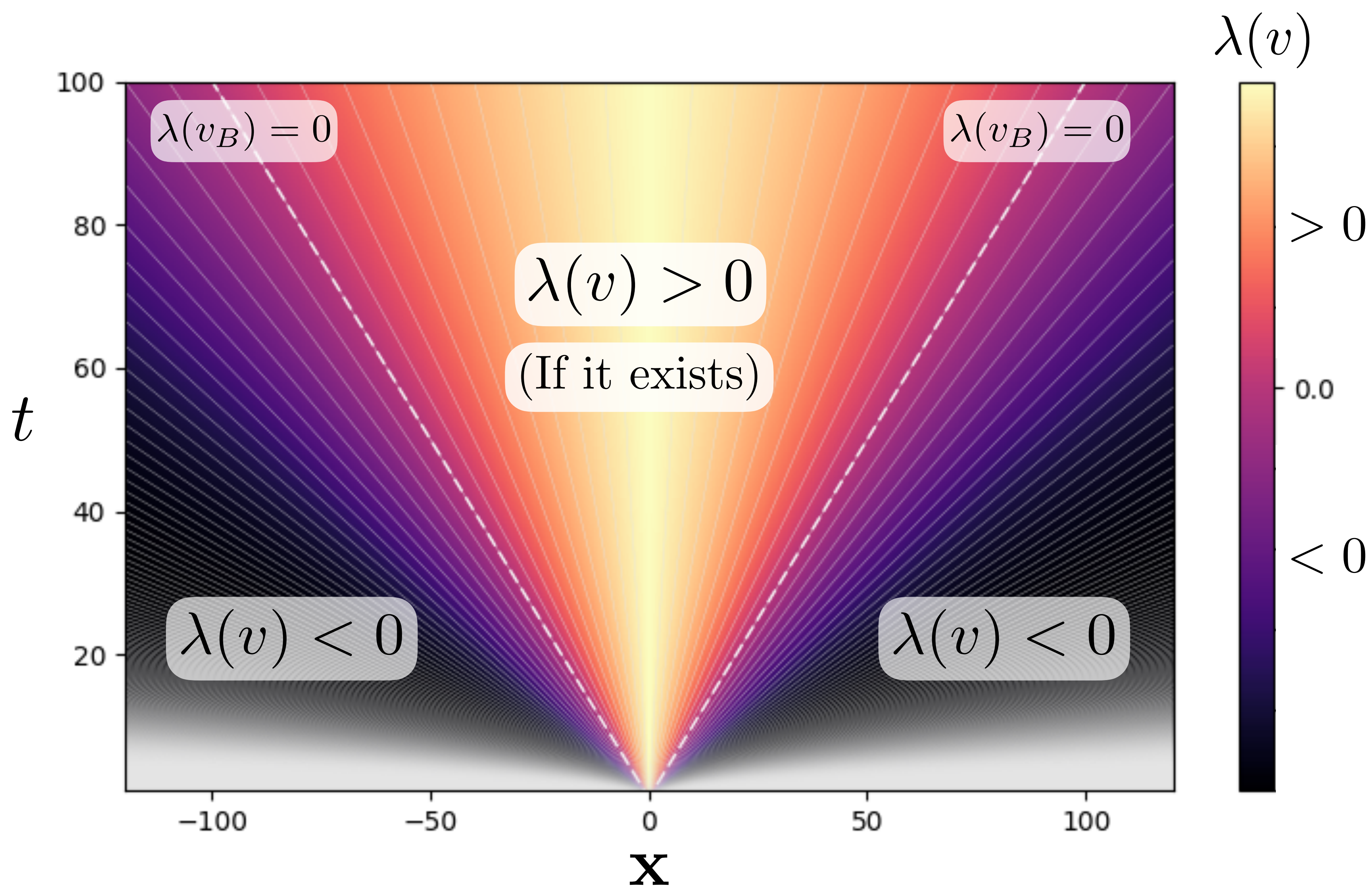
VK, Huse Nahum 2018

Velocity dependent Lyapunov

$$C(\mathbf{x}, t) \sim e^{\lambda(\mathbf{v})t} \quad \text{for} \quad \mathbf{x} = \mathbf{v}t$$

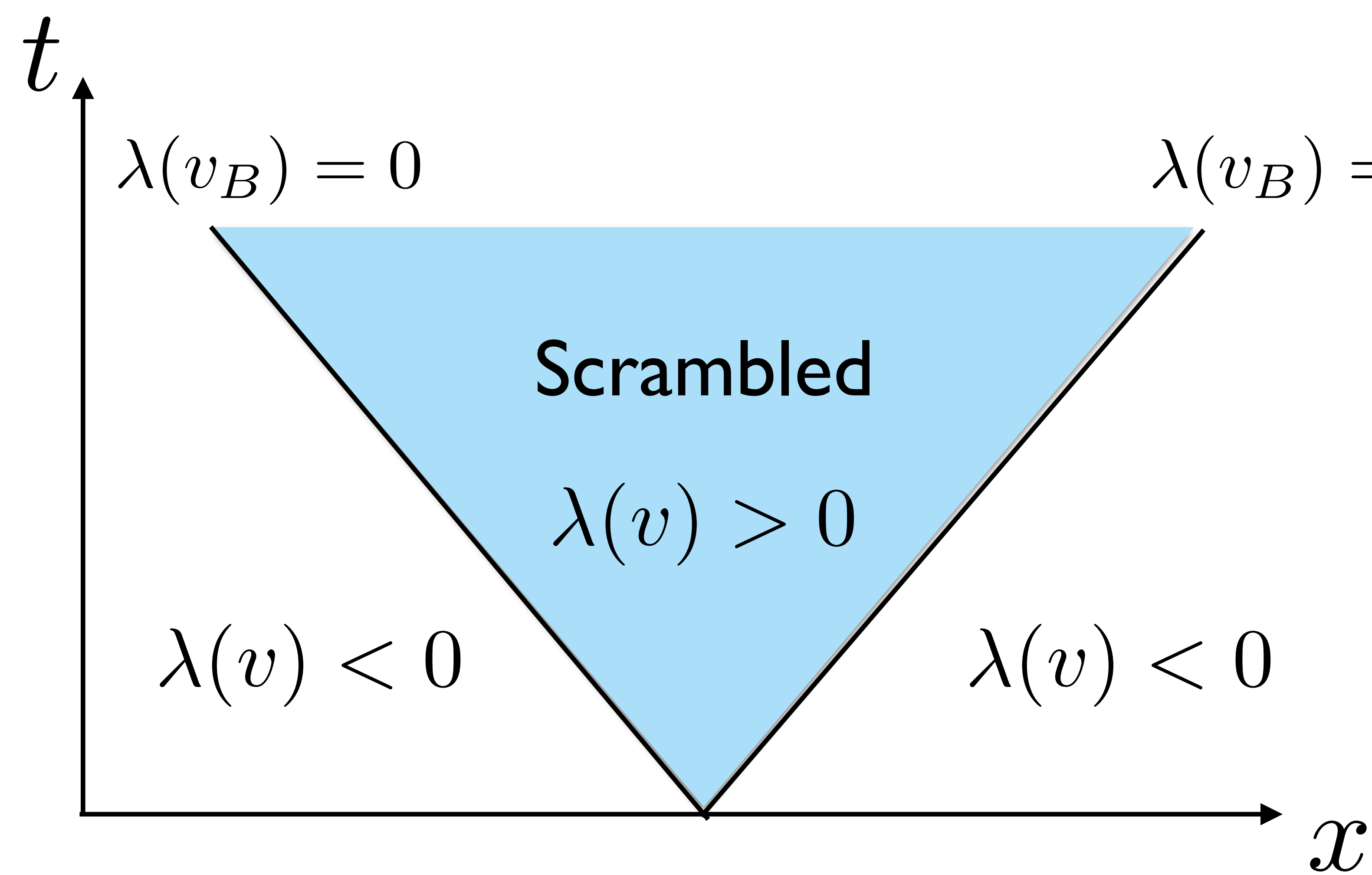
- Spatiotemporal structure of chaos organized along "rays".
- *All* local quantum systems show negative $\lambda(v)$ *outside* the light-cone: exponential *decay* of correlations outside the light-cone. Follows from Lieb Robinson bounds.
- Only large N/semi-classical systems display positive $\lambda(v)$ *inside* the light-cone. No such exponentially growing regime for strongly interacting "fully" quantum systems with local Hilbert space ~ O(1).
- Many qualitative similarities between integrable and non-integrable systems in growth of $C(x_0, t)$ outside the light cone. Thus, operator spreading dynamics, while illuminating for many purposes, may not be the best diagnostic for ``chaos" in strongly quantum systems.

Velocity dependent Lyapunov exponents



$$C(\mathbf{x}, t) \sim e^{\lambda(\mathbf{v})t} \quad \text{for} \quad \mathbf{x} = \mathbf{v}t$$

Classical chaos



Classically, C(x,t) grows or decays in time along rays with a velocity dependent Lyapunov exponent

$$C(x = vt, t) \sim e^{\lambda(v)t}$$

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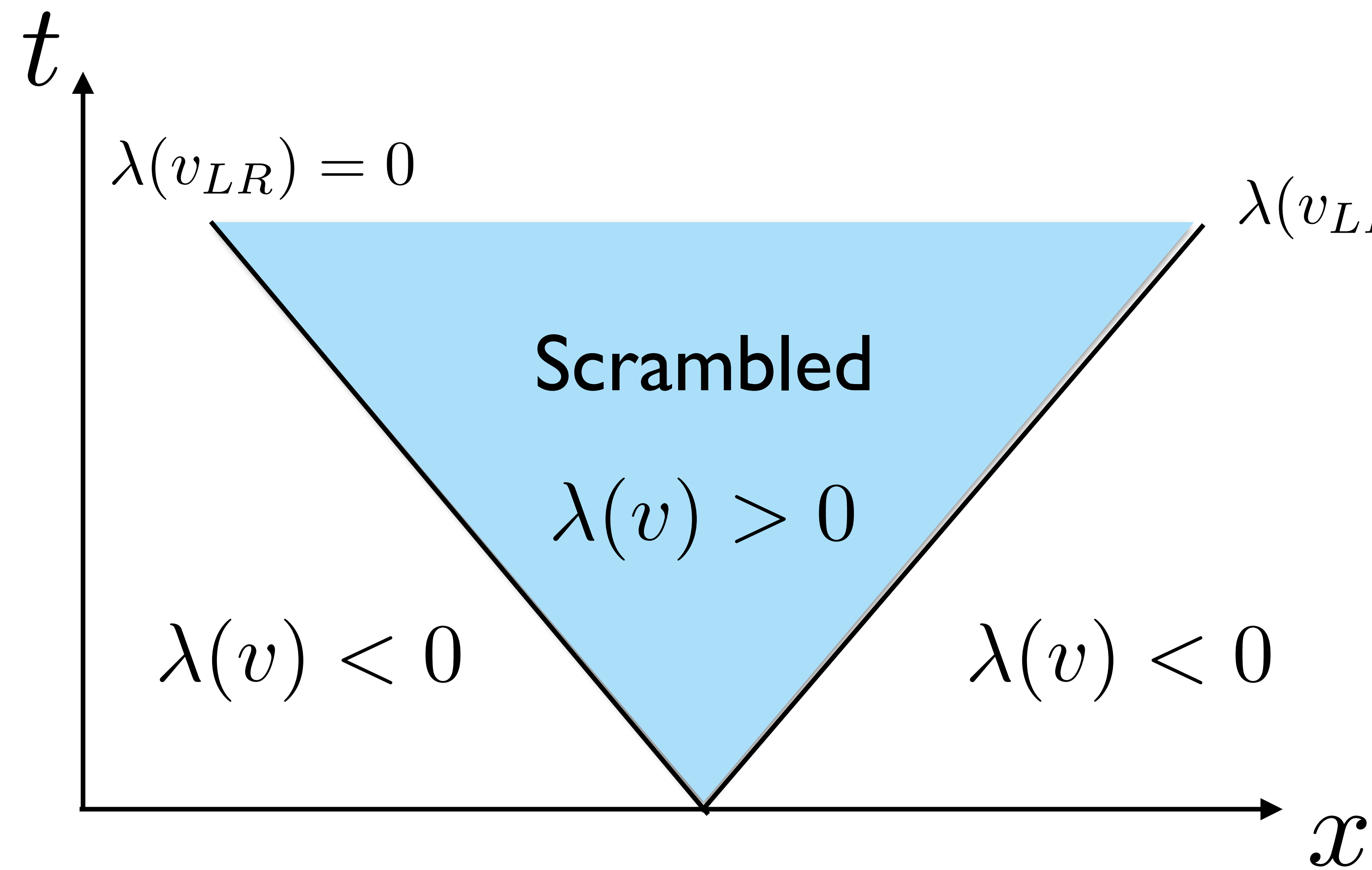
Lieb-Robinson 1972, Deissler, Kaneko 1986

Quantum chaos: large N/ semiclassical



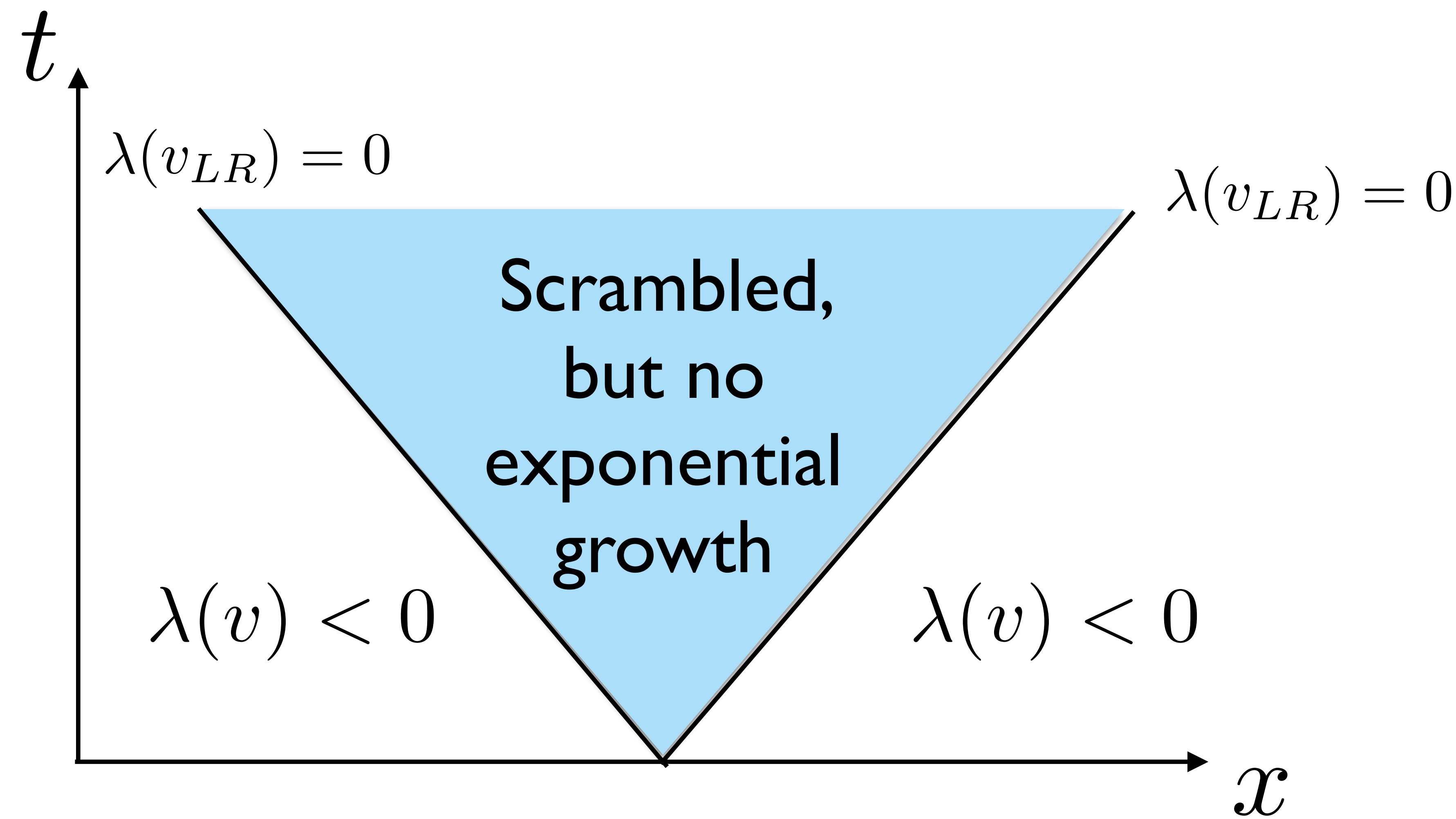
Large N/ semiclassical quantum models show exponential regime:

$$C(x,t) \sim \frac{1}{N^2} e^{\lambda_L (t - |x|/v_B)}$$

e.g. SYK chain (Gu, Qi, Stanford 2016), weakly interacting diffusive metals (Patel et. al. 2017, Aleiner et. al 2016)

VK, Huse Nahum 2018

“Strongly quantum chaos”



No exponentially growing regime with positive Lyapunov exponents seems to exist (yet?) for “strongly quantum” many-body chaos.

VK, Huse Nahum 2018

Lots of interesting open directions for understanding the dynamics of operator spreading, quantum entanglement, thermalization…!