Why do we call it pressure?

F. Pegoraro

Physics Department "Enrico Fermi", Pisa University
Largo Bruno Pontecorvo 3, I-56127 Pisa, Italy
francesco.pegoraro@unipi.it

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Plasmas are different from gases and fluids

In most physical regimes of interest the dynamics of a plasma is intrinsically kinetic:

it requires a *phase space* description involving a particle distribution function that depends on position $\mathbf{x}$, momentum $\mathbf{p}$ and time t,

instead of the simpler *configuration space* description involving mean quantities that depend only on $\mathbf{x}$ and t.

In these regimes the concept of pressure, as we know it in gases and in fluids, does not appear to be applicable.

Nevertheless a quick look at the literature is sufficient to show that this concept is widely used, although with some caveats and with some necessary generalizations.

Scheme of the lectures

- First lecture: kinetic plasma descriptions and the definition of a "pressure-like" quantity. The *moment equations* and the closure problem.
  Finding heuristic closures for linear waves: Langmuir and ion-acoustic waves. Closures in magnetized plasmas.
- Second lecture: physical mechanisms that can lead to an anisotropic pressure: single particle and collective effects. Experimental observations in the solar wind.
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Pressure in fluids and gases

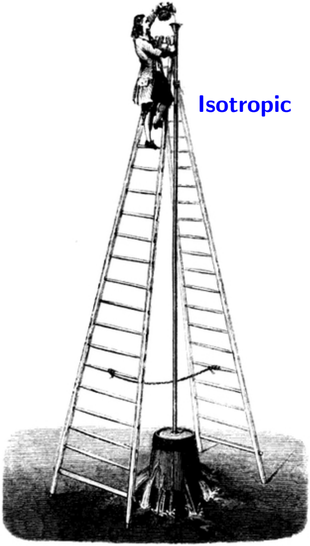


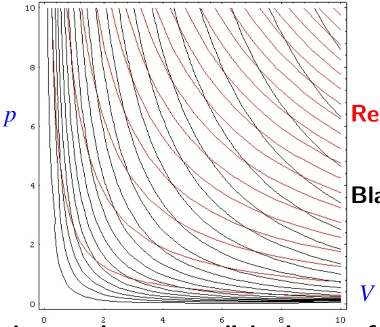
Fig. 45.—Hydrostatic paradox. Pascal's experiment.

Isotropic

Equation of state: $p = p(V,T)$

for ideal gases $p = nRT$

Dynamics along thermodynamic transformations



Red: isotherm

Black: adiabats

Sound waves in a gas: adiabatic transformations

Fluid-gas ordering

- The relaxation time, due e.g. to molecule collisions in a gas, is the shortest time in the theory. Analogously, the mean free-path (essentially the distance between two successive collisions in a gas) is the shortest distance in the theory.
- The dynamics of the gas is described by introducing an expansion parameter that is defined either as the ratio between the mean free path and the spatial scale of the phenomenon under investigation or as the ratio between the relaxation time and the dynamical time.
- To zero order we have global thermodynamic equilibrium, adiabatic equation of state, Euler-type equations of motion.
  To next order we have local thermodynamic equilibrium, dissipative effects such as thermal conductivity and viscosity and Navier-Stokes type equations of motion[1].

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Plasma ordering

- The relaxation time, due e.g. to Coulomb collisions, is longer than the period of the charge density (Langmuir) waves. Analogously, the mean free-path is longer than the Debye length.
  The ratio between the Coulomb collision frequency ν_{coll} and the plasma frequency[2] ω_{pe} scales as the plasma parameter
  $$g \sim (n\lambda_d^3)^{-1} \ll 1$$
- The dynamics of the plasma is described by an expansion in the parameter g.
- To zero order we obtain the *collisionless Vlasov equation* coupled to the Maxwell equations that have the plasma charge density and current density (obtained self-consistently from the Vlasov equation) as sources.
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[2] This ratio can be easily as small or smaller than 10^{-8}.

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Vlasov equation and its velocity moments

Define the distribution function $f = f(\mathbf{x}, \mathbf{v}, t)$ which obeys the Vlasov equation[3]

$$\frac{\partial}{\partial t} f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m} (\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0, \tag{1}$$

where the species index has been dropped.
Take velocity moments $\int d^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t)\,(1, \mathbf{v}, \mathbf{v}\mathbf{v}, \ldots)$

$$\int d^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t) = n(\mathbf{x}, t), \quad \frac{\partial}{\partial t} n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \tag{2}$$

$$\int d^3\mathbf{v}\, \mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t) = n(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t). \tag{3}$$

[3] written here in the non relativistic limit. C.g.s. units are used throughout the presentation.

Moments of the Vlasov equation

The second moment obeys the equation[4]

$$m\frac{\partial}{\partial t}(n\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(nm\mathbf{u}\mathbf{u}+\mathbf{\Pi}) = (nq)\,(\mathbf{E}+\frac{\mathbf{u}}{c}\times\mathbf{B}), \qquad (4)$$

where

$$\mathbf{\Pi}(\mathbf{x},t) = \int d^3\mathbf{v}\, f(\mathbf{x},\mathbf{v},t)\,(\mathbf{v}-\mathbf{u}(\mathbf{x},t))\,(\mathbf{v}-\mathbf{u}(\mathbf{x},t)) \qquad (5)$$

is a symmetric tensor. In general

$$\mathbf{\Pi}(\mathbf{x},t) \neq p(\mathbf{x},t)\,\boldsymbol{I}, \qquad (6)$$

i.e. the “pressure tensor” cannot be reduced to a scalar function (times the identity matrix $\boldsymbol{I}$).

[4] It can be reduced to the standard Euler form by subtracting Eq.(18) multiplied times $\mathbf{u}$.

The closure problem

• Within the gas ordering, to leading order, the distribution function f is isotropic (in the local rest frame). Then the pressure is a scalar and the thermodynamic limit allows us to close the fluid equations by assuming a law $p = p(n)$ that relates the dynamics of the pressure to that of the density[5].

• Within the plasma ordering the only information we can obtain on the distribution function f is by solving the Vlasov equation itself, which makes Eq.(20) essentially useless.

From the Vlasov equation we can derive the moment equation for $\mathbf{\Pi}$ but it leads to a 3-index tensor that "generalizes" the standard heat flux and for which we have no expression in terms of the lowest order moments, and so on, if we continue deriving higher order moments.

In the second lecture I will use the moment equation for $\mathbf{\Pi}$ and assume that heat flux vanishes.

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[5] In an incompressible fluid the pressure ceases to be an independent dynamical variable

The closure problem

The moment equations (often somewhat erroneously called fluid equations) appear to be similar to the true fluid equations but they are conceptually very different as they *do not form a system of closed equations*.

The difference does not rise at the level of Eq.(18) that expresses the conservation of the number of particles[6], but at the level of the momentum density continuity equation in the expression of the second order moment contribution to the momentum density flux.

The closure problem is amplified in the case of a relativistic plasma (a plasma where the particle velocities are not small with respect to the velocity of light) as it enters also the definition of the inertia term that is no longer simply given by the sum of the particle rest masses.

[6] In the absence of chemical reactions and ionization processes

Ad hoc closures

In the fluid ordering $\nu_{\rm coll}/\omega_{pe} >> 1$ the closure is in a sense "universal" being ensured by the local thermodynamic equilibrium conditions.
In the plasma ordering $\nu_{\rm coll}/\omega_{pe} << 1$ we lack such a general closure condition, indeed we lack a closure altogether, but in given subdomains of parameter space we can find *ad hoc closures* based on different expansion parameters. These expansions are not general and are easily violated during the evolution of the system. Often they are used as[7] "models" more than as valid limits.

In a linear wave the wave phase velocity can be compared with the particle velocities.

In a magnetized plasma the particle cyclotron frequency, and the particle gyro-radius, can be compared to the inverse time scale or to the length scale of the phenomenon under consideration.

[7] "Models" are very useful when exploring a new phenomenon as they can identify the "major players". Clearly, after the exploratory phase the investigation has to be refined using physically valid equations.

Simple examples of *ad hoc* closures

Closure conditions can be devised on the basis of such comparisons. *These closures do not reflect a property of the plasma but only those of a specific phenomenon in the plasma*.

Linear phase velocity closures (unmagnetized plasma)

"Cold", "warm" plasma closures: $\omega/(kv_{th}) << 1$

$\rightarrow$ neglect the pressure tensor altogether, no need for a closure (dispersion relation for longitudinal (Langmuir) waves $\omega^2 = \omega_{pe}^2$).

$\rightarrow$ first thermal correction[8] : impose a 1-D adiabatic closure per the pressure tensor (dispersion relation for longitudinal (Langmuir) waves $\omega^2 = \omega_{pe}^2 + 3\, k^2 v_{the}^2$).

[8] This closures can be validated by solving in the appropriate limit the linearized Vlasov equation.

Simple examples of *ad hoc* closures

Linear phase velocity closures (unmagnetized plasma)

"Hot" plasma closure[9]: $\omega/(kv_{the}) >> 1$

$\rightarrow$ neglect electron inertia and adopt an isothermal electron closure. It leads to the linearized Boltzmann electron response $\tilde{n}/n_0 \sim e\tilde{\varphi}/T_e$. Dispersion relation for quasineutral ionacoustic waves $\omega^2 = k^2 c_s^2$, with $c_s^2 = T_e/m_i$.

It is not at all clear how to extend these closures to a finite amplitude regime where the particle oscillation velocity comes into play (or the ratio between the electrostatic potential energy and the particle temperature) and where harmonics (shorter scales) are produced by the nonlinearities and are accompanied e.g., by the steepening of the wave profile.

[9] With cold ions

Failures of *ad hoc* closures

Particle populations can appear to be split into subpopulations with different dynamics: e.g., circulating and trapped particles (depending on the ratio between the particle kinetic energy and the fluctuating electrostatic potential energy).
A closure for each subpopulation ? Can be done[10] but it the wave amplitude changes with time, how do you treat the particles that get trapped or untrapped, etc... ?

The "ad hoc" closures fail when they are really needed i.e. in the nonlinear regimes!

One can resort to the "salvific" word "model" or better solve numerically the full nonlinear Vlasov equation[11] guided by the indications that the "models" can give.

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[10] See e.g. some models of ionacoustic shocks

[11] This can be done now on powerful supercomputer even for high dimensionality problems and is rather straightforward for 1-D Langmuir and ionacoustic waves.

Closures in magnetized plasmas

If the collision frequency is smaller than the cyclotron frequency Ω_c (due e.g. to a large scale magnetic field) the particle dynamics along magnetic field lines differs from the perpendicular dynamics[12].

This means that one construct hybrid kinetic-moment equations by taking velocity moments separately with respect to the perpendicular velocities and to the parallel velocity[13].

This mean that different closures need be devised for the parallel and for the perpendicular components[14] of the pressure tensor.

In the parallel direction we have essentially velocity-type closures (with $\omega/k \to (\omega + s\Omega_c)/k_{||}$). In the perpendicular direction the cold plasma limit corresponds instead to $k_\perp \rho_{th} << 1$, with ρ_{th} the particle gyroradius (Larmor radius) computed with the thermal velocity.

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[12] In the nonrelativistic limit they are decoupled.

[13] This separation between the parallel and the perpendicular dynamics is the starting point of the derivation of the se called gyro-kinetic and drift-kinetic equations

[14] Disregarding mixed parallel-perpendicular terms.

Closures in magnetized plasmas

A consequence of the different parallel and perpendicular dynamics is that the particle distribution function is inherently anisotropic.

On timescales longer than the cyclotron period the pressure tensor can be taken to be isotropic in the perpendicular plane due to the rapid gyration of the particle (gyrotropic pressure).

This is at the basis of the so called double adiabatic closure where the pressure tensor is written in the form[15]

$$\mathbf{\Pi} = p_\perp \boldsymbol{I} + (p_{||} - p_\perp)\,\mathbf{b}\,\mathbf{b}, \qquad \text{with} \quad \mathbf{b} = \mathbf{B}/B, \tag{7}$$

where $(p_{||}$ and $p_\perp)$ obey (in the ideal MHD framework) the closure equations

$$\frac{d}{dt}\left(\frac{p_{||}B^2}{n^3}\right) = 0, \qquad \frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = 0. \tag{8}$$

[15] G. Chew, F. Goldberger, and F. Low, Proceedings of the Royal Society of London A: 236, 112 (1956).

Closures in magnetized plasmas

The equation for the perpendicular pressure can be interpreted as the conservation of the particle magnetic moment $|\mathbf{v}_\perp|^2/B$.
The equation for the parallel pressure follows[16] from the conservation of the "action" of a plasma element moving along a magnetic line (and of the magnetic flux). A more general version of CGL equations (in index notation) is

$$\frac{dp_{||}^{\alpha}}{dt^{\alpha}} + p_{||}^{\alpha}\frac{\partial u_k^{\alpha}}{\partial x_k} = -2p_{||}^{\alpha} b_l b_k \frac{\partial u_l^{\alpha}}{\partial x_k} \tag{9}$$

$$\frac{dp_{\perp}^{\alpha}}{dt^{\alpha}} + 2p_{\perp}^{\alpha}\frac{\partial u_k^{\alpha}}{\partial x_k} = p_{\perp}^{\alpha} b_l b_k \frac{\partial u_l^{\alpha}}{\partial x_k}. \tag{10}$$

These equations can apply to separate species (α).

In the next lecture it will be indicated how these equations can be derived by the moment equation for the pressure tensor $\mathbf{\Pi}$.

[16] See R.M. Kulsrud, in Handbook of Plasma Physics, M.N. Rosenbluth and R.Z. Sagdeev Ed., Vol.1, p.115, North Holland Publ., (1983).

Shear Alfvèn waves and fire-hose instability

Magnetohydrodynamic (single fluid) description with double adiabatic equation of state.

Dispersion relation of shear Alfvèn waves with isotropic plasma pressure

$$\omega^2 = k_{||}^2 B^2/(4\pi n m_i) = k_{||}^2 c_A^2$$

where c_A is the "magnetic sound" (Alfvèn) velocity.

Corresponding dispersion relation with an anisotropic plasma obeying the double adiabatic equation of state

$$\omega^2 = k_{||}^2 (B^2/4\pi + p_\perp - p_{||})/(n m_i)\,.$$

It corresponds to an instability, $\omega^2 < 0$, when $B^2/4\pi + p_\perp < p_{||}$: parallel pressure (if not balanced by perpendicular pressure) counteracts the parallel magnetic tension which is the restoring force of shear Alfvèn waves.

Processes that make the pressure tensor anisotropic

The presence of anisotropy is the rule rather than the exception.

Some single and multi particle effects in a magnetized plasma

Energy gain: Joule heating (acceleration) along magnetic field lines, perpendicular heating in an e.m. laser pulse, wave resonances (a curious case: anomalous Doppler effect) Radio-frequency heating.

Energy loss: cyclotron (synchrotron) radiation,

Energy transport along field lines in the presence of "temperature" gradients,

Particle propagation in an inhomogeneous (turbulent) magnetic field: different dependence of the parallel and perpendicular equations of state on B. Plasma compression and expansion, (solar wind problem) Selective particle confinement: magnetic traps and mirrors.

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A measurement and a problem

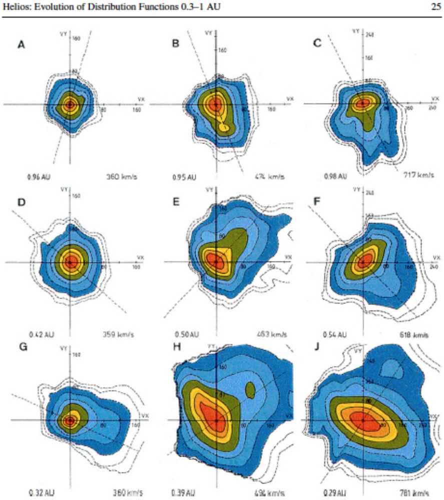


Fig. 1 Proton velocity distribution functions for three types of solar wind: Slow (*left column*), intermediate-speed (*middle*), and fast (*right*). The heliocentric distance decreases from top to bottom as indicated in the respective frames. Increasingly strong deviations from a Maxwellian occur at smaller distances from the Sun, with proton beams along the field (*dashed lines*) and large anisotropies perpendicular to the field in the core

Proton velocity distribution
in the solar wind (fast right)
largest anisotropy closer to the sun

E. Marsch, Space Sci. Rev., **172**, 23 (2012)

A measurement and a problem

$$\frac{d}{dt^\alpha}\left(\frac{P_\perp^\alpha}{n^\alpha B}\right)=0 \quad \dashrightarrow \quad P_\perp^\alpha \sim n^\alpha B \sim \frac{1}{R^4}$$

$$B(R) \sim \frac{1}{R^2} \qquad n^\alpha \sim \frac{1}{R^2}$$

(if the spiral deformation of the solar magnetic field is negelected)

$$\frac{d}{dt^\alpha}\left(\frac{P_{||}^\alpha B^2}{(n^\alpha)^3}\right)=0 \quad \dashrightarrow \quad P_{||}^\alpha \sim \frac{(n^\alpha)^3}{B^2} \sim \frac{1}{R^2}$$

Pressure anisotropy in the solar wind as predicted by ihe double adiabatic closure

$$\frac{P_\perp^\alpha}{P_{||}^\alpha} \sim \frac{1}{R^2} \qquad \beta_\perp^\alpha \equiv \frac{8\pi P_\perp^\alpha}{B^2} \sim R^0 \qquad \beta_{||}^\alpha \equiv \frac{8\pi P_{||}^\alpha}{B^2} \sim R^2$$

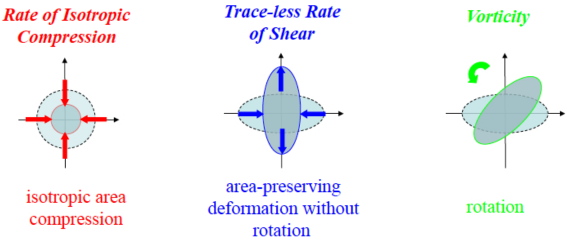
The Chew-Goldberger-Low (CGL) relations [1] predict that the plasma ions should become anisotropic in the sense of $T_{\|} > T_\perp$ if the particle motion is adiabatic and collisionless; here, T is the ion temperature parallel and perpendicular to the background magnetic field. However, Coulomb collisions and pressure-anisotropy instabilities act to pitch-angle scatter the plasma back towards isotropy [2]. At 1 AU, the measured most probable value of the proton temperature anisotropy is $T_\perp/T_{\|} \approx 0.89$ (Fig. 1, top panel below). If CGL were valid, this would imply a proton temperature anisotropy of $T_\perp/T_{\|} \gtrsim 200$ at 5 solar radii.

S.D. Bale *et al.* PRL, **103**, 211101 (2009)
see also
P. Hellinger *et al.* GRL, **33**, L09101 (2006)

Processes that make the pressure tensor anisotropic

I will discuss within the moment description a "kinetic" effect that leads to pressure anisotropy in the presence of a plasma velocity shear (D. Del Sarto, F. Pegoraro MNRAS **475**, 181 (2018)).

This treatment has important limitations but it clearly identifies the main features of the process putting together velocity shear (actually the symmetric part $[(\partial u_i)/(\partial x_j) + (\partial u_j)/(\partial x_i)]/2$ of the velocity strain tensor that characterizes the 'deformation" of the velocity field) and the shape and dynamics of the pressure tensor.



Processes that make the pressure tensor anisotropic

The approach to the investigation of the anisotropization process due to the plasma velocity distribution obeys two related motivations:

1) the extension to higher orders of the so called finite Larmor radius corrections to the double adiabatic pressure tensor[17].

2) The evidence obtained with kinetic (hybrid) simulations[18] of a correlation between plasma vorticity and generation of pressure anisotropy.

As consistent with the double adiabatic closure strong magnetic field is assumed. Furthermore the treatment I will present is essentially 2-D (in the plane perpendicular to the almost straight magnetic field)

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[17] The inclusion of *gyroviscous* Finite-Larmor-Radius (FLR) corrections related to the components of the gradient velocity tensor breaks the gyrotropic symmetry of the CGL pressure tensor (A.F. Kaufman, Phys fluids **3**, 619 (1960). There is a huge literature (see in particular the classic K. V. Roberts and J. B. Taylor Phys. Rev. Lett. **8**, 197 (1962)) on this subject which been revived in connection with the recent developments in gyrokinetic theory and simulations.

[18] L. Franci, *et al.* Ap.J, **833**, 91 (2016), T.N Parashar, W.H. Matthaeus Ap.J, **832**, 57 (2016), F. Valentini *et al.* New J. Phys. **18**, 125001 (2016),

Moment equations for the pressure tensor

From Vasov equation, for a given species, in index notation:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(n u_i) = 0 \tag{11}$$

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = \frac{q}{mc}(cE_i + \varepsilon_{ilm} u_l B_m) - \frac{1}{mn}\frac{\partial \Pi_{ik}}{\partial x_k} \tag{12}$$

$$\frac{\partial \Pi_{ij}}{\partial t} + \frac{\partial Q_{kij}}{\partial x_k} + \frac{\partial}{\partial x_k}(u_k \Pi_{ij}) + \frac{\partial u_i}{\partial x_k}\Pi_{kj} + \frac{\partial u_j}{\partial x_k}\Pi_{ik} - \frac{q}{mc}\left(\varepsilon_{ilm}\Pi_{lj}B_m + \varepsilon_{jlm}B_m\Pi_{il}\right) = 0 \tag{13}$$

Index notation used with ε_{jlm} the Levi-Civita symbol (vector product).
If a closure condition for Q_{ijk} is given, the system of fluid equations above is closed once it is coupled to the equations for the e.m. fields.

Moment equations for the pressure tensor

$$\frac{\partial E_i}{\partial x_i} = \frac{1}{4\pi}(n^e q^e + n^i q^i), \qquad \frac{\partial B_i}{\partial x_i} = 0, \qquad \frac{\partial B_i}{\partial t} = -c\varepsilon_{ijk}\frac{\partial E_k}{\partial x_j} \tag{14}$$

$$\varepsilon_{ijk}\frac{\partial B_k}{\partial x_j} = \frac{4\pi}{c}J_i + \frac{1}{c}\frac{\partial E_i}{\partial t} \qquad J_i \equiv n^e q^e u_i^e + n^i q^i u_i^i \tag{15}$$

and satisfies the energy conservation equation

$$\frac{\partial}{\partial t}\left\{\sum_\alpha\left[\frac{m^\alpha n^\alpha}{2}(u^\alpha)^2 + \frac{\mathrm{tr}\{\mathbf{\Pi}^\alpha\}}{2}\right] + \frac{B^2}{8\pi} + \frac{E^2}{8\pi}\right\} = \tag{16}$$

$$= -\nabla\cdot\left\{\sum_\alpha\left[\mathbf{Q}^\alpha + \mathbf{u}^\alpha\cdot\mathbf{\Pi}^\alpha + \mathbf{u}^\alpha\left(\frac{\mathrm{tr}\{\mathbf{\Pi}^\alpha\}}{2} + \frac{n^\alpha m^\alpha (u^\alpha)^2}{2}\right)\right] + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right\}$$

with the heat flow vector $Q_i^\alpha \equiv Q_{ijk}^\alpha \delta_{jk}/2$.

Extended MHD equations with a pressure tensor $\mathbf{\Pi}$

Sum in the usual way the electron and the ion equations using the quasineutrality condition to obtain a single fluid theory (as in the standard double adiabatic MHD equations).

Close the system by setting to zero the heat flux tensor.
This is the weakest assumption although at least for the plasma dynamics perpendicular to the background magnetic field, which I will consider in the following part of the presentation, it can be shown that it is reasonable.

Tensor notation: $\mathbf{\Pi}$ has components Π_{ij},
$(\mathbf{\Pi} \times \mathbf{b})_{ij}$ means $\varepsilon_{ilk} \Pi_{lj} b_k$, while $(\mathbf{\Pi} \cdot \nabla \mathbf{u})_{ij}$ means $\Pi_{il} \partial_l u_j$, etc.
The symbol ^{T} means transpose. Clearly $\mathbf{\Pi}^T = \mathbf{\Pi}$ (symmetric tensor).

Extended MHD equations with a pressure tensor $\boldsymbol{\Pi}$

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{u}) = 0, \tag{17}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \Omega_c \frac{\boldsymbol{J}\times\boldsymbol{b}}{ne} - \frac{\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}}{mn}, \tag{18}$$

$$\frac{\partial \boldsymbol{\Pi}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{\Pi}) + \boldsymbol{\Pi}\cdot\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\Pi}\cdot\boldsymbol{\nabla}\boldsymbol{u})^T - \Omega_c(\boldsymbol{\Pi}\times\boldsymbol{b} + (\boldsymbol{\Pi}\times\boldsymbol{b})^T) = 0, \tag{19}$$

$$\boldsymbol{J} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left\{\left(\boldsymbol{u} - \frac{\boldsymbol{J}}{ne}\right)\times\boldsymbol{B}\right\}. \tag{20}$$

Here $\Omega_c \equiv e|\boldsymbol{B}|/(mc)$ is the ion cyclotron frequency, $\boldsymbol{b}$ the unit vector along the local magnetic field and $\boldsymbol{\nabla}\cdot(\boldsymbol{u}\boldsymbol{\Pi}) \equiv (\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{\Pi} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{\Pi}$.

Extended MHD equations with a pressure tensor $\boldsymbol{\Pi}$

The evolution of the pressure tensor described by Eq.(19) is determined by the contribution of the two linear operators

$$\mathscr{L}_{\boldsymbol{u}}(\boldsymbol{\Pi}) \equiv \boldsymbol{\nabla} \cdot (\boldsymbol{u}\boldsymbol{\Pi}) + \boldsymbol{\Pi} \cdot \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\Pi} \cdot \boldsymbol{\nabla}\boldsymbol{u})^T, \tag{21}$$

$$\mathscr{M}_{\boldsymbol{B}}(\boldsymbol{\Pi}) \equiv \Omega_c(\boldsymbol{\Pi} \times \boldsymbol{b} + (\boldsymbol{\Pi} \times \boldsymbol{b})^T), \tag{22}$$

their actions on $\boldsymbol{\Pi}$ involves the time scales $\tau_H \equiv |\boldsymbol{\nabla}\boldsymbol{u}|^{-1}$ and $\tau_B \equiv \Omega_c^{-1}$.
The CGL closure is obtained by setting $\mathscr{M}_{\boldsymbol{B}}(\boldsymbol{\Pi}) = 0$

Taking $\mathbf{b}$ in a fixed direction (say along z in a 2D configuration with a uniform external field) the operator $\mathscr{M}_{\boldsymbol{B}}$ corresponds to a rotation in the x-y plane. The operator $\mathscr{L}_{\boldsymbol{u}}$ consists of different contributions.
We take $u_z = 0$ and in the following we only consider the dynamics in the x-y plane i.e. the corrections to the CGL pressure tensor that is isotropic in this plane : *gyrotropic distribution*.

Extended MHD equations with a pressure tensor $\mathbf{\Pi}$

The velocity strain tensor $\nabla \boldsymbol{u}$ in this 2D configuration[19] consists of a compressional part

$$C_{\perp,ij} \equiv -\frac{1}{2}\frac{\partial u_k}{\partial x_k}\delta_{ij}, \qquad (k = x,y), \tag{23}$$

of the trace-less rate of shear that gives the (incompressible) distortion (rate of shear) of the velocity distribution

$$D_{\perp,ij} \equiv \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \quad (i,j = x,y), \tag{24}$$

and of the vorticity

$$W_{\perp,ij} \equiv \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right], \quad (i,j = x,y). \tag{25}$$

[19] An analogous splitting could be done in 3D. Note that compression in the perpendicular plane is different from volume compression

Generation of a non-gyrotropic pressure tensor

Using a matrix notation where $[\boldsymbol{A},\boldsymbol{B}] = \boldsymbol{AB} - \boldsymbol{BA}$ and $\{\boldsymbol{A},\boldsymbol{B}\} = \boldsymbol{AB} + \boldsymbol{BA}$ we obtain

$$\frac{d\boldsymbol{\Pi}_\perp}{dt} = [\mathbf{B}_\perp + \mathbf{W}_\perp, \boldsymbol{\Pi}_\perp] - \{\mathbf{D}_\perp, \boldsymbol{\Pi}_\perp\} + 4C_\perp \boldsymbol{\Pi}_\perp. \tag{26}$$

We split $\boldsymbol{\Pi}_\perp$ into a gyrotropic and an agyrotropic (non-gyrotropic) part

$$\boldsymbol{\Pi}_\perp = \mathrm{tr}(\boldsymbol{\Pi}_\perp)\mathbf{I}_\perp/2 + \boldsymbol{\Pi}_\perp^{ng}, \qquad \text{and obtain}$$

$$\frac{1}{2}\frac{d}{dt}\mathrm{tr}(\boldsymbol{\Pi}_\perp) = -\mathrm{tr}(\mathbf{D}_\perp \boldsymbol{\Pi}_\perp) + 2C_\perp \mathrm{tr}((\boldsymbol{\Pi}_\perp), \tag{27}$$

$$\frac{d\boldsymbol{\Pi}_\perp^{ng}}{dt} = [\mathbf{B}_\perp + \mathbf{W}_\perp, \boldsymbol{\Pi}_\perp^{ng}] - \{\mathbf{D}_\perp, \boldsymbol{\Pi}_\perp^{ng}\} + 4C_\perp \boldsymbol{\Pi}_\perp^{ng}, \tag{28}$$

$$+\mathbf{I}_\perp \mathrm{tr}(\mathbf{D}_\perp \boldsymbol{\Pi}_\perp^{ng}) - \mathbf{D}_\perp \mathrm{tr}(\boldsymbol{\Pi}_\perp).$$

Non-gyrotropic pressure tensor in a shear flow

Eq.(28) shows (the red term) that a non-null rate of shear $\mathbf{D}_\perp$ can generate agyrotropy on a time scale $\tau_{an} \sim ||\mathbf{D}_\perp||^{-1}$ from an initial isotropic state ($\mathbf{\Pi}_\perp^{ng} = 0$) while the action of vorticity simply adds up to that of the magnetic field.

In this 2D configuration, the evolution of the parallel component $P_{||} = \Pi_{ij} b_i b_j$ is given by $dP_{||}/dt = 4C_\perp P_{||}$.

The correlation between a non-gyrotropic distribution and vorticity is only indirect as it follows from the correlation between fluid vorticity and rate of shear $\mathbf{D}$

as shown e.g., in a sheared flow of the form $u_x(y)$

$$\nabla \boldsymbol{u} = \frac{\partial u_x}{\partial y} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right] + \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] = \mathbf{W}_\perp + \mathbf{D}_\perp. \tag{29}$$

Anisotropy-driven instabilities.

Anisotropy (both gyrotropic and non-gyrotropic) modifies existing instabilities[20] and leads to new ones.

These instabilities influence the evolution of the anisotropy generating mechanism, for example the shear flow discussed in the second lecture.

An example is provided by the Weibel instability[21] or by the fire-hose and mirror instabilities (not discussed here) whose thresholds have been supposed to fix the boundaries in the parameter space of the ion gyrotropic anisotropy measured in the solar wind[22].

In a different context a Weibel-type instability on the electron scales, called the current filamentation instability, has been studied in the context of ultraintense laser plasma interactions[23].

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20 See e.g. reconnecting instabilities for which the change of magnetic topology is allowed or enhanced by pressure anisotropy

21 E. W. Weibel, Phys. Rev. Lett., **2**, 83 (1959)

22 See among others P. Hellinger *et al.*, Geophys. Res. Lett., **33**, L09101 (2006)

23 See among others F. Pegoraro, *et al*, Phys. Scr., **T63**, 262 (1996) and more recent articles including the extension to relativistic plasmas

Magnetic field generation and the Weibel instability

Magnetic fields represent a fundamental feature of laboratory and space plasmas. At low frequencies and long spatial scales, magnetic fields emerge as the dominant factor in the dynamics of a plasma as a consequence of the effective cancellation of the electric forces due to plasma quasi-neutrality. Conversely, at high frequencies and shorter spatial scales, magnetic fields play an increasingly important role when the particle velocities approach the speed of light.

The Weibel instability (and its beam-plasma counterpart: the current filamentation instability) is an electromagnetic instability that generates a magnetic field in the presence of particle phase-space anisotropies.

The Weibel instabilitiy is of primary importance in the astrophysical context: e.g.for the formation (or the seeding at small spatial scales) of cosmological magnetic fields or the development of collisionless shocks.

Magnetic field generation and the Weibel instability

A direct link between electron pressure anisotropy and the generation of magnetic field is shown by inserting the moment equation for the electron momentum[24]

$$m_e n \left[\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla)\mathbf{u}_e\right] = -\nabla \cdot \mathbf{\Pi}_e - ne\left[\mathbf{E} + \frac{\mathbf{u}_e}{c} \times \mathbf{B}\right] \qquad (30)$$

into Faraday's law $\nabla \times \mathbf{E} = -(1/c)\partial \mathbf{B}/\partial t$ and by neglecting for simplicity the inertial terms. If $\mathbf{\Pi}_e = p_e \boldsymbol{I}$, and if p_e satisfies a barotropic closure (so that $\nabla \times [(1/n)(\nabla p)] = 0$) magnetic flux conservation applies (in Eq.(30) in the electron or in the single fluid plasma in MHD.

If the pressure tensor $\mathbf{\Pi}_e$ *is anisotropic* $\nabla \times [(1/n)(\nabla \cdot \mathbf{\Pi}_e)] \neq 0$ *and magnetic flux conservation is violated.* It can be destroyed (magnetic reconnection) or it can be generated (Weibel instability).

[24] as obtained from Eqs.(19,18)

The way collective excitations work: Weibel instability and magnetic field generation

Collisions would eventually make the particle distribution function isotropic if it were not for faster mechanisms that reinforce anisotropy[25].

Collective excitations take part of the role of the collisions, but in general they do not lead to thermalization: the process is much more complex.

• In an anisotropic plasma, because of the magnetic part of the Lorentz force, a transverse e.m. mode can propagate with a phase velocity smaller[26] than c and can thus interact with the plasma particles.

• Then the anisotropic degrees of freedom in the particle distribution can be thought of as thermal baths at different "temperatures" with the instability putting the two baths in contact and extracting work (the magnetic field energy) in the process as in a "thermal"" machine.

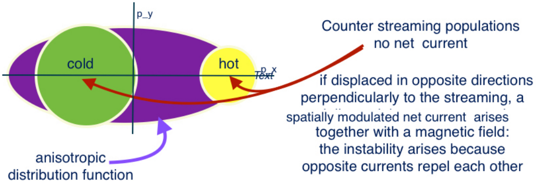
[25] Some of them, such as synchrotron radiation in a magnetized plasma or the effect of the velocity shear, were mentioned and discussed in the second lecture.

[26] In an isotropic plasma transverse modes $\omega^2 = k^2c^2 + \omega_{pe}^2$ have phase velocity larger than the light speed c

Weibel Instability

An approximate but intuitive explanation of the magnetic field generation can be derived from a virtual displacement argument borrowed from the theory of the closely related current filamentation instability

Assume $T_y \gg T_x, T_z$ and split the electron distribution function into two parts corresponding to positive and to negative values of v_y, Displace them in opposite directions along x. The opposite current densities that are formed are modulated along x and produce a magnetic field along z.
Opposite currents repel, the initial displacement is reinforced, the instability can develop and the magnetic field along z can grow.



Kinetic Development of the Weibel Instability

Consider high frequency modes evolving on electron time scales in a collisionless plasma and use the Vlasov-Maxwell system of equations for the electron distribution function f_e taking immobile ions. Velocities are normalized to the speed of light and times to ω_{pe}^{-1}

$$\frac{\partial f_e}{\partial t} + \mathbf{v}.\frac{\partial f_e}{\partial \mathbf{x}} + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_e}{\partial \mathbf{v}} = 0.$$

Simplified geometry: 1D-2V configuration: all quantities depend on x and time only, the particle velocities and the electric field have x-y components and the magnetic field is along z.

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x}, \qquad -\frac{\partial B_z}{\partial x} = \frac{\partial E_y}{\partial t} + J_y, \qquad \frac{\partial^2 \phi}{\partial^2 x} = -\rho,$$

ρ, J are the charge and current densities, ϕ is the electrostatic potential. Initial distribution function: $f_M = n/(\pi\sqrt{T_x T_y})\exp\left(-\, v_x^2/T_x - v_y^2/T_y\right)$.

Kinetic Development of the Weibel Instability

From the original Weibel's article

$$f_0 = \frac{n}{u_0^2 u_3 (2\pi)^{3/2}} \exp\left[-\frac{v_0^2}{2u_0^2} - \frac{v_3^2}{2u_3^2}\right], \qquad (5)$$

$$k^2 - \omega^2 = \omega_p^2 \left\{ A - \left(A \frac{\omega \pm \omega_c}{u_3 k} + \frac{\omega}{u_3 k} \right) \phi\left(\frac{\omega \pm \omega_c}{u_3 k} \right) \right\}, \qquad (6)$$

where

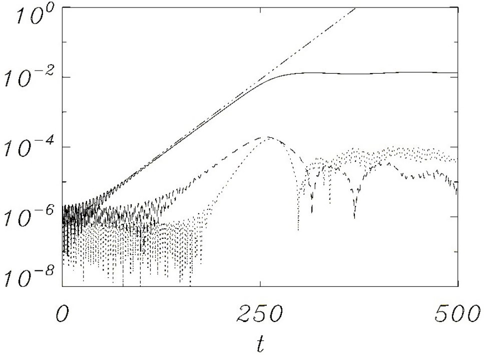
$$\omega_p^2 = ne^2/m, \quad \omega_c = eB_0/m, \quad A = (u_0/u_3)^2 - 1, \qquad (7)$$

and

$$\phi(z) = \exp(-\tfrac{1}{2}z^2) \int_{-i\infty}^{z} \exp(\tfrac{1}{2}\xi^2)\, d\xi. \qquad (8)$$

Onset, linear growth, beginning of its saturation phase

Time evolution of the Fourier components of the most unstable $k = k_{max} = 1$ magnetic and inductive electric field mode, $B_{z,k=1}$ and $E_{y,k=1}$, solid and dashed line, and of the most unstable $k = 2.3$ longitudinal electric field $E_{x,k=2.3}$ (dotted line). The dash-three dotted line represents the linear growth rate of the magnetic field. (L. Palodhi *et al.*.PPCF, **51**, 125006, (2009)).



Coupling to Langmuir waves

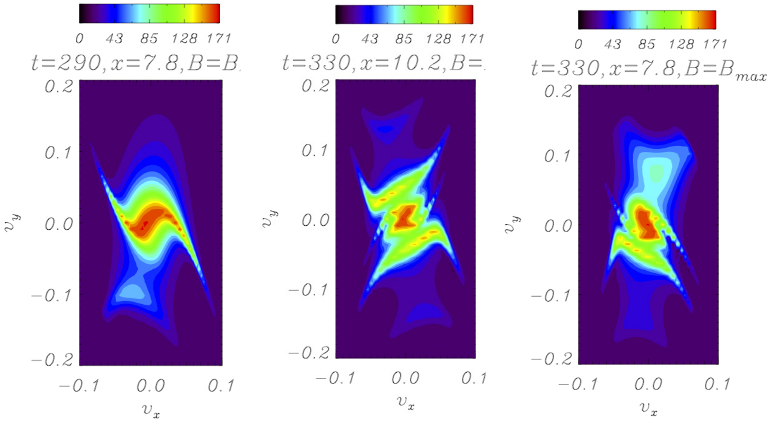
The longitudinal electric field $E_{x,k}$ arises from the coupling between the Weibel instability and the Langmuir waves due to the electron density modulation induced by the spatial modulation of $B_{z,k}^2$, and thus grows in time at twice the growth rate of the magnetic field.

During the linear phase the evolution of the electron distribution function in velocity space is characterized by a differential rotation in velocity space at the points where $|B_z|$ generated by the Weibel instability has a maximum and by a Y- shaped deformation with axis along v_y where t $|B_z|$ vanishes and the inductive electric field $|E_y|$ is largest.

Deformation of the e lectron distribution function

Initial deformation of the electron distribution function

how can we define a pressure tensor ?



Resonant Langmuir waves

As the Weibel instability enters its fully nonlinear phase the winding of the distribution function becomes tighter until it becomes "multi-armed". Positive slopes in v_x in are formed. Although these slopes evolve in time, they can give rise to the resonant excitation of Langmuir waves with phase velocities much smaller than those of the Langmuir waves driven by the nonlinear coupling.

This new destabilizing process leads to a highly structured electron density distribution along x, while B_z remains spatially regular even at late times.

“Different” Plasmas

PHYSICAL REVIEW D 84, 056003 (2011)

Non-Abelian plasma instabilities: SU(3) versus SU(2)

Andreas Ipp and Anton Rebhan

Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria

Michael Strickland

Department of Physics, Gettysburg College, Gettysburg, Pennsylvania 17325, USA

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We present the first 3 + 1 dimensional simulations of non-Abelian plasma instabilities in gauge-covariant Boltzmann-Vlasov equations for the QCD gauge group SU(3) as well as for SU(4) and SU(5). The real-time evolution of instabilities for a plasma with stationary momentum-space anisotropy is studied using a hard-loop effective theory that is discretized in the velocities of hard particles. We find that the numerically less expensive calculations using the group SU(2) essentially reproduce the nonperturbative dynamics of non-Abelian plasma instabilities with higher rank gauge groups provided the mass parameters of the corresponding hard-loop effective theories are the same. In particular, we find very similar spectra for the turbulent cascade that forms in the strong-field regime, which is associated with an approximately linear growth of energy in collective fields. The magnitude of the linear growth however turns out to increase with the number of colors.

Chromo-Weibel instability

ELSEVIER

Physics Reports 359 (2002) 355–528

The quark–gluon plasma: collective dynamics and hard thermal loops

Jean-Paul Blaizot[*,1], Edmond Iancu[1]

Service de Physique Théorique[2], CE-Saclay 91191 Gif-sur-Yvette, France

Conclusions: Why do we call it pressure?

When needed, use concepts such as pressure, reduced moment equations etc. as powerful investigation tools
but keep in mind they are convenient tools, not exact treatments, and do not take their predictions too far.

You can resort to high dimensionality fully kinetic computer simulations. These are getting more and more powerful all the time and already allow us to explore regimes that would be almost impossible to treat with other means.
But keep in mind that, finally, you must *understand* and understanding requires language, and language in its turn requires models that need be able to collect the numerical results and put them in a single logical frame, and models again are based on simplified tools such as pressure.

CGL derivation

$$\frac{\partial}{\partial t}\Pi_{ij} + \underbrace{\mathscr{L}_{\mathbf{u}}(\Pi_{ij})}_{|\nabla\mathbf{u}|\equiv\tau_H^{-1}} + \underbrace{\mathscr{D}_{\mathbf{Q}}(\Pi_{ij})}_{\tau_Q^{-1}} = \underbrace{\mathscr{M}_{\mathbf{B}}(\Pi_{ij})}_{\Omega_{c\alpha}\equiv\tau_B^{-1}} \tag{31}$$

$$\mathscr{L}_{\mathbf{u}}(\Pi_{ij}) \equiv \frac{\partial}{\partial x_k}(u_k\Pi_{ij}) + \Pi_{kj}\frac{\partial u_i}{\partial x_k} + \Pi_{ik}\frac{\partial u_j}{\partial x_k}$$

$$\mathscr{D}_{\mathbf{Q}}(\Pi_{ij}) \equiv \frac{\partial}{\partial x_k}Q_{ijk}, \qquad \mathscr{M}_{\mathbf{B}}(\Pi_{ij}) \equiv \frac{q}{m}\left(\varepsilon_{ilm}\Pi_{lj}B_m + \varepsilon_{jlm}\Pi_{il}B_m\right)$$

The standard CGL closure is obtained in the limit of a sufficiently strong magnetic field and/or sufficiently weak velocity strain. In this closure the diagonal block shape of the pressure tensor

$$\Pi^0_{ij} = p_\perp\delta_{ij} + (p_{||} - p_\perp)b_ib_j \tag{32}$$

($b_i \equiv B_i/|\mathbf{B}|$ being the local direction of the magnetic field) is obtained by solving the tensor equation $\mathscr{M}_{\mathbf{B}}(\Pi^0_{ij}) = 0$, which corresponds to the zeroth-order equilibrium solution of Eq.(31).

CGL derivation

The CGL double adiabatic equations are then written for $p_{||}^{\alpha}$ and $p_{\perp}^{\alpha}$, by solving to next order[27]

$$\frac{\partial}{\partial t}\Pi_{ij}^{0} + \mathscr{L}_{\mathbf{u}}(\Pi_{ij}^{0}) = 0 \tag{33}$$

contracting it by δ_{ij} and by $b_i b_j$ and using[28] $|\mathbf{b}| = 1$.

$$\frac{d}{dt}(\Pi_{ij}\delta_{ij}) + (\Pi_{ij}\delta_{ij})\frac{\partial u_k}{\partial x_k} + 2\Pi_{ik}\frac{\partial u_i}{\partial x_k} = 0$$

$$\frac{\partial}{\partial t}(\Pi_{ij}b_i b_j) + \frac{\partial}{\partial x_k}\left(u_k \Pi_{ij} b_i b_j\right) - \Pi_{ij}\frac{\partial}{\partial t}\left(b_i b_j\right) + b_i b_j\left(\Pi_{ik}\frac{\partial u_j}{\partial x_k} + \Pi_{kj}\frac{\partial u_i}{\partial x_k}\right) = 0$$

27 After projecting out the term $\mathscr{M}_{\mathbf{B}}(\Pi_{ij}^{1}) = 0$

28 For a nondiagonal form of the pressure tensor we must use an explicit equation for $d\mathbf{b}/dt$.

Current filamentation instability in the context of laser plasma interaction

The transport of the fast electrons as a collimated beam is only possible by means of a "return" current able to maintain global charge neutrality as well as to compensate locally for the fast electron current.

Similarity with Weibel instability: counter propagating electron beams can be assimilated to an anisotropic distribution function (no magnetic field is initially present). $\mathbf{\Pi}_e$ arises from the relative motion of the two cold populations. Counter propagating equal current beams currents lead to the development of fast transverse electromagnetic (current-filamentation) instabilities + longitudinal electrostatic (two-stream) instabilities (not present for more standard anisotropic distributions).

Heuristically, the current-filamentation instability is driven by the magnetic repulsion of the transversally displaced opposite currents.

It is the leading instability in relativistic conditions and generates strong "quasi-static magnetic fields (with spatial scale of the order of some electron skin-depths).

Filamentation instability: Linear dispersion relation

In the case of two counter propagating electron populations, the transverse electromagnetic current filamentation instability is coupled to the two stream electrostatic instability that develops along the beams' direction.
Assuming the ions to be at rest and to provide a uniform neutralizing background, the linear dispersion relation can be obtained by linearizing the relativistic equations for the two counter-streaming cold electron populations together with Maxwell's equations (in normalized units):

$$\frac{\partial n_\alpha}{\partial t} = \nabla \cdot \mathbf{j}_\alpha, \qquad \frac{\partial \mathbf{p}_\alpha}{\partial t} = -\mathbf{u}_\alpha \cdot \nabla \mathbf{p}_\alpha - (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}), \tag{34}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \sum_\alpha \mathbf{j}_\alpha, \tag{35}$$

with $\mathbf{u}_\alpha = \mathbf{p}_\alpha/(1+p_\alpha^2)^{1/2}$, and $\mathbf{j}_\alpha = -n_\alpha \mathbf{u}_\alpha, \quad \alpha = 1,2.$
Consider: a homogeneous plasma with velocities along the x direction $u_{0,\alpha}$, such that the net current density is zero $\sum_\alpha n_{0,\alpha} u_{0,\alpha} = 0$, and a perturbation with frequency ω and wavevector $\mathbf{k} = (k_x, k_y)$, such that the perturbed magnetic field, arising from the separation along y of the oppositely directed currents along x, is in the z direction.

Filamentation instability: Linear dispersion relation

With $\Omega_\alpha = \omega - k_x u_{0,\alpha}$ and $\Gamma_\alpha = (1-u_{0,\alpha}^2)^{-1/2}$, the linear dispersion relation is

$$(1-\Omega_2^{-2})\left[k_x^2(1+\Omega_4^{-2}) - \omega^2(1-\Omega_1^{-2}) - 2\omega k_x \Omega_3^{-2}\right] \tag{36}$$

$$+k_y^2\left[(1-\Omega_1^{-2})(1+\Omega_4^{-2}) + \Omega_3^{-4}\right] = 0, \qquad \text{with}$$

$$\Omega_1^{-2} = \sum_\alpha \frac{n_{0,\alpha}}{\Gamma_\alpha \Omega_\alpha^2},\ \Omega_2^{-2} = \sum_\alpha \frac{n_{0,\alpha}}{\Gamma_\alpha^3 \Omega_\alpha^2},\ \Omega_3^{-2} = \sum_\alpha \frac{n_{0,\alpha} u_{0,\alpha}}{\Gamma_\alpha \Omega_\alpha^2},\ \Omega_4^{-2} = \sum_\alpha \frac{n_{0,\alpha} u_{0,\alpha}^2}{\Gamma_\alpha \Omega_\alpha^2}.$$

For $k_y = 0$, no magnetic field is produced and the electrostatic two-stream instability amplifies the electric field E_x with a growth rate obtained by solving the equation $1-\Omega_2^{-2} = 0$. For $k_x = 0$, the dispersion relation reduces to

$$\omega^2(1-\Omega_2^{-2})(1-\Omega_1^{-2}) - k_y^2\left[(1-\Omega_1^{-2})(1+\Omega_4^{-2}) + \Omega_3^{-4}\right] = 0, \tag{37}$$

which contains two oscillatory solutions and one purely growing electromagnetic instability (the current filamentation instability) which amplifies the magnetic field B_z with a growth rate that is linear on k_y for $k_y d_e < 1$ (in dimensional units) and becomes approximately constant and of order ω_{pe} for $k_y d_e > 1$ when the velocity on the two counterstreaming beams is close to the velocity of light.