Large-Scale Flow and Structure Formation in Stellar Atmospheres - II

Nana L. Shatashvili [1,2] ***with***

S. M. Mahajan [3] & V.I. Berezhiani [2,4]

(1) I. Javakhishvili Tbilisi State University (TSU), Georgia
(2) TSU Andronikashvili Institute of Physics, Georgia
(3) Institute for Fusion Studies, The University of Texas, Austin, USA
(4) School of Physics, Free University of Tbilisi, Georgia

Based On:

1. S.M. Mahajan, R.Miklaszevski, K.I. Nikol'skaya & N.L. Shatashvili. ***Phys. Plasmas*. 8,** 1340 (2001); ***Adv. Space Res.*, 30,** 345 (2002).
2. S.M. Mahajan, K.I. Nikol'skaya, N.L. Shatashvili & Z. Yoshida. ***The Astrophys. J.*** **576**, L161 (2002)
3. S.M. Mahajan, N.L. Shatashvili, S.V. Mikeladze & K.I. Sigua. ***The Astrophys. J.*** **634**, 419 (2005)
4. S.M. Mahajan, N.L. Shatashvili, S.V. Mikeladze & K.I. Sigua. ***Phys. Plasmas.*** **13**, 062902 (2006)
5. V.I. Berezhiani, N.L. Shatashvili, S.M. Mahajan. ***Phys. Plasmas*, 22**, 022902 (2015).
6. N.L. Shatashvili, V.I. Berezhiani, S.M Mahajan. ***Astrophys.Space Sci-es***, **361(2)**, 70(2016).
7. V.I. Berezhiani, N.L. Shatashvili, N.L. Tsintsadze. ***Physi. Scr.*** **90**, 068005 (2015)
8. A.A. Barnaveli, N.L. Shatashvili. ***Astrophys. Space. Sci.*** **362:**164 (2017).

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Compact Astrophysical Objects with Degenerate Electrons

BB class of equilibria have been studied for both relativistic and nonrelativistic plasmas, most investigations are limited to ”dilute” or **non-degenerate plasmas: the constituent particles are assumed to obey the classical Maxwell-Boltzman statistics**.

Question: **how such states would change/transform if the plasmas were highly dense and degenerate (*mean inter-particle distance is << de Broglie thermal wavelength*) - their energy distribution is dictated by Fermi-Dirac statistics**.

Notice: **at very high densities, particle Fermi Energy can become relativistic & degeneracy pressure may dominate thermal pressure.**

Such **highly dense/degenerate plasmas are found in several astrophysical and cosmological environments as well as in the laboratories devoted to inertial confinement and high energy density physics**; in the latter intense lasers are employed to create such extreme conditions.

Model

The natural habitats for dense/degenerate matter: Compact astrophysical objects like white and brown dwarfs, neutron stars, magnetars with believed characteristic electron number densities $\sim 10^{26} - 10^{32}\ cm^{-3}$ **, formed under extreme conditions.**

We develop the simplest model in which the effect of quantum degeneracy on the nature of the BB class of equilibrium states can be illustrated; fundamental role of another quantum effect – spin vorticity – on BB states was studied in Mahajan *et al* (2011, 2012).

We choose **a model hypothetical system** ***(relevant to specific aspects of a white dwarf (WD))*** of **a two-species neutral plasma with non-degenerate non relativistic ions, and degenerate relativistic electrons embedded in a magnetic field.**

It **is assumed that, despite the relativistic mass increase, the electron fluid vorticity is negligible compared to the electron cyclotron frequency (*****such a situation may pertain, for example, in the pre-WD state of star evolution, and in the dynamics of the WD atmosphere*****).**

The study of the degenerate electron inertia effects on the Beltrami States in dense neutral plasmas will be shown later.

Model Details

For an **ideal isotropic degenerate Fermi gas of electrons** at temperature T_e the relevant thermodynamic quantities – *the pressure* $\mathcal{P}_e$ & *the proper internal energy density* $\mathcal{E}_e$ (the corresponding enthalpy $w_e = \mathcal{E}_e + \mathcal{P}_e$) , *per unit volume* – can be calculated to be

$$\mathcal{P}_e = \frac{m_e^4 c^5}{3\pi^2\hbar^3} f(P_F), \qquad \mathcal{E}_e = \frac{m_e^4 c^5}{3\pi^2\hbar^3}\left[P_F^3\left(\left(1+P_F^2\right)^{1/2}-1\right) - f(P_F)\right], \quad (1)$$

$$8f(P_F) = 3\sinh^{-1} P_F + P_F\left(1+P_F^2\right)^{1/2}\left(2P_F^2-3\right) \quad (2)$$

$P_F = p_F/m_e c$ **is the normalized Fermi momentum of electrons**

Fermi Energy in terms of P_F **is** $\epsilon_F = m_e c^2\left[\left(1+P_F^2\right)^{1/2}-1\right]$.

P_F is related to the **rest-frame electron density** n_e via $p_F = m_e c\,(n_e/n_c)^{1/3}$.

$n_c = 5.9 \times 10^{29} cm^{-3}$ **- critical number-density at which the Fermi momentum equals** $m_e c$ - defines the onset of the relativistic regime.

The electron plasma is treated as the completely degenerate gas – their thermal energy is much lower than their Fermi energy $(n_e T_e/\mathcal{P}_e \ll 1)$.

Distribution function of electrons remains locally Juttner-Fermian which for ***0*-temperature case leads to the just density dependent thermodynamical quantities** $\mathcal{E}_e(n_e)$, $\mathcal{P}_e(n_e)$ & $w_e(n_e)$.

Electron plasma dynamics is isentropic, obeys relation:

$$d(w_e/n_e) = (d\mathcal{P}_e)/n_e.$$

Model Equations

Equation of motion for degenerate electron fluid reduces to:

$$\frac{\partial}{\partial t}\left(\sqrt{1+P_F^2}\,\mathbf{p}_e\right) + m_e c^2 \nabla\left(\sqrt{1+P_F^2}\,\gamma_e\right) =$$

$$= -e\mathbf{E} - \frac{e}{c}\mathbf{V}_e \times \mathbf{B} + \frac{e}{c}\mathbf{V}_e \times \nabla \times \left(\sqrt{1+P_F^2}\,\mathbf{p}_e\right) \qquad (3)$$

With $\mathbf{p}_e = \gamma_e m_e \mathbf{V}_e$ being electron hydrodynamic momentum

Under our assumption of negligible electron fluid vorticity the last term can be negligible.

For the non-degenerate ion fluid we have the equation of motion written as (m_i - proton mass):

$$m_i\left[\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \cdot \nabla)\mathbf{V}_i\right] = -\frac{1}{N_i}\nabla p_i + e\mathbf{E} + \frac{e}{c}\mathbf{V}_i \times \mathbf{B}\,. \qquad (4)$$

Simplest model **- non relativistic ions & inertialess electrons** $N_e \simeq N_i = N$ **- there are two independent Beltrami conditions** (*aligning ion & electron generalized vorticities along their respective velocities*):

$$\mathbf{b} + \nabla \times \mathbf{V} = d\,N\,\mathbf{V}\,, \quad \mathbf{b} = a\,N\left[\mathbf{V} - \frac{1}{N}\nabla \times \mathbf{b}\right]\,, \quad \mathbf{b} = e\mathbf{B}/m_i c \qquad (5)$$

Bernoulli Condition

Density is normalized to N_0 (the corresponding rest-frame density is n_0)

Magnetic field is normalized to some ambient measure B_0

All **velocities are measured in terms of** corresponding Alfvén speed $V_A = B_0/\sqrt{4\pi N_0 m_i}$

All **lengths [times] are normalized to the skin depth** λ_i $[\lambda_i/V_A]$

where $\lambda_i = c/\omega_{pi} = c\sqrt{m_i/4\pi N_0 e^2}$

The ***Beltrami conditions*** **(5) must be supplemented by the Bernoulli constraint to define an equilibrium state (the stationary solution of the dynamical system):**

$$\nabla\left(\beta_0 \ln N + \mu_0\sqrt{1+P_F^2}\,\gamma + \frac{V^2}{2}\right) = 0 \qquad (6)$$

Where β_0 **is the ratio of thermal pressure to magnetic pressure**, $\mu_0 = m_e c^2/m_i V_A^2$

and **for the electron fluid Lorentz factor we put** $\gamma_e \simeq \gamma(\mathbf{V})$.

Bernoulli condition (6) is an expression of the balance of all remaining potential forces when Beltrami conditions (5) are imposed on the two-fluid equilibrium equations.

$P_F = p_F/m_e c = (NN_0/n_c\gamma)^{1/3}$ $[= (Nn_0/n_c)^{1/3}]$ **is a function of density. (5-6) is a complete system of equations.**

$[\nabla\cdot(N\mathbf{V}) = 0]$ **- Equilibrium Continuity Eq. ,** $[\nabla\cdot\mathbf{b} = 0]$ **are automatically satisfied.**

New class of Double Beltrami Equlibria
sustained by Electron Degeneracy Pressure

1) **The Beltrami conditions reflect the simple physics:** (i) the inertia-less (despite the relativistic increase in mass) degenerate electrons follow the field lines, (ii) while the ions, due to their finite inertia, follow the magnetic field modified by the fluid vorticity.

The combined field $\mathbf{b} + \nabla \times \mathbf{V}$ - an expression of magneto-fluid unification, may be seen either as an effective magnetic field or an effective vorticity.

2) The **Beltrami conditions (5) are not directly affected by the degeneracy effects** in the current approximation neglecting the electron inertia. These are precisely the two conditions that define the Hall MHD states. **In the highest density regimes Fermi momentum** (& hence the Lorentz factor $\gamma(\mathbf{V})$) **may be so large that the effective electron inertia will have to be included in (5).**

3) In this minimal model, **electron degeneracy manifests only through the Bernoulli condition (6).** The degeneracy induced term ~ μ_0 would go to unity (whose gradient is zero), and would disappear in the absence of the degeneracy pressure. **For significant P_F, the degeneracy pressure can be >> thermal pressure (measured by β_0).**

Degenerate electron gas can sustain a qualitatively new state: a nontrivial Double Beltrami – Bernoulli equilibrium at zero temperature. ***In the classical zero-beta plasmas, only the relatively trivial, single Beltrami states are accessible***.

4) It is trivial to eliminate $\boldsymbol{b}$ in Eqs. (5) to obtain

$$\frac{1}{N}\nabla\times\nabla\times\mathbf{V}+\nabla\times\left[\left(\frac{1}{aN}-d\right)N\,\mathbf{V}\right]+\left(1-\frac{d}{a}\right)\mathbf{V}=0\ , \tag{7}$$

which, coupled with (6), provides us with a closed system of four equations in four variables *(N,V)*.

Once this is solved with appropriate boundary conditions, one can invoke 1st eq. of (5) to calculate $\boldsymbol{b}$. See solution for similar math. problem in Mahajan *et al* (2001).

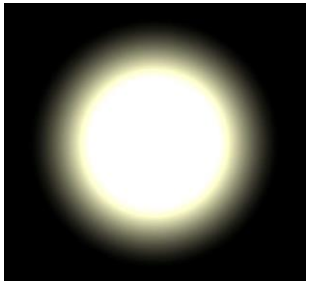
5) **The Bernoulli condition (6) introduces a brand new player in the equilibrium balance**; the spatial variation in the electron degeneracy energy (~ μ_0) could increase or decrease the plasma $\boldsymbol{\beta_0}$ or the fluid kinetic energy (measured by V^2) in the corresponding region. Thus, **Fermi energy could be converted to kinetic energy; it could also forge a re-adjustment of the kinetic energy from a high-density/low-velocity plasma to a low-density/high-velocity plasma**. Similar energy transformations, mediated through classical gravity, were discussed in Mahajan *et al* (2002, 2005, 2006).

Possible extensions of model:

- **When electron fluid degeneracy is very high and one can not neglect inertia effects in their vorticity, the order of BB states is likely to rise** (the triple BB states have been studied in 2008).

- **For the supper-relativistic electrons extension will be the introduction of Gravity, which could balance the highly degenerate electron fluid pressure**. Gravity (Newtonian) effects in the BB system have been investigated in the solar physics context (e.g Mahajan *et.al* (2002, 2005, 2006).

Illustrative Example - White Dwarfs



A possible application of the "degenerate" BB states may be found in stellar physics.

Star collapses + cools down: the density of lighter elements increases affecting the total pressure / enthalpy of unit fluid element – first order departure from the classical e-i plasma.

Beyond the hot, pre-white dwarf stage, photon cooling dominates and gravitational contraction is dramatically reduced as the interior equation of state hardens into that of a strongly degenerate electron gas.

Mechanical and thermal properties separate; the degenerate electrons provide the dominant pressure, while the thermal motions of the ions make a negligible contribution to the mechanical support; the roles of electrons and ions are reversed in their contribution to the overall energy.

Recent studies show that **a significant fraction of White Dwarfs are found to be magnetic with typical fields strengths below 1KG. Massive and cool White Dwarfs,** interestingly, **are found with much higher fields detected.** Recent investigations have uncovered **several cool, magnetic, polluted hydrogen atmosphere (DAs) white dwarfs.**

A simple example: **if degenerate BB states could shed some light on the physics of WDs?**

Considering High B-field WDs, we assume: degenerate electron densities ~ (10^{25}-10^{29}) cm^{-3} ;

Magnetic fields ~ (10^{5} - 10^{9}) G , Temperatures ~ (40000-6000) K . Alfvén speed $V_A \sim (10^4 - 10^6)$ cm/s,

➔ $\beta_0 \sim (10^6 - 10^0)$ & $\mu_0 \sim (10^{10} - 10^6) >> 1$. **Ion skin-depth** $\lambda_i \sim (10^{-5} - 10^{-7})\,cm$ - very short.

Illustrative Example - White Dwarfs – *large-scale flows*

For this class of systems, **2nd term (*degeneracy pressure*) in (6) >> 1st term (*thermal pressure*).**

Neglecting the 1st term, and remembering that for non relativistic flows (*essential at ion speeds*)

***γ* (*V*) ~ 1** , **Bernoulli Condition with inclusion of classical (Newtonian) gravity (justified by observations for WDs) implies**

$$\mu_0 \sqrt{1+P_F^2} - \frac{R_A}{R} + \frac{V^2}{2} = const \qquad (8)$$

***const* measures the main energy content of the fluid**; the Beltrami conditions (5) remain the same;

R - radial distance from the center of WD normalized to its radius R_W **[~ (0 .008-0.02)R_{sun}]** ;

$R_A = GM_W / R_W V_A^2$ (here G is the gravitational constant and M_W - WD mass).

Since P_F is a function of Fermi energy (and hence, of density), **we assume that at some distance R_*** (corresponding to density maximum), P_F **reaches its maximum value** P_{F*} .

Taking the corresponding minimum velocity to be zero ($V_* \sim 0$), we find $const = \mu_0 \sqrt{1+P_{F*}^2} - R_A/R_*$.

Magnitude of velocity is now determined to be

$$|\mathbf{V}| \sim \sqrt{2\mu_0}\,\kappa(P_F) \qquad (9)$$

with

$$\kappa(P_F) = \left[\left(\sqrt{1+P_{F*}^2} - \sqrt{1+P_F^2}\right) - \frac{R_A}{\mu_0}\left(\frac{1}{R_*} - \frac{1}{R}\right)\right]^{1/2}.$$

Results – *Outer Layers of WD-s*

Dimensionless coefficient $R_A / \mu_0 << 1$ measures relative strength of gravity versus degenerate pressure term.

For WD-s with Mass $M_W \sim (0.8 - 0.25)\, M_{sun}$ & radius $R_W \sim (0.013 - 0.02) R_{sun}$,

$R_A / \mu_0 \sim (0.2 - 0.04) << 1$; **less massive the WD, the smaller is the coefficient .**

DB structure scales are small compared to R_W in outer layers of the WD (where model applies).

The gravity contribution to the flow velocity can be neglected

at specific distance of outer layers of WD-s with $R \geq R_*$ & $(R - R_*)/ R_* << 1$, $R_* \leq 1$.

Gravity contribution determines the radial distance in WD 's outer layer over which the "catastrophic" acceleration of flow may appear (due to the magneto-fluid coupling).

In the regions where the flows are insignificant (at very short distances from the WD 's surface) gravity controls the stratification but as we approach the flow "blow-up "distances (the flow becomes strong) the self-consistent magneto-Bernoulli processes take over & control density / velocity stratification.

Calculating the maximum flow velocity, occurring at $\kappa (P_F)$ maximum (density minimum), needs a detailed knowledge of the system.

If $\sqrt{2\mu_0}\,\kappa(P_F) > 1$ **the generated flow is locally super-Alfvénic in contradistinction to the non-degenerate, thermal pressure dominated plasma,** when the maximal velocity due to the magneto-Bernoulli mechanism be locally sub-Alfvénic (when local plasma β <1 as in the Solar Atmosphere).

This example shows that **the electron degeneracy effects can be both strong, and lead to interesting predictions like the anticorelation between the density and flow speeds.**

The richness introduced by electron-degeneracy to the Beltrami-Bernoulli states could help us better understand compact astrophysical objects. When star contracts, its outer layers keep the multi-Structure character although density in structures becomes defined by electron degeneracy pressure.

Important conclusion for future studies - when studying the evolution of the atmospheres/outer layers of compact objects, flow effects can not be ignored. Knowledge of the effects introduced by flows (observed in stellar outer layers) acquired for classical plasmas can be used when investigating the dynamics of White Dwarfs and their evolution.

The possibility of the existence of DB relaxed states in plasmas with degenerate electrons (met in astrophysical conditions) is found. **Non degenerate double BB states guarantee scale separation phenomenon ➔ provide energy transformation pathways for various astrophysical phenomena** (erruptions, fast / transient outflow & jet formation, ***B***-field generation, structure formation, heating & etc.), **such pathways could be explored for degenerate case with degeneracy pressure providing an additional energy source.**

Astrophysical Objects with Degenerate Electrons & Positrons *& Ion Fraction*

Magnetospheres of rotating neutron stars are believed to contain e-p plasmas produced in the cusp regions of the stars due to intense electromagnetic radiation. Since protons or other ions may exist in such environments, **three-component e-p-i plasmas can exist in pulsar magnetospheres**.

Positron component could have a variety of origins:

(1) positrons can be created in the interstellar medium due to interaction of atoms & cosmic ray nuclei,

(2) they can be introduced in a Tokamak e-i plasma by injecting bursts of neutral positronium atoms (e+e−), which are then ionized by plasma.

The annihilation usually occurs at much longer characteristic time scales compared with the time in which the collective interaction between the charged particles takes place.

The natural habitats for dense/degenerate matter: Compact astrophysical objects like white and brown dwarfs, neutron stars, magnetars with believed characteristic electron number densities $\sim 10^{26} - 10^{32}\ cm^{-3}$, formed under extreme conditions.

Model

We develop the simplest model in which the effect of quantum degeneracy as well as the mobility of heavier ions on the nature of the BB class of equilibrium states can be illustrated.

We choose **a model hypothetical system *(relevant to specific aspects of a white dwarf (WD))*** of **a three-species neutral plasma of degenerate relativistic electron-positron plasma with small fraction of non-degenerate classical mobile ions.**

The new BB equilibrium is defined by: two relativistic Beltrami conditions (one for each dynamic degenerate species), one non-relativistic Beltrami condition for ion fluid, an appropriate Bernoulli condition, and Ampere's law to close the set. This set of equations will lead to what may be called a *quadruple Beltrami system*.

The ions, though a small mobile component, play an essential role, they create an asymmetry in the electron-positron dynamics (to maintain charge neutrality, there is a larger concentration of electrons than positrons) and that asymmetry introduces a new and very important dynamical scale. This scale, though present in a classical non-degenerate plasma, turns out to be degeneracy dependent and could be vastly different from its classical counterpart.

Presence of mobile ions leads to "effective mass" asymmetry in electron & positron fluids, which, coupled with degeneracy-induced inertia, manifests in the existence of Quadruple Beltrami fields.

Model Equations - 1

Charge neutrality in an e-p-i plasma of degenerate electrons (−), positrons (+) and a small mobile ion component, forces the following density relationships

$$N_0^- = N_0^+ + N_{i0} \implies \frac{N_0^+}{N_0^-} = 1 - \alpha \qquad \text{with} \quad \alpha = \frac{N_{i0}}{N_0^-}, \tag{10}$$

The equation for ion dynamics is standard. **The e(p) dynamics** will be described by the relativistic degenerate fluid equations: ***the Continuity***

$$\frac{\partial N^\pm}{\partial t} + \nabla \cdot \left(N^\pm \mathbf{V}_\pm\right) = 0, \tag{11}$$

and ***the Equation of Motion***

$$\frac{\partial}{\partial t}\left(G^\pm \mathbf{p}_\pm\right) + m_\pm c^2 \nabla\left(G^\pm \gamma_\pm\right) = q_\pm \mathbf{E} + \mathbf{V}_\pm \times \mathbf{\Omega}_\pm \tag{12}$$

where $\mathbf{p}_\pm = \gamma_\pm m_\pm \mathbf{V}_\pm$ - is hydrodynamic momentum, $n^\pm = N^\pm/\gamma_\pm$ is the rest-frame particle density ($N^\pm$ denotes laboratory frame number density), the degeneracy effects manifest through the "effective mass" $G^\pm = w_\pm/n^\pm m_\pm c^2$, where $w_\pm$ is the enthalpy per unit volume. For fully degenerate relativistic e(p) plasma its general expression transfers to

$w_\pm \equiv w_\pm(n)$ and $w_\pm/n^\pm m_e c^2 = (1 + (R^\pm)^2)^{1/2}$ $R^\pm = p_{F\pm}/m_\pm c$

The mass factor is then $G^\pm = [1 + (n^\pm/n_c)^{2/3}]^{1/2}$ for arbitrary $n_\pm/n_c$

Model Equations - 2

On taking the *curl of these equations, one can cast them* into an ideal vortex dynamics

$$\frac{\partial}{\partial t}\mathbf{\Omega}_{\pm} = \nabla \times (\mathbf{V}_{\pm} \times \mathbf{\Omega}_{\pm}), \qquad \textit{where} \qquad \mathbf{\Omega}_{\pm} = (q_{\pm}/c)\,\mathbf{B} + \nabla \times (\tilde{G}^{\pm}\mathbf{p}_{\pm}). \quad (13)$$

We emphasize that the so called **plasma approximation** for a degenerate e(p) assembly is valid if their average kinetic energy $(\sim\epsilon_F^{\pm})$ is larger than the interaction energy $(\sim e^2(n_0^{\pm})^{1/3})$.

This condition **is fulfilled for a sufficiently dense fluid when**

$$n_0^{\pm} \gg (2m_- e^2/(3\pi^2)^{2/3}\hbar^2)^3 = 6\ 3\times10^{22}\ \mathrm{cm}^{-3}\ ;$$

such a condition would imply $R^{\pm} \gg 4.76 \times 10^{-3}$.

The low frequency dynamics is, now, closed with **Ampere's law**

$$\nabla \times \mathbf{B} = \frac{4\pi e}{c}\left[(1-\alpha)N^{+}\mathbf{V}_{+} - N^{-}\mathbf{V}_{-} + \alpha N_i \mathbf{V}_i\right], \quad (14)$$

The small static/mobile ion population, represented by α and V_i , creates an asymmetry between the currents contributed by the electrons and positrons. This will be the source of a new scale-length that turns out to be much larger than the intrinsic electron and positron scale lengths (skin depths).

Equilibrium States in Relativistic E-P-I Degenerate Plasma

Density is normalized to electrons N_0 (the corresponding rest-frame density is n_0)

Magnetic field is normalized to some ambient measure B_0

All **velocities are measured in terms of** corresponding Alfvén speed $V_A = V_A^- = B_0/\sqrt{8\pi n_0^- m^- G_0^-}$

All **lengths [times] are normalized to the skin depth** $\lambda_{\rm eff}[\lambda_{\rm eff}/V_A]$.

where $\lambda_{\rm eff} \equiv \lambda_{\rm eff}^- = \frac{1}{\sqrt{2}}\frac{c}{\omega_p^-} = c\sqrt{\frac{m^- G_0^-}{8\pi n_0^- e^2}}$, $G_0^\pm(n_0^\pm) = [1 + (R_0^\pm)^2]^{1/2}$, $R_0^\pm = \left(\frac{n_0^\pm}{n_c}\right)^{1/3}$

The intrinsic skin depths, the natural length scales of the dynamics, are generally much shorter compared to the system size. For the degenerate electron fluid, the effective mass goes to

$G_0^-(n_0^-) = 1 + \frac{1}{2}(\frac{n_0^-}{n_c})^{2/3}$ for $(R_0^- \ll 1)$ and to $G_0^-(n_0^-) = (\frac{n_0^-}{n_c})^{1/3}$ for $(R_0^- \gg 1)$

Following the well-known procedure we obtain **the set of equilibrium equations for the degenerate system** (***the primary difference is in the physics of*** G_-).

The ***Beltrami conditions***:

$$\mathbf{B} \pm \nabla \times (G^\pm \gamma_\pm \mathbf{V}_\pm) = a_\pm \frac{n^\pm}{G^\pm}(G^\pm \gamma_\pm \mathbf{V}_\pm), \qquad (15)$$

aligning the Generalized vorticities along their velocity fields, and the ***Bernoulli conditions***

$$\nabla(G^\pm \gamma_\pm \pm \varphi) = 0 \quad \Longrightarrow \quad G^+\gamma_+ + G^-\gamma_- = const. \qquad (16)$$

And ***Ion fluid Beltrami Condition***

$$\mathbf{B} + \zeta \nabla \times \mathbf{V}_i = \alpha a_i n_i \mathbf{V}_i, \quad \text{where } \zeta = \left[G_0^- \frac{m^-}{m_i}\right]^{-1} \qquad (17)$$

The Quadruple Beltrami System

An appropriate tedious manipulation of the set Eqs. (14)–(16), leads us to an explicit **Quadruple Beltrami equation** obeyed by the Ion Fluid Velocity $\boldsymbol{V}_{\mathbf{i}}$ ***(the Beltrami index is measured by the highest number of curl operators)****. Written schematically as*

$$\nabla \times \nabla \times \nabla \times \nabla \times \mathbf{V}_i - b_1' \nabla \times \nabla \times \nabla \times \mathbf{V}_i + b_2' \nabla \times \nabla \times \mathbf{V}_i - b_3' \nabla \times \mathbf{V}_i + b_4' \mathbf{V}_i = 0. \quad (18)$$

Equation (18) was derived in the incompressible approximation, and for $\gamma_+ \sim \gamma_- \equiv 1$

The *b* coefficients are functions of effective masses, Beltrami & system characteristic parameters.

Incompressibility assumption is expected to be adequate for outer layers of compact objects, though, compressibility effects can be significant e.g. in the atmospheres of pre-compact stars (Berezhiani et al. 2015). Ion fluid velocity & the magnetic field are related to

e-p plasma average bulk fluid velocity $\mathbf{V} = \frac{1}{2}\left[(1-\alpha)\mathbf{V}_+ + \mathbf{V}_-\right].$ (19)

through

$$\mathbf{V} = \eta\left(2\beta G_0^+ \nabla \times \nabla \times \mathbf{B} - \left[a_+(1-\alpha)\beta - a_-\right]\nabla \times \mathbf{B}\right) + \eta\left(\left[1+(1-\alpha)\beta\right]\mathbf{B}\right) - \alpha\beta\nabla \times \mathbf{V}_i + \frac{\alpha}{2}\left[a_+(1-\alpha)\beta - a_-\right]\mathbf{V}_i \quad (20)$$

with

$$\eta \equiv \left[a_+(1-\alpha)\beta + a_-\right]^{-1} \text{ and } \beta = G_0^- / G_0^+. \quad (21)$$

The **quadruple Beltrami (18) can be factorized as**

$$(curl - \mu_1)(curl - \mu_2)(curl - \mu_3)(curl - \mu_4)\mathbf{V}_i = 0, \qquad (22)$$

where μ_i -s define coefficients in Eq. (22) & are functions of α , β , n_0^- and the degeneracy-determined G_0^+ .

The general solution of Eq. (13) is a sum of four Beltrami fields $\boldsymbol{F_k}$ (solutions of Beltrami Equations $\nabla \times \mathbf{F}_k = \mu_k \mathbf{F}$) while eigenvalues (μ_k) of the *curl operator are* the solutions of the fourth order equation

$$\mu^4 - b_1' \mu^3 + b_2' \mu^2 - b_3' \mu + b_4' = 0. \qquad (23)$$

An examination of the various *b* coefficients of (14), for the most relevant limit ***α* << 1** , reveal:

though the inverse scales, determined by b_1' , b_2', and b_3' , do get somewhat modified by α <<1 corrections, it is **the inverse scale associated with b_4' that is most profoundly affected; being ~ *α* , it tends to become small, i**.e., ***the corresponding scale length becomes large as α approaches zero; this scale length becomes strictly*** **infinite for α=0, and disappears reducing (23) to a triple Beltrami system**.

Thus, **the ion contamination-induced asymmetry may lead to the formation of macroscopic structures through creating an intermediate/large length scale, much larger than the intrinsic scale skin depths, and less than the system size.**

The possible significance and importance of this somewhat natural mechanism (a small ion contamination is rather natural) for creating Macro-structures in astrophysical objects, could hardly be overstressed.

Notice: this mechanism operates for all levels of degeneracy **(the range of R_0^- was irrelevant).**

Illustrative Examples – White Dwarfs - Large Scale

This **new macroscopic scale can be "determined" by dominant balance arguments**; as scale gets larger, $|\nabla|$ gets smaller, and the dominant balance will be between the last terms of (14), yielding ($\zeta \gg 1$):

$$L_{\rm macro} = \frac{|b_3'|}{|b_4'|} = \frac{A}{\alpha} \qquad \text{where} \quad A = \zeta \frac{|(a_+ - a_-)[1 - \frac{\alpha}{\zeta}(G_0^+)] + \frac{\alpha}{\zeta} a_i [(G_0^+) - a_+ a_-]|}{|a_i (a_+ - a_-) - a_+ a_-|} \qquad (24)$$

Assuming that: $\alpha G_0^+ / \zeta = \alpha\beta (G_0^+)^2 \frac{m^-}{m_i} \leq \alpha \ll 1$ [*e(p)* plasma density is within $(10^{25}\text{–}10^{32})\ \mathrm{cm}^{-3}$]]

we can simplify ***A*** when both $a_+ \ll a_i$ and $a_- \ll a_i$. .

(i) When **a₊ ≠ a₋** the simplified expression

$$L_{\rm macro} \sim \frac{\zeta}{\alpha} \left| \frac{1}{a_i} + \frac{\alpha}{\zeta} \frac{(G_0^+) - a_+ a_-}{a_+ - a_-} \right| \qquad (25)$$

for $a_i \leq \zeta$ satisfies $L_{\rm macro} \gg 1$.

(ii) When $a_+ \sim a_- = a \neq (G_0^+)^{1/2}$ $\quad L_{\rm macro} \sim \frac{a_i}{a^2} |(G_0^+) - a^2| \gg 1$ for all $a_i \gg a$. (26)

Without ion contamination (α = 0), the degenerate e-p system is still capable of creating length scales larger than the non-relativistic skin depths through the degeneracy-enhanced inertia of the light particles.

Notice **that even with equal effective masses ($G^- = G^+ \equiv G(n)$ at equal electron-positron temperature), inertia change due to degeneracy can cause asymmetry in e(p) fluids.**

Illustrative Examples – Meso Scale

Even in the absence of ions (***L*macro → infinity**), the Beltrami states could be characterized by what could be called meso-scales—the temperature & degeneracy-boosted effective skin depths $\lambda^{\pm}_{eff}$ larger than λ [$\lambda_{\rm eff}^{\pm}/\lambda = \sqrt{G_0^{\pm}} > 1$ and $1 < \sqrt{G_0^{\pm}} < 5.6$ for densities $(10^{25} - 10^{32})\mathrm{cm}^{-3}$].

For pure compressible e-p plasma, if $\nabla[G^{\pm}(n^{\pm})]$ is at a much slower rate than the spatial derivatives of ***B*** and $\boldsymbol{V}_{\pm}$, we can write following relation:

$$\left(\frac{G}{n}\right)\nabla\times\left(\frac{1}{n}\right)\nabla\times\nabla\times\mathbf{V}-\kappa_1\left(\frac{1}{n}\right)\nabla\times\nabla\times\mathbf{V}+\kappa_2\nabla\times\left(\frac{G}{n}-a_+a_-\right)\left(\frac{n}{G}\right)\mathbf{V}-\kappa_3\mathbf{V}=0. \qquad (27)$$

Estimation for the large scale l_{meso} in case of pure degenerate e-p plasma, derived from the dominant balance, gives:

$$l_{\rm meso}=\frac{|\kappa_2|}{|\kappa_3|}|(G/n)-a_+a_-|=2\frac{|(G/n)-a_+a_-|}{|a_+-a_-|}\gg 1 \quad \text{if} \quad a_+=a_-=a\neq\left(\frac{G(n)}{n}\right)^{1/2}. \qquad (28)$$

Hence, ***whenever the local density satisfies this condition there is a guaranteed scale separation in the degenerate e-p plasma with at least one large scale present.***

At the same time: for larger scale to exist we do need an entirely different mechanism — a dynamic ion-species with a much lower density and higher rest mass (justified by observations for many astrophysical objects plasmas) — this scale corresponds to the ion skin depth enhanced, dramatically, by low density $[\lambda_i = (\alpha m_-/m_i)^{-1/2}\,\lambda \gg \lambda]$.

Scale Hierarchy

This work registers a major departure from e-i system leading to the most important result — by studying BB states in an e-p-i (*small dynamic ion contamination added to a primarily e-p plasma*), we demonstrated the creation of a new macroscopic length scale L_{macro} lying between *the system* size and relatively small intrinsic scales (measured by the skin depths) of the system.

(1) For a pure electron-positron plasma, the equilibrium is *triple Beltrami with the following fundamental three* scales: system size L , and the two intrinsic scales (electron and positron *skin depths).*

(2) **The e-p skin depths**, microscopic in a non degenerate plasma, **can become much larger due to degeneracy effects and could be classified as meso-scales,** l_{meso} .

(3) **When a dynamic low density ion species is added, the equilibrium becomes *quadruple Bertrami with a new* additional scale,** L_{macro}**.** Although the exact magnitude of this scale is complicated, its origin is entirely due to the ion contamination; this scale disappears as the ion concentration α goes to zero. **Both the larger ion mass and low density contribute towards boosting** L_{macro} **.**

(4) The meso-scale l_{meso} cannot become very large but for some special constraints on the Bertrami parameters, for instance, if $a_- \neq a_+$ and both $a_\pm \ll 1$, or the condition (28) is satisfied.

Discussion & Summary

We derived *Quadruple [Triple] Beltrami* relaxed states in e-p-i plasma with classical ions, and degenerate electrons and positrons. Such a mix is often met in both astrophysical and laboratory conditions.

The presence of the mobile ion component has a striking qualitative effect; it converts, what would have been, a *triple Beltrami state to a new quadruple Beltrami state. In* the process, it adds structures at a brand new macroscopic scale L_{macro} (absent when ion concentration is zero) that is much larger than the intrinsic skin depth ($\lambda = c\sqrt{\frac{m^-}{8\pi n_0^- e^2}}$) of the lighter components.

Though primarily controlled by the mobile ion concentration, L_{macro} **also takes cognizance of the electron and positron inertias that could be considerably enhanced by degeneracy**.

The creation of these new intermediate scales (between the system size, and λ) adds immensely to the richness of the structures that such an e-p-i plasma can sustain; many more pathways become accessible for energy transformations.

Such pathways could help us better understand a host of quiescent as well as explosive astrophysical phenomena — eruptions, fast/transient outflow and jet formation, magnetic field generation, structure formation, heating etc.

Flow Generation / Acceleration Due to Magneto-fluid Coupling

A possible application of the ”degenerate” BB states was found in stellar physics [BSM 2015] where we have considered High B-field WDs and assumed: degenerate electrons densities ~ $(10^{25}$-$10^{29})$ cm^{-3} ; Magnetic fields ~ $(10^5 - 10^9)$ G , Temperatures ~ (40000-6000) K . Alfvén speed $V_A \sim (10^4$-$10^6)$ cm/s, ➔ $\beta_0 \sim (10^6 - 10^0)$ & $\mu_0 \sim (10^{10} - 10^6) >> 1$. **Ion skin-depth** $\lambda_i \sim (10^{-5} - 10^{-7})cm$ - very short.

It was shown that Gravity contribution determines the radial distance in WD ’s outer layer over which the ”catastrophic” acceleration of flow may appear (due to the magneto-fluid coupling).

For the special class of magnetic **WDs BSM 2015 predicted that the electron degeneracy effects can be both strong and lead to the anti-correlation between density and flow speeds— the generated flow gets locally super-Alfvénic in contradistinction to non-degenerate, thermal pressure dominated Solar Atmosphere plasma** (with local plasma β <1) for which the maximal velocity due to the magneto-Bernoulli mechanism was found to be only locally sub-Alfvénic.

When star contracts, its outer layers keep the multi-Structure character although density in structures becomes defined by electron degeneracy pressure.

Steady State Considerations

Assuming: **at some height of magnetic WDs surface there exist fully ionized magnetized plasma structures such that the quasi-equilibrium two-fluid model of BSM 2015 will capture the physics of flow or/and magnetic field amplification.**

Corresponding equilibrium state equations are given by (5) (stating that for the non-relativistic ions & inertialess electrons there are two independent Beltrami Conditions):

$$\boldsymbol{b} = a\,N\,[\,\boldsymbol{V} - N^{-1}\,\nabla\times\boldsymbol{b}\,]\;,\quad \boldsymbol{b} + \nabla\times\boldsymbol{V} = d\,N\,\boldsymbol{V}\;, \qquad (29)$$

where $\boldsymbol{b} = e\boldsymbol{B}/mic$ and it was assumed, that electron and proton laboratory-frame densities are nearly equal— $N_e \approx N_i = N$ (rest-frame density $n_{e,i} = N_{e,i}/\gamma\,(\mathbf{V}_{e,i}\,)$, $\gamma\,(\mathbf{V}_{e,i}\,)$ - Lorentz factor);

a & d are dimensionless constants related to the two invariants:

Magnetic helicity $h_1 = \int(\boldsymbol{A}\cdot\boldsymbol{b})\,d^3x$ & Generalized helicity $h_2 = \int(\boldsymbol{A}+\boldsymbol{V})\cdot(\boldsymbol{b}+\nabla\times\boldsymbol{V})\,d^3x$ of the system with $\boldsymbol{A}$ being the dimensionless vector potential.

Normalized variables: fluid velocity $\boldsymbol{V}$ & current $\boldsymbol{J} = \nabla\times\boldsymbol{b}$ when electron and ion speeds are given by $\boldsymbol{V}_e = \boldsymbol{V} - (1/N)\,\nabla\times\boldsymbol{b}$, and $\boldsymbol{V}_i = \boldsymbol{V}$, respectively.

Notice: the electron vorticity is primarily magnetic ($\boldsymbol{b}$) while the ion vorticity has both kinematic and magnetic parts ($\boldsymbol{b}+\nabla\times\boldsymbol{V}$).

Flow Generation / Acceleration - extension

We construct the detailed solutions of (29, 8) with gravity taken into account for concrete parameters relevant to magnetic WDs to show the explicit effects of degeneracy on two-fluid BB structures when star contracts and cools down.

Rewriting equations with inclusion of classical gravity for nonrelatvistic flows

($\gamma(\mathbf{V}) \sim 1$ justified by observations):

$$\boldsymbol{b} = a\, N\, [\, \boldsymbol{V} - \kappa\, N^{-1}\, \nabla \times \boldsymbol{b}\,]\ , \quad \boldsymbol{b} + \kappa\, \nabla \times \boldsymbol{V} = d\, N\, \boldsymbol{V}\ , \tag{30}$$

$$\nabla \left[\, \beta_0 \ln N + \mu_0 (1 + P_F^{\,2})^{1/2} - R_A R^{-1} + 0.5\, V^{\,2} \right] = 0 \tag{31}$$

R - radial distance from the center of WD normalized to its radius $\boldsymbol{R_W}$ **[~ (0 .008-0.02)*R* *sun*]**

$\boldsymbol{R_A = G M_W / R_W V_A^{\,2}}$ (here G is the gravitational constant and M_W - WD mass).

Dimensionless parameter $\boldsymbol{\kappa = \lambda_i / R_w}$

For above parameters neglecting the first term related to ion fluid pressure, (30-31) give:

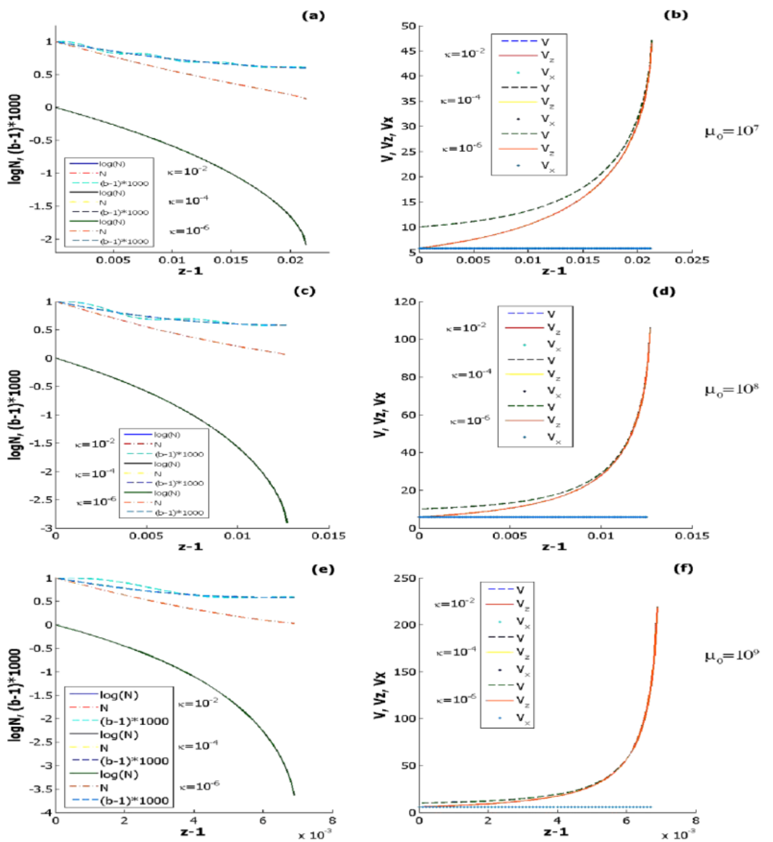
$$\kappa^2 N^{-1}\, \nabla \times \nabla \times \boldsymbol{V} + \kappa\, \nabla \times [\, (a^{-1} N^{-1} - d\,) N\, \boldsymbol{V}\,] + [1 - d a^{-1}\,]\, \boldsymbol{V} = 0, \tag{32}$$

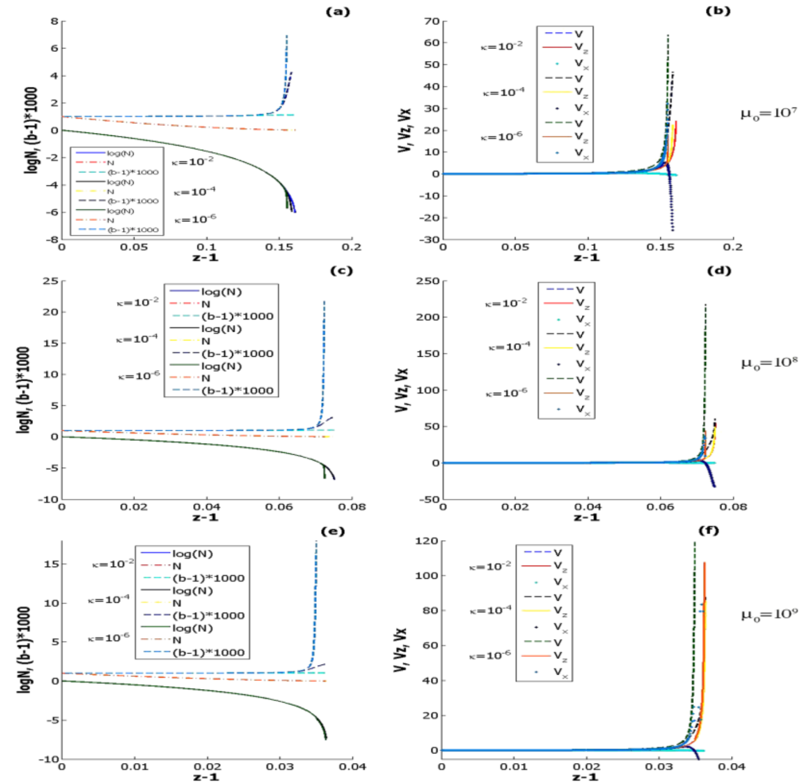
$$N = [a_0^{-2} \mu_0^{-2} (\, [\, R_A R^{-1} - R_A R_0^{-1}] - 0.5\, [\boldsymbol{V}^{\,2} - \boldsymbol{V_0}^{2}] + 0.5\, \alpha\, \mu_0^{\,2})^{\,2} - a_0^{-2}\,]^{\,3/2} \tag{33}$$

where: $a_0 = (n_0/n_c)^{1/3}$ and $\alpha = (\,1 + a_0^{\,2} N_0^{\,2/3}\,)^{1/2}$; ***subscript "0" is used for the height from stellar surface where the boundary conditions are applied .***

Flow Generation / Acceleration – *White Dwarfs' Atmospheres*

To illustrate the coupling process we picked up several sets of our runs. **Density, magnetic fields and velocity vs height for n_0 = 10^{26} cm^{-3} [$a_0 = (1/6)^{1/3} \times 10^{-1}$] & $a = d = 0.1$ (*left*) and n_0 = 10^{27} cm^{-3} [$a_0 = (1/6)^{1/3} \times 10^{-2/3}$] & $a = d = 10$ (*right*) are given for various values of μ_0**

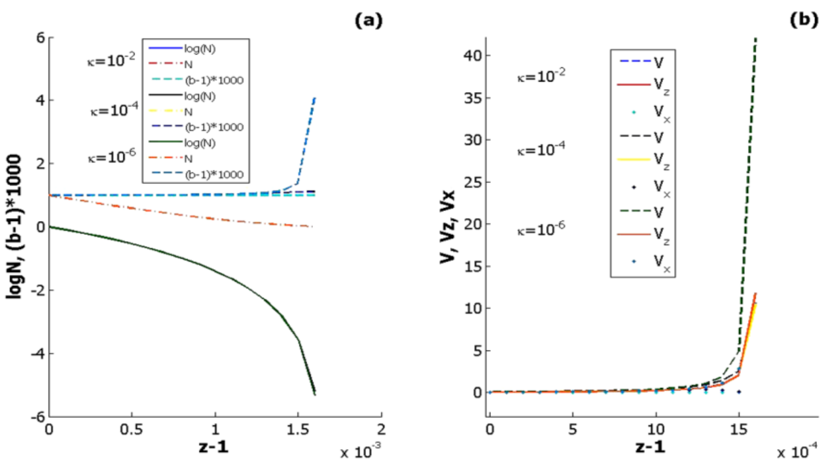


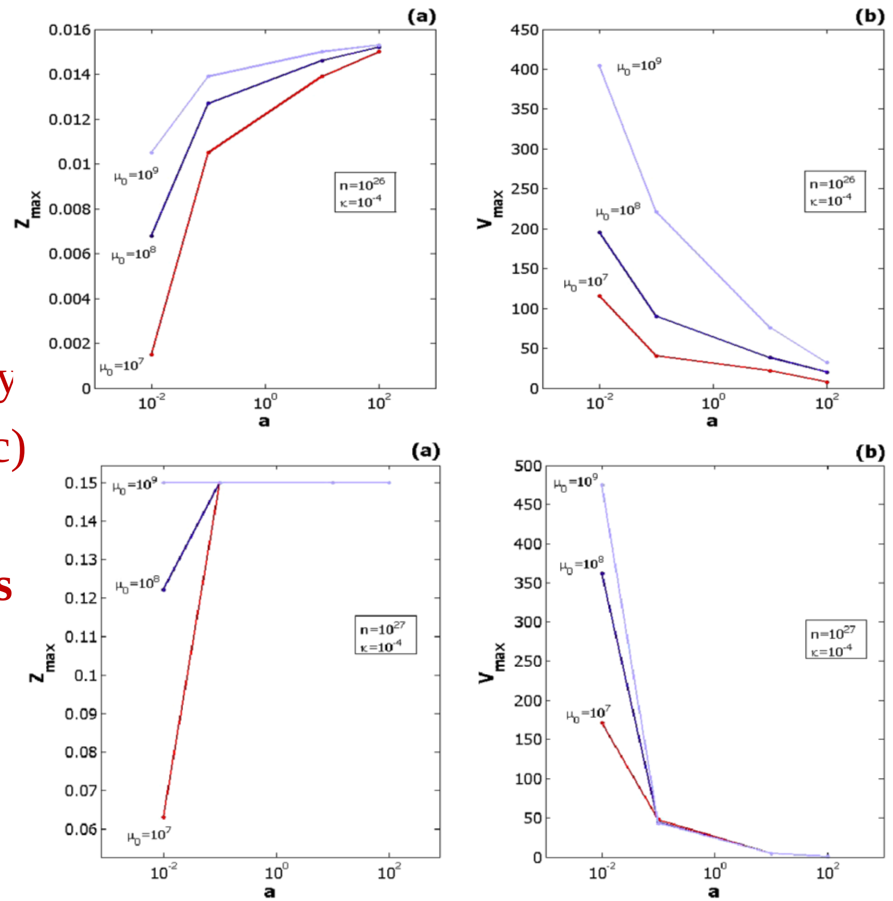


There is a clear evidence of magnetic field amplification together with flow acceleration

Flow Generation / Acceleration – *White Dwarfs' Atmospheres 2*

We have also verified the analytical prediction of BSM 2015 for maximal flow speed (achieved after acceleration) to be super-Alfvénic - even for unrealistic Hall term strength parameter $\kappa \sim 10^{-6}$ at specific boundary conditions we see the tendency of accelerated flow to become locally almost super-Alfvénic (starting from sub-Alfvénic) $n_0 = 10^{25}$ cm^{-3} [$a_0 = (1/6)^{1/3} \times 10^{-4/3}$] and $a = d = 10$ at $\mu_0 = 10^9$. **Initial flow with 0.15 km/s speed accelerates 450 times reaching 60 km/s .**





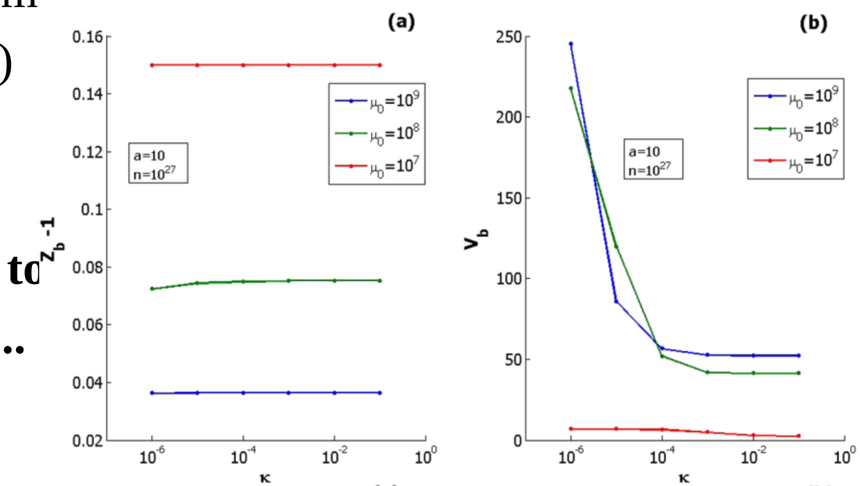
Blow-up distance and blow-up velocity versus DB parameter $a \sim d$ for initial density $n_0 = 10^{26}$ cm^{-3}(***top***) / $n_0 = 10^{27}$ cm^{-3}(***bottom***)

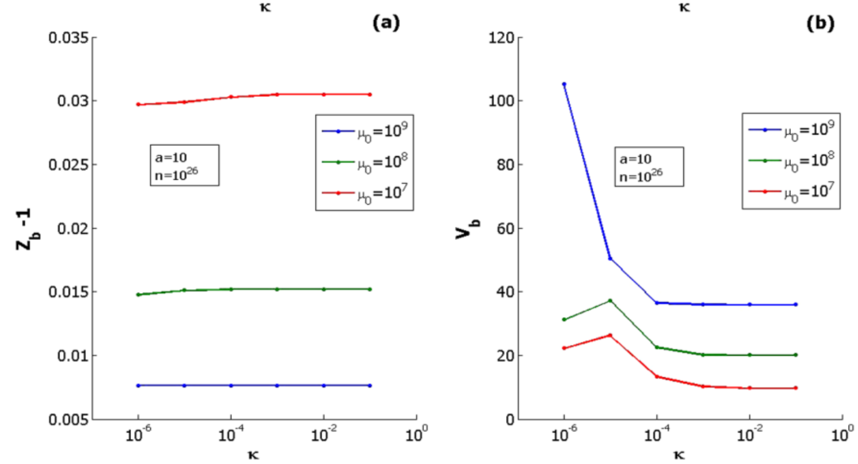
Flow Generation / Acceleration – *White Dwarfs' Atmospheres 3*

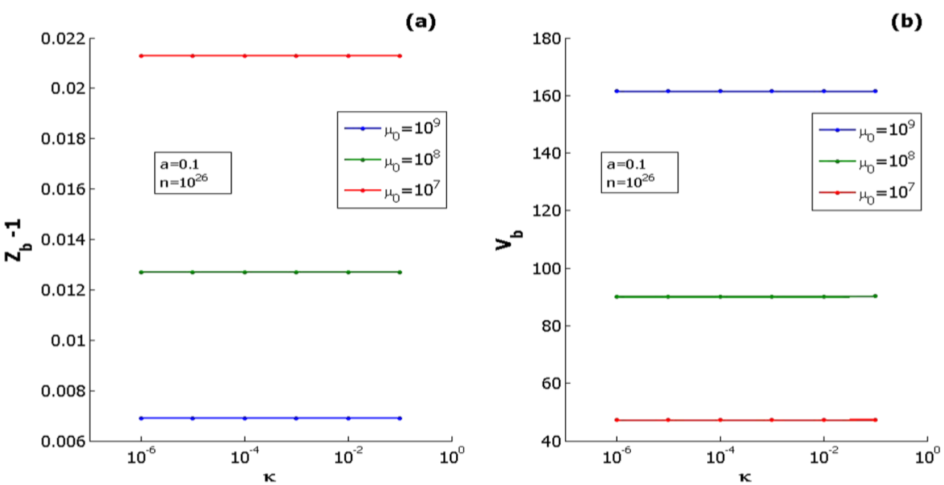
Blow-up (**a**) distance and (**b**) velocity vs Hall-term strength κ for $n_0 = 10^{26}$ cm^{-3} and $a = d = 0.1$ (***left***) and $n_0 = 10^{27}$ cm^{-3} , $a = d = 10$ (***top right***), $n_0 = 10^{26}$ cm^{-3} , $a = d = 10$ (***bottom right***).

For fixed μ_0 blow-up process is more sensitive to changes in κ at higher μ_0 (lower initial *B*-field).. The smaller the μ_0 lower is the final speed and smaller is the blow-up distance.

For fixed μ_0 blow-up process is less sensitive to changes in Hall-term strength κ







There is a clear tendency for initial flow to become Super-Alfvénic at blow-up

Conclusions

The mechanism for flow generation in dense degenerate stellar atmospheres is suggested when the electron gas is degenerate and ions are assumed to be classical.

It is shown, that **there is a catastrophe in such system—fast flows are generated due to magneto-fluid coupling near the surface. Distance over which acceleration appears is determined by the strength of gravity and degeneracy parameter.**

Application of this mechanism for White Dwarfs' atmospheres is examined and appropriate physical parameter range for flow generation/acceleration is found; possibility of the super-Alfvénic flow generation is shown; *the simultaneous possibility of flow acceleration and magnetic field amplification for specific boundary conditions is explored;* in some cases initial background flow can be accelerated 100 and more times leading to transient jet formation while the Magnetic field amplification is less strong.

We extended the studies of BSM 2015 and SBM 2016 and showed that the degeneracy effects are significant for specific class of dense stellar atmospheres/outer layers dynamics, specifically, for the structure formation phenomena there —

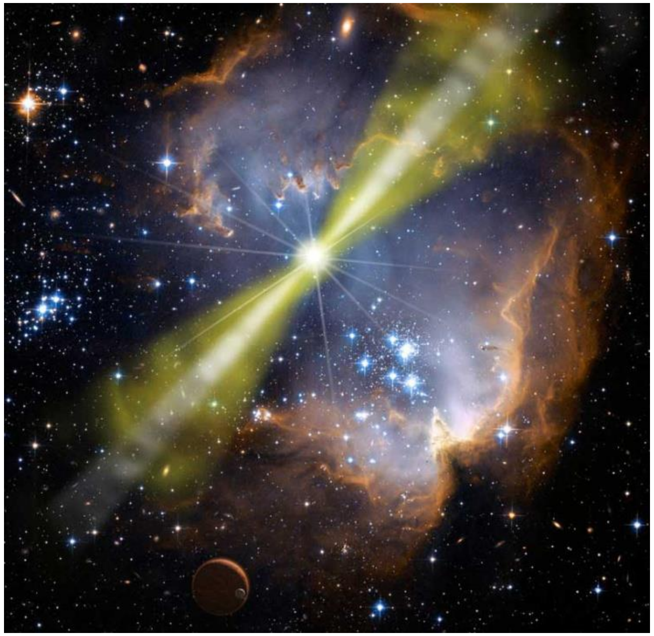
We suggest that when studying the evolution of the compact objects flow effects cannot be ignored since their catastrophic generation close to the surface may determine the further evolution of stars and their atmospheres.

Compact Astrophysical Objects with Degenerate Electron-positron plasma

Recent observations as well as the modern theoretical considerations indicate on the existence of super-dense electron-positron plasmas in a variety of astrophysical environments.

The presence of the e-p plasma is also argued in the ***MeV epoch of the early Universe***.

Intense ***e-p pair creation takes place during the process of gravitational collapse of massive stars***; it is shown that in certain circumstances the gravitational collapse of the stars may lead to the charge separation with the field strength exceeding the Schwinger limit resulting in e-p pair plasma creation with estimated **density to be ~ 10^{34} *cm*$^{-3}$**.

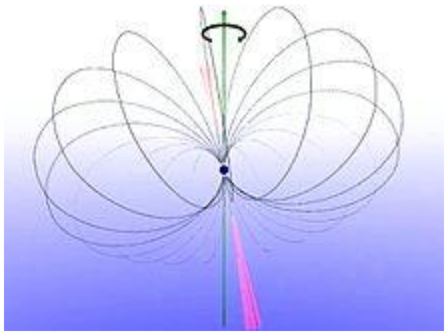


Bright GRB occurring in a star-forming region. Energy from the explosion is beamed into two narrow, oppositely directed jets.

The e-p plasma density can be in the range (10^{30} – 10^{37}) *cm*$^{-3}$ of the ***gamma-ray burst (GRB) source***.

Generation of a high density e-p plasma is augmented by production of intense pulses of *X- and Gamma-rays*.

Compact Astrophysical Objects - 2



The natural habitats for dense/degenerate matter:

Compact astrophysical objects like **white and brown dwarfs, neutron stars, magnetars** with believed characteristic **electron number densities** $\sim 10^{26} - 10^{32}\ cm^{-3}$, formed under extreme conditions.

In *BSM (2015)* we developed the simplest model in which the effect of quantum degeneracy on the nature of the Beltrami-Bernoulli class of equilibrium states was illustrated. We choose **a model hypothetical system *(relevant to specific aspects of a white dwarf (WD))*** of a two-species neutral plasma with non-degenerate non relativistic ions, and degenerate relativistic electrons embedded in a magnetic field.

The electron fluid vorticity was assumed negligible compared to the electron cyclotron frequency (*such a situation may pertain, for example, in pre-WD state of star evolution, and in the dynamics of the WD atmosphere*).

SMB (2016) studied the ***degenerate e-p inertia effects as well as the mobile ion species on Beltrami States in dense neutral plasmas*** and found the **existence of meso- and large-scales adding immensely to the richness of the structures that such medium can sustain; many more pathways become accessible for energy transformations.**

When star contracts, its outer layers keep the multi-Structure character although density in structures becomes defined by electron degeneracy pressure.

Model

For a highly compressed state, the plasma behaves as a degenerate Fermi gas provided that the averaged inter-particle distance is smaller than the thermal de Broglie wavelength. As the ***density increases, the Fermi energy of the particles*** $\epsilon_F^{\pm} = \hbar^2 (3\pi^2 n^{\pm})^{2/3}/2m_e$ ***becomes larger than the interaction energy*** $(\sim e^2 (n_0^{\pm})^{1/3})$ - mutual interaction of particles becomes unimportant—**plasma becomes more ideal**.

Condition is fulfilled for a sufficiently dense fluid when

$$n_0^{\pm} >> (2m_e e^2/(3\pi^2)^{2/3}\hbar^2)^3 : 6.3\text{x}10^{22}\text{cm}^{-3}$$

When density increases further, particle's relativistic motion shall be taken into account, leading to the relativistic Fermi energy of the particles in the following form $\epsilon_F^{\pm} = m_e c^2 [(1+(R^{\pm})^2)^{1/2} - 1]$ where $R^{\pm} = p_F^{\pm}/m_e c$, $p_F^{\pm}$ — is the Fermi momentum related to the rest-frame particle density by the following relation $p_F^{\pm} = m_e c (n^{\pm}/n_c)^{1/3}$;

here, $n_c = 5.9\times10^{29}\text{cm}^{-3}$ is the normalizing critical number-density.

When $n^{\pm} \gg n_c$ plasma turns out to be ultra-relativistic even for non-relativistic temperature $T^{\pm} \ll \epsilon_F^{\pm}$.

Pair plasmas with such densities cannot be in complete thermodynamic equilibrium with the photon gas into which it annihilates. Equilibrium is reached within the time-period related mainly to the electron-positron annihilations. Subsequently, thermodynamic equilibrium between pairs and photons (with zero chemical potential) will be achieved. ***Plasma becomes optically thick with steady state pair density defined by plasma temperature.***

On the other hand, ***for high density optically thin plasma*** [*Berezhiani, Shatashvili, Tsintsadze (2015)*], ***e-p annihilation time becomes considerably short*** ($\tau_{ann} \approx 0.3\times10^{-16}$ *s* for the densities ~10^{30} cm^{-3}). However, it is still larger than corresponding densities plasma oscillations characteristic time-scale [~ ω_e^{-1}] ; e.g, **for densities (10^{30} - 10^{35}) cm^{-3}**, we found that $\tau_{ann}\,\omega_e^{-1}$ is in the range [60 - 7] and **collective plasma high frequency oscillations have enough time to manifest themselves**.

To understand the dynamics of intense X- and Gamma-ray pulses emanating from the compact astrophysical objects as well as to study the nonlinear interactions of intense laser pulses and dense degenerate plasmas, ***it is important to investigate the wave self-modulation and soliton formation phenomena in dense e-p plasmas***.

The existence of stable localized envelope solitons of EM radiation has been suggested as a potential mechanism for the production of micro-pulses in active galactic nuclei (AGN) & pulsars.

Localized solitons created in the plasma-dominated era are also invoked to explain the observed inhomogeneities of the visible universe.

The existence of soliton-like electromagnetic distributions in a fully degenerate e-p plasma was shown in BST 2015 applying relativistic HD & Maxwell equations. For a c.p. wave: soliton solutions exist both in relativistic & nonrelativistic degenerate plasmas; possibility of cavitation.

EM Solitons In Degenerate Relativistic e-p Plasma

Fluid equations can be reduced to following

$$\frac{\partial}{\partial t}(G\mathbf{p}) + m_e c^2\ \nabla(G\gamma) = q\ \mathbf{E} + \mathbf{V}\times\boldsymbol{\Omega} \tag{34}$$

$$\frac{\partial}{\partial t}\boldsymbol{\Omega} = \nabla\times(\mathbf{V}\times\boldsymbol{\Omega}), \tag{35}$$

where $\boldsymbol{\Omega} = (q/c)\mathbf{B} + \nabla\times(G\mathbf{p})$, $\mathbf{p} = \gamma m_e \mathbf{V}$.

Expressing EM fields by the vector and scalar potentials (with $\nabla\cdot\mathbf{A} = 0$) Maxwell equations can be written as

$$\frac{\partial^2\mathbf{A}}{\partial t^2} - c^2\Delta\mathbf{A} + c\,\frac{\partial}{\partial t}(\nabla\varphi) - 4\pi c\,\mathbf{J} = 0 \tag{36}$$

$$\Delta\varphi = -4\pi\,\rho. \tag{37}$$

From equation (34) it follows that if the generalized vorticity is initially zerc ($\boldsymbol{\Omega} = 0$) everywhere in space, it remains zero for all subsequent times. **We assume that before the EM radiation is 'switched on' the generalized vorticity of the system is zero**. == > (34) now takes the form

$$\frac{\partial}{\partial t}\left(G\mathbf{p} + \frac{q}{c}\mathbf{A}\right) + \nabla\left(m_e c^2 G\,\gamma + q\,\varphi\right) = 0. \tag{38}$$

Equations (36)-(37) in one-dimensional case. Assuming that all quantities vary only with one spatial coordinate z and in time t the transverse component of the equation of motion (38) is immediately integrı $\mathbf{p}_\perp = -\,q\,\mathbf{A}_\perp/(c\,G).$

Gauge condition $A_z = 0$ ➔ **longitudinal motion doesn't depend on charge sign**

($\gamma = \left[1 + (p_\perp^2 + p_z^2)/m_e^2 c^2\right]^{1/2}$)

The EM pressure gives equal longitudinal momenta to both the electrons and positrons ➔ no charge separation

$$n_e = n_p = n \quad , \quad \varphi = 0$$

Thus for longitudinal motion we have **equation of mot**ion $\frac{\partial}{\partial t} G\, p_z + m_e c^2 \frac{\partial}{\partial z} G\, \gamma = 0$ (39)

and **Continuity Equation** $\frac{\partial}{\partial t}\gamma n + \frac{\partial}{\partial z}\left(n\gamma V_z\right) = 0.$ (40)

$J_z = 0 \quad \mathbf{J}_\perp = (2ne^2/c\ G)\mathbf{A}_\perp$ substituting this into (3) we get

$$\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - c^2 \frac{\partial^2 \mathbf{A}_\perp}{\partial z^2} + \Omega_e^2 \left(\frac{n}{n_0}\frac{G_0}{G}\right)\mathbf{A}_\perp = 0, \qquad (41)$$

where $\Omega_e = (8\pi e^2 n_0/(m_e G_0)\ )^{1/2}$ is the Langmuir frequency. For stationary localized solution

$$e\mathbf{A}_\perp/m_e c^2 = (1/2)(\mathbf{x} + i\mathbf{y})A(z)\exp(-i\omega t) + c.\,c., \qquad (42)$$

where $A(z)$ is a real valued amplitude, $p_z = 0$. Straightforward algebra leads to

$$n = n_0\left(1 - A^2/R_0^2\right)^{3/2} \qquad (43)$$

$$G = G_0\left[1 - A^2/\left(1 + R_0^2\right)\right]^{3/2} \qquad (44)$$

It follows from equation (43) that our considerations remain valid provided $A \leq R_0$; *the plasma density decreases in the* area of EM field localization and if at certain point of this area $A \rightarrow R_0$ *then the plasma density becomes zero (* $n \rightarrow 0$ *),* hence, at that point the cavitation takes place.

Equation (41) reduces to

$$\frac{d^2a}{d\eta^2} - \lambda a + f\left(a^2\right)a = 0, \qquad (45)$$

where the **nonlinearity function** is given by

$$f\left(a^2\right) = 1 - \frac{\left(1 - a^2\right)^{3/2}}{\left(1 - \epsilon^2 a^2\right)^{1/2}}. \qquad (46)$$

here $\eta = z(\Omega_e/c)$ is a dimensionless constant and $\lambda = 1 - \omega^2/\Omega_e^2$, $a = A/R_0$ $\epsilon^2 = R_0^2/(1 + R_0^2)$.

For small intensities $(a \ll 1)$ nonlinearity function is $f = (3 - \epsilon^2)a^2/2$ while $f \to 1$ for $a \to 1$.

Note **that the saturation character of nonlinearity is related to plasma cavitation** . Since $0 \leqslant f \leqslant 1$ **equation (45) admits the soliton solutions for all allowed intensities of EM field** $(0 \leqslant a^2 \leqslant 1)$ provided that $0 \leqslant \lambda \leqslant Max[f] = 1$ [.

The paramerer λ is the nonlinear 'frequency shift' and it has the meaning of the reciprocal of the square of the characteristic width of the soliton.

The general solution of equation (45) cannot be expressed in terms of the elementary function except for the **ultra-relativistic degenerate plasma case**, i.e. for $R_0 \gg 1$ $(\epsilon \to 1)$.

In this case $f = a^2$ and for $\lambda_{(\epsilon=1)} = a_m^2/2$ **the soliton solution of (12) takes the simple form**

$$a = a_m \, sech\left(\frac{a_m}{\sqrt{2}}x\right). \qquad (47)$$

which exists for $a_m = \left(A_m/R_0\right) \leqslant 1$ $(\lambda_{(\epsilon=1)} \leqslant 0.5)$.

Conclusions for Soliton Solutions

In t*he relativistic degenerate plasma the* amplitude of EM soliton can become relativistically strong $A_m >> 1$. *In the region of the soliton localization the e–p* plasma density decreases considerably while for $A_m \rightarrow R_0$ the plasma cavitation takes place.

We have shown that fully degenerate electron–positron plasma supports the existence of stationary soliton solution in over-dense plasma ($\omega < \Omega_e$). *In relativistic degenerate e–p plasma the* intensity of EM field can be relativistically strong while for norelativistic degeneracy case the soliton intensity is always nonrelativistic.

It is also shown that the cavitation of plasma can occur in both the relativistic and nonrelativistic degenerate plasmas. The generalization for the case of moving soliton is straightforward and is beyond the intended scope of the present paper.

The 1D model of present study can be generalized for 2D and 3D problems similarly in either so-called ‘pancake’ regime of propagation*, or in so-called beam-regime of propa*gation. Preliminary analysis shows, that in such cases a nonlinear Schrödinger equation with saturating nonlinearity similar to (46) can be derived implying that the generation of stable multi-dimensional localized solutions —‘the light bullets’ or the solitary filaments — is possible.

Model Equations for the Self-Guiding of EM Beams in *Degenerate e-p plasma of Gamma-Ray Sources*

W***e apply the Fluid-Maxwell model to investigate the possibility of self-trapping of intense EM pulse in the transparent degenerate (e-p) plasma in a limit of narrow pulse*** $L_\perp << L_\parallel$ ($L_\parallel$ & $L_\perp$ are characteristic longitudinal & transverse spatial dimensions of field) **to demonstrate the formation of stable 2D solitonic structures.** The basic set of Maxwell-Fluid equations reads as [BST 2015]

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - c^2 \Delta \mathbf{A} + c\frac{\partial}{\partial t}(\nabla\varphi) - 4\pi e c\left(N^+\mathbf{V}^+ - N^-\mathbf{V}^-\right) = 0, \tag{48}$$

$$\Delta\varphi = -4\pi e\left(N^+ - N^-\right), \tag{49}$$

$$\frac{\partial}{\partial t}\left(G^\pm \mathbf{p}^\pm \pm \frac{e}{c}\mathbf{A}\right) + \nabla\left(m_e c^2 G^\pm \gamma \pm e\,\varphi\right) = 0, \tag{50}$$

$$\frac{\partial}{\partial t}N^\pm + \nabla\left(N^\pm \mathbf{V}^\pm\right) = 0, \tag{51}$$

$\boldsymbol{p}^\pm = \gamma^\pm \boldsymbol{V}^\pm$ - hydrodynamical momentum of particles; $\gamma^\pm$ - relativistic factor; $N^\pm$ - density in lab-frame

$G^\pm = [1 + (n^\pm/n_c)^{2/3}]^{1/2}$ - ***"effective mass" depends on plasma rest-frame density*** $n^\pm = N^\pm/\gamma^\pm$ and is valid for arbitrary strength of relativity defined by the ratio $n^\pm/\gamma^\pm$.

Introducing Generalized momentum $\Pi^\pm = G^\pm \mathbf{p}^\pm$ and generalized Relativistic factor $\Gamma^\pm = G^\pm \gamma^\pm$
In terms of normalized quantities:

Model Equations - 2

$$\tilde{t} = \omega t\,, \quad \tilde{\mathbf{r}} = \frac{\omega}{c}\mathbf{r}\,, \quad \tilde{\mathbf{A}} = \frac{e\mathbf{A}}{m_e c^2}\,, \quad \tilde{\varphi} = \frac{e\varphi}{m_e c^2}\,,$$

$$\tilde{\mathbf{\Pi}}^{\pm} = \frac{\mathbf{\Pi}^{\pm}}{m_e c}\,, \quad \tilde{n}^{\pm} = \frac{n^{\pm}}{n_0} \quad \text{and} \quad \tilde{N}^{\pm} = \frac{N^{\pm}}{n_0}.$$

We obtain the following ***dimensionless equations*** (tilde is suppressed below):

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta\mathbf{A} + \frac{\partial}{\partial t}(\nabla\varphi) - \varepsilon^2(\mathbf{J}^+ - \mathbf{J}^-) = 0, \tag{52}$$

$$\Delta\varphi = \varepsilon^2(N^- - N^+), \tag{53}$$

$$\frac{\partial}{\partial t}(\mathbf{\Pi}^{\pm} \pm \mathbf{A}) + \nabla(\Gamma^{\pm} \pm \varphi) = 0, \tag{54}$$

$$\frac{\partial N^{\pm}}{\partial t} + \nabla \cdot \mathbf{J}^{\pm} = 0, \tag{55}$$

with $\mathbf{J}^{\pm} = n^{\pm}\mathbf{\Pi}^{\pm}/\Gamma^{\pm}$ and $\Gamma^{\pm} = \sqrt{(G^{\pm})^2 + (\mathbf{\Pi}^{\pm})^2}$. Here $\varepsilon = \omega_e / \omega << 1$, ω - is EM field frequency; $\omega_e = \sqrt{4\pi e^2 n_0/m_e}$.

$G^{\pm} = \sqrt{1 + R_0^2 (n^{\pm})^{2/3}}$ [with $R_0 = \left(\frac{n_0}{n_c}\right)^{1/3}$] ***is valid for entire range of physically allowed densities.***

Model Equations - 3

We apply below the method of multiple scale expansion of the equations in the small parameter ε **.**

All physical variables $(Q = A, \varphi, \Pi^{\pm}, \Gamma^{\pm}, N^{\pm}, G^{\pm})$ can be expanded as

$$Q = Q_{\{0\}}(\xi, x_1, y_1, z_2) + \varepsilon Q_{\{1\}}(\xi, x_1, y_1, z_2)\,, \tag{56}$$

where $(x_1, y_1, z_2) = (\varepsilon x, \varepsilon y, \varepsilon^2 z)$ and $\xi = z - bt$; $(b^2 - 1) \sim \varepsilon^2$.

We assume that EM field is circularly polarized: (57)

$$\mathbf{A}_{\{0\perp\}} = \frac{1}{2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})A\exp(i\xi/b),$$

with $\boldsymbol{A}$ being a slowly varying envelope of EM beam.

To the lowest order in ε , following the standard procedure, we get

$\mathbf{\Pi}^{\pm}_{\{0\perp\}} = \pm\mathbf{A}_{\{0\perp\}}$; $\varphi_{(0)} = 0$; $\Gamma^{\pm}_{\{0\}} = \Gamma_0 = const$; Thus, *Poisson eq.* $\varphi_{\{0\}} = 0$ gives: $N^{\pm}_{\{0\}} = N_{\{0\}}$.

Here $\Gamma^{\pm}_{\{0\}} = \sqrt{G^2_{\{0\}} + |A|^2}$ with $G_{\{0\}} = \sqrt{1 + R_0^2 n^{2/3}_{\{0\}}}$; since fields vanish at infinity the factor $\Gamma_0 = \sqrt{1 + R_0^2} \equiv const.$ Taking into account that $n_{\{0\}} = N_{\{0\}}/\gamma = \frac{G_{\{0\}}}{\Gamma_0} N_{\{0\}}$ and solving relations $\Gamma^{\pm}_{\{0\}} = \Gamma_0$, we obtain the expression for density ($\delta = R_0/\sqrt{1 + R_0^2}$) :

$$N_{\{0\}} = \frac{1}{\delta^3\sqrt{1 - |A|^2/\Gamma_0^2}}\left(\delta^2 - |A|^2/\Gamma_0^2\right)^{3/2}, \tag{58}$$

(58) is valid for arbitrary level of δ provided that $|A|^2 < R_0^2$.

Nonlinear Schrödinger equation

For ***slowly varying envelope***, Maxwell equation (52) reduces to ($\nabla_\perp^2 = \partial_{x_1}^2 + \partial_{y_1}^2$)

$$2i\frac{\partial A}{\partial z_2} + \nabla_\perp^2 A + \sigma A - A\frac{2N_{\{0\}}}{\Gamma_0} = 0\,, \tag{59}$$

Where $\sigma = (b^2-1)/b^2\varepsilon^2$; if $b = \omega/kc$, ***the*** $\sigma - 1/\Gamma_0 = 0$ ***is nothing but the dispersion relation*** $\omega^2 = k^2c^2 + 2\Omega_e^2$, where $\Omega_e = \omega_e/\sqrt{\Gamma_0}$ modified plasma frequency due to degeneracy. Introducing $a = A/R_0$, dropping subscript for (z_2, x_1, y_1), making renormalization: $z = 2z/\Gamma_0$; $r_\perp = \sqrt{2/\Gamma_0}\,r_\perp$, →

$$2i\frac{\partial a}{\partial z} + \nabla_\perp^2 a + f\left(|a|^2\right)a = 0\,, \tag{60}$$

with

$$f\left(|a|^2\right) = 1 - \frac{\left(1-|a|^2\right)^{3/2}}{\left(1-\delta^2|a|^2\right)^{1/2}}\,. \tag{61}$$

Nonlinear Schrödinger equation* - *NSE (60) is with nonlinearity function f by (61) ;

f is a growing function of $|a|$ attaining its maximum value $f = 1$ at $|a| = 1$.

For $|a| \ll 1$ nonlinearity function reduces to $f \simeq \beta\,|a|^2$, where $\beta = 0.5(3-\delta^2)$ varies within (1.5 -1) for an arbitrary level of degeneracy $(0 < \delta < 1)$.

For weak degeneracy level $\delta \ll 1$ nonlinearity function $f \simeq 1 - (1-|a|^2)^{3/2}$;

For relativistic degeneracy $\delta \to 1$ $f \simeq |a|^2$ at $|a| \le 1$.

Solitary Solutions

The NSE with various type saturating nonlinearities has been studied thoroughly in the past; our Eq. (60) [with (61)] proves to be similar qualitatively though showing quantitative difference. First, ***one has to establish the existence and the stability of the self-trapped solutions of Equation (60).***

Using the **axially symmetric solution ansatz** in the form $a = U(r)\exp(i\lambda z)$, where $r = (x^2 + y^2)^{1/2}$ and $\boldsymbol{\lambda}$ is the so called *propagation constant*, the ordinary nonlinear differential equation for the radially dependent envelope ***U(r)*** reduces to

$$\nabla_{rr}^2 U - \lambda U + f(U^2)\,U = 0\,, \qquad \text{with} \qquad \nabla_{rr}^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \qquad (62)$$

We ***consider lowest order nodeless (ground state) solution of (62)*** with maximum U_m at the center ($r = 0$) and monotonically decreasing ($U \rightarrow 0$) as $r \rightarrow \infty$.

Maximal value U_m is determined by eigenvalue λ , which satisfies $0 < \lambda < f_m$, where f_m *is a maximal value of the nonlinearity function.*

The numerical simulation study of (62) with (61) for arbitrary level of degeneracy parameter δ shows:

Despite the value of $f_m = 1$ the allowed range of λ is significantly narrow $\lambda < \lambda_c < 1$. U_m is a growing function of λ attaining its maximal value $U_m = 1$ at $\lambda = \lambda_c$. λ_c decreases with $\boldsymbol{\delta}$ from its maximum = 0.2912 at $\delta \ll 1$ to 0.2055 for $\delta \gg 1$.

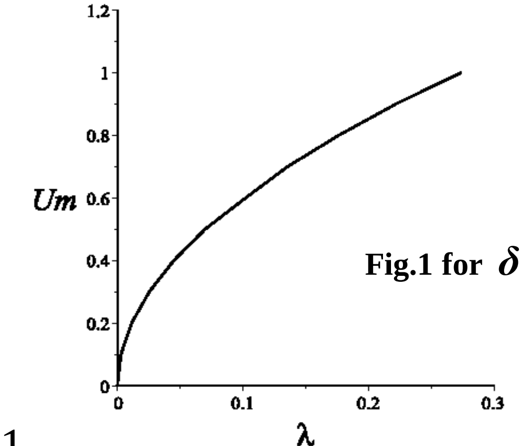


Fig.1 for δ = 0.5

Solitary Solutions - 2

Important characteristics of obtained solitary solutions are the so called ***beam "power"*** defined by

$$P = 2\pi \int_0^\infty dr\, r\, U^2(r,\lambda)$$

Numerical simulations show that **for the arbitrary level of degeneracy** δ **, power** P **is a growing function of** λ **and** U_m [such behavior for δ = 0.5 is presented in Fig. 3 where P versus U_m is plotted] ;

consequently, ***solitary solutions are stable against small perturbations.***

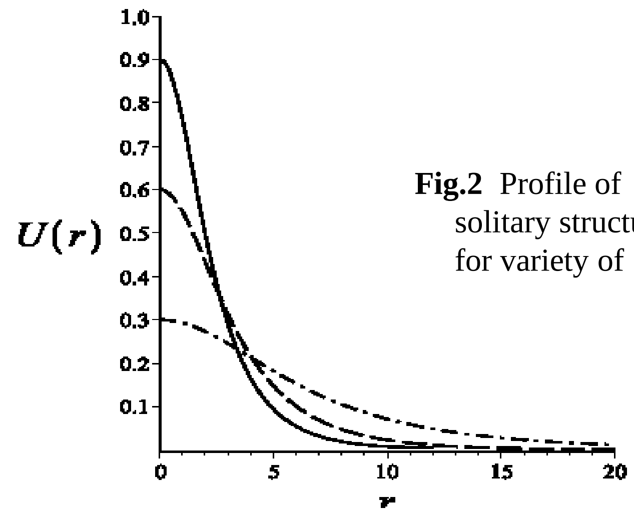


Fig.2 Profile of solitary structure for variety of U_m

For $\lambda \rightarrow 1$ [$U_m \rightarrow 0$] power $P \rightarrow P_c$,

while **the maximal allowed value of power is achieved at** $\lambda = \lambda_c$ **[** $U_m = 1$ **] ;**

here, P_c **is a critical power**.

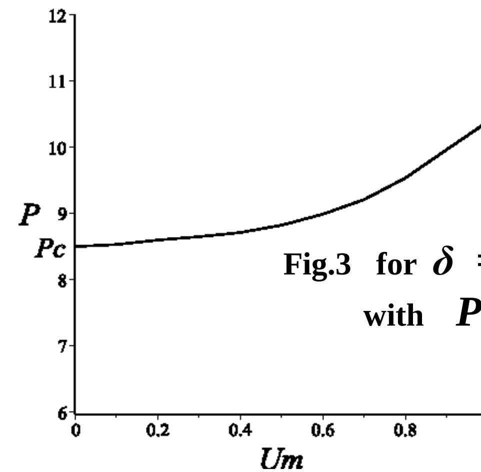


Fig.3 for δ = 0.5 **with** $P_c = 8.5$

Thus, **a self-trapped propagation of solitary beam could be formed in such plasmas provided** $P \geq P_c$ **.**

Critical Power of Self-trapped Beam

Critical power of self-trapped beam P_c depends on degeneracy parameter λ . This relation can be obtained analytically assuming that for the small amplitude case ($\lambda << 1$, $f \approx \beta U^2$) with obvious change of variables

$$[U(r) = \sqrt{\frac{\lambda}{\beta}} g(\sqrt{\lambda} r)]$$

Equation (48) reduces to $\nabla^2_{rr} g - g + g^3 = 0,$ where $g(r)$ is a stationary Townes mode.

Knowing the Power of ground state Townes mode $P_g = 2\pi \int_0^\infty dr\, r\, g^2(r) = 11.69$, one can find the critical power to be

$$P_c = \beta^{-1} P_g = 23.37/(3 - \delta^2).$$

For the nonrelativistically degenerate case $P_c = 7.79$

while **for the super-relativistic degenerate case $P_c = 11.69$.**

Critical power in dimensional units can be written in a following convenient form:

$$[P_c] = 33\,\chi\,\frac{\omega^2}{\Omega_e^2}\,\mathrm{GW}, \qquad \text{where} \qquad \chi = R_0^2/(3 + 2R_0^2). \qquad (49)$$

A physically justified range of allowed plasma densities is presumably within (10^{24} - 10^{34}) cm^{-3}
[$R_0 \sim 1.19 \times 10^{-2} - 25.69$] ;
corresponding critical power P_c for self-trapped solution to occur is within

$$(1.6 \times 10^{-3} \div 16.5)\,\frac{\omega^2}{\Omega_e^2} \quad \mathrm{GW}.$$

Summary and Discussion

Note: that recent progress in creating dense e-p plasmas in Laboratory conditions and achievements in the development of free electron powerful X-ray sources indicate that **in the future the generation of the optically thin e-p plasmas can be expected with the solid state densities in the range of (10^{23} – 10^{28}) cm^{-3} and above.**

For the X-ray pulse with wave-length ~3nm interacting with $n \sim 10^{24}$ cm^{-3} density plasma, the critical power becomes 194 MW. ***We emphasize:*** in case of the cold non-degenerate classical plasma (temperature is zero), such an effect of the existence of self-trapped solitary structures is absent.

Analysis of the stability study of obtained solitary solutions confirms (through direct simulation of derived equations) ***that like in the case of other type saturating nonlinearities, the ground state solution is stable***; even for Gaussian profile initial radial distribution of field with power & amplitude close to ground state solution field quickly relaxes to the equilibrium shape structure; ***exception is for the initial profile with amplitude ~1 or for the initial amplitude far from ground state solution.***

In certain cases when the initial Gaussian profile amplitude field is far from the equilibrium one the amplitude $|A|$ of the evolving field has a tendency to reach values ≥ 1 implying that cavitation (complete expulsion of plasma from field localization area) will take place.

Results

We showed that:

The stable self-trapped solitary 2D structures exist for the arbitrary level of degeneracy.

We have found the critical power for the self-guided propagation.

The results of the given study can be applied to understand:

- **the radiation properties of astrophysical gamma-ray sources**
- **as well as may be useful to design the future laboratory experiments.**

The results can be useful to understand the dynamics of x-ray pulses emanating from the compact astrophysical objects as well as to study the nonlinear interactions of intense laser pulses and dense degenerate plasmas that are relevant for the next-generation intense laser–solid density plasma experiments.