Vorticities in relativistic plasmas: from waves to reconnection

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- ***Part I: Waves in relativistic plasmas***
- ***Part II: Electro–Vortical formulation***
- ***Part III: Generalized Connetion and Reconnection***

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Part I:
VORTICITY AND WAVES IN RELATIVISTIC PLASMAS

- ***Vortical model for relativistic plasmas***
- ***Circular polarized waves***

Relativistic Plasma equations

- the rest-frame density of the fluid n.
- the energy density ϵ, pressure p, enthalpy density $h = \epsilon + p$, and temperature T.
- relativistic velocities and the Lorentz factor $\gamma = (1 - \mathbf{v}^2)^{-1/2}$.
- coupled to Maxwell equations via the current density $n\gamma\mathbf{v}$.

Plasma fluid equation for specie j

$$m_j\gamma_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla\right)(f_j\gamma_j\mathbf{v}_j) = q_j\gamma_j\left(\mathbf{E} + \mathbf{v}_j \times \mathbf{B}\right) - \frac{1}{n_j}\nabla p_j$$

Continuity equation

$$\frac{\partial(\gamma_j n_j)}{\partial t} + \nabla \cdot (\gamma_j n_j \mathbf{v}_j) = 0$$

$$f \equiv \frac{h}{mn} = f(T)$$

And an equation of state for pressure and density.

We re-write the fluid equation as...

Let us assume constant rest-frame density n and constant temperature

$$m_j f_j \frac{\partial(\gamma_j \mathbf{v})}{\partial t} - m_j f_j \mathbf{v}_j \times \nabla \times (\gamma_j \mathbf{v}_j) = q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \frac{1}{2} \nabla (\mathbf{v}_j \cdot \mathbf{v}_j)$$

where we have used $\mathbf{a} \times (\nabla \times \mathbf{b}) = (\nabla \mathbf{b}) \cdot \mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$

Now, we notice

$$m_j f_j \frac{\partial(\gamma_j \mathbf{v}_j)}{\partial t} = q_j \left[\mathbf{E} + \mathbf{v}_j \times \left(\mathbf{B} + \frac{m_j f_j}{q_j} \nabla \times (\gamma_j \mathbf{v}_j) \right) \right] - \frac{1}{2} \nabla (\mathbf{v}_j \cdot \mathbf{v}_j)$$

it appears the interesting field

$$\Omega_j = \mathbf{B} + \frac{m_j f_j}{q_j} \nabla \times (\gamma_j \mathbf{v}_j) = \nabla \times \mathbf{P}_j$$

that will be a generalized vorticity with the potential [the canonical momentum]

$$\mathbf{P}_j = \mathbf{A} + \frac{m_j f_j}{q_j} \gamma_j \mathbf{v}_j$$

Generalized vorticity equation

Taking the curl of the previous equation

$$\frac{m_j f_j}{q_j} \frac{\partial \nabla \times (\gamma_j \mathbf{v}_j)}{\partial t} = \nabla \times \mathbf{E} + \nabla \times (\mathbf{v}_j \times \Omega_J)$$

and remembering that $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ we obtain

$$\frac{\partial \Omega_j}{\partial t} = \nabla \times (\mathbf{v}_j \times \Omega_j)$$

The plasma dynamics becomes simplified in terms of the Generalized vorticity!

$$\Omega_j = \mathbf{B} + \frac{m_j f_j}{q_j} \nabla \times (\gamma_j \mathbf{v}_j)$$

Maxwell equations

$\mathbf{E}, \mathbf{B}$ electric and magnetic fields

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\
\frac{\partial \mathbf{E}}{\partial t} + \sum_i q_i n_i \gamma_i \mathbf{v}_i &= \nabla \times \mathbf{B} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{E} &= \sum_i q_i n_i \gamma_i
\end{aligned}$$

From Maxwell equations we obtain...

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\
\frac{\partial \mathbf{E}}{\partial t} + \sum_i q_i n_i \gamma_i \mathbf{v}_i &= \nabla \times \mathbf{B} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{E} &= \sum_i q_i n_i \gamma_i
\end{aligned}
$$

$$\nabla \times (\nabla \times \mathbf{B}) + \frac{\partial^2 \mathbf{B}}{\partial t^2} = \sum_i q_i \nabla \times (n_i \gamma_i \mathbf{v}_i)$$

Vorticity and helicity

The vorticity field is any psedovector that is the rotational (curl) of a vector field (potential).

The vorticity field has associated a quantity called helicity

For example, the magnetic helicity is

$$h = \int \mathbf{A} \cdot \mathbf{B} \, d^3x$$

such that

$$\begin{aligned} \frac{\partial h}{\partial t} &= \int \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} \, d^3x + \int \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} \, d^3x \\ &= \int (-\mathbf{E} - \nabla\phi) \cdot \mathbf{B} \, d^3x - \int \mathbf{A} \cdot \nabla \times \mathbf{E} \, d^3x \\ &\equiv -2 \int \mathbf{E} \cdot \mathbf{B} \, d^3x - \int (\phi\mathbf{B} + \mathbf{E} \times \mathbf{A}) \cdot \, d^2\mathbf{x} \\ &\equiv -2 \int \mathbf{E} \cdot \mathbf{B} \, d^3x \end{aligned}$$

is not always conserved!

Plasma fluid generalized helicity

The helicity associated to the relativistic plasma fluid (for constant density and pressure) is

$$h = \int \mathbf{P} \cdot \Omega \; d^3x$$

which satisfies

$$\begin{aligned} \frac{\partial h}{\partial t} &= \int \frac{\partial \mathbf{P}}{\partial t} \cdot \Omega \; d^3x + \int \mathbf{P} \cdot \frac{\partial \Omega}{\partial t} \; d^3x \\ &= \int (\mathbf{v} \times \Omega) \cdot \Omega \; d^3x + \int \mathbf{P} \cdot [\nabla \times (\mathbf{v} \times \Omega)] \; d^3x \\ &\equiv 0 \end{aligned}$$

the Generalized Helicity is conserved[2]

[2] Mahajan & Yoshida, Phys. Plasmas **18**, 055701 (2011).

If pressure is not constant...

$$\frac{\partial \Omega}{\partial t} = \nabla \times (\mathbf{v} \times \Omega) + \frac{1}{n^2} \nabla n \times \nabla p$$

the last term is so-called Biermann battery. It can generate vorticity from plasma thermodynamical inhomogenities.

- The conservation of helicity establishes topological constraints. It can forbid the creation (destruction) of vorticity in plasmas.
- We can see that the generalized helicity remains unchanged in ideal dynamics. This conservation implies serious contraints on the origin and dynamics of magnetic fields.
- Otherwise, the nonideal effects can change the helicity. For example, if gradients of pressure and temperature have different directions [Biermann battery].
- An anisotropic pressure tensor may also generate vorticity.

Dimensionless system. Positive ($q = e$) and negative ($q = -e$), two-fluids plasma

Magnetic fields are normalized to background magnetic field B_0 (measured in rest frame), time to Ω_0, distance to Ω_0^{-1}, with the generalized cyclotron

$$\Omega_0 = \frac{eB_0}{(m_+ f_+ + m_- f_-)c}$$

The equations are now

$$\frac{\partial \Omega_\pm}{\partial t} = \nabla \times (\mathbf{v}_\pm \times \Omega_\pm)$$

$$\Omega_\pm = \mathbf{B} \pm \mu_\pm \nabla \times (\gamma_\pm \mathbf{v}_\pm) \, ; \qquad \mu_\pm = \frac{m_\pm f_\pm}{m_+ f_+ + m_- f_-}$$

$$\nabla \times (\nabla \times \mathbf{B}) + \frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{1}{U_{A0}^2} \nabla \times (\gamma_+ \mathbf{v}_+ - \gamma_- \mathbf{v}_-)$$

with the normalzied Alfven speed $U_{A0} = B_0 / \sqrt{4\pi n_0 (m_+ f_+ + m_- f_-)}$

Exact propagation circularly polarized waves

No background flow, background magnetic field in $\hat{z}$. Transverse waves propagating in $\hat{z}$ direction, with constant frequency and constant wavevector. Hence, $\mathbf{v} \cdot \hat{z} = 0$ and $\mathbf{B} \cdot \hat{z} = 0$

$$\mathbf{v}_{\pm} = \frac{v_{\pm}}{2}\left[(\hat{x} + i\hat{y})e^{ikz - i\omega t} + c.c.\right] \; ; \qquad \mathbf{B} = \frac{B}{2}\left[(\hat{x} + i\hat{y})e^{ikz - i\omega t} + c.c.\right]$$

where $v_{\pm}$ and B are constant amplitudes. Notice that

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}_{\pm} \cdot \mathbf{v}_{\pm}}} = \frac{1}{\sqrt{1 - v_{\pm}^2}}$$

is now constant.

The system is reduced to

$$\omega B + \omega k\, \mu_{+}\gamma_{+}v_{+} = kv_{+}$$

$$\omega B - \omega k\, \mu_{-}\gamma_{-}v_{-} = kv_{-}$$

$$(k^2 - \omega^2)B = \frac{k}{U_{A0}^2}(\gamma_{+}v_{+} - \gamma_{-}v_{-})$$

Dispersion relation for pair plasmas[3]

Consider $m_+ = m_- = m$ and $f_+ = f_- = f$. Then $\mu_+ = \mu_- = 1/2$.

$$\omega_\pm^2 = \frac{k^2}{2} + \frac{2}{U_{A0}^2} + \frac{2}{\gamma_+^2\gamma_-^2} \pm 2\left(\left[\frac{k^2}{4} + \frac{1}{U_{A0}^2} + \frac{1}{\gamma_+^2\gamma_-^2}\right]^2 - \frac{k^2}{\gamma_+^2\gamma_-^2}\right)^{1/2}$$

High–frequency modes in physical units

$$\omega_+^2 \approx c^2k^2 + \frac{\omega_p^2}{f} + \frac{\Omega_c^2}{f^2\gamma_+^2\gamma_-^2}$$

Low–frequency modes in physical units

$$\omega_-^2 \approx \frac{V_A^2k^2}{f\gamma_+^2\gamma_-^2}\left(1 + \frac{c^2k^2}{\omega_p^2} + \frac{V_A^2}{c^2f\gamma_+^2\gamma_-^2}\right)^{-1}$$

with $\omega_p = \sqrt{8\pi n_0 e^2/m}$, $\Omega_c = eB_0/(mc)$, and $V_A = B_0/\sqrt{8\pi n_0 m}$.

[3] Mahajan & Lingam, Phys. Plasmas **25**, 072112 (2018).

Amplitude–dependent dispersion relation

High–frequency wave cut–off

$$c^2k^2 = \omega^2 - \omega^2_{cut-off}$$

$$\omega^2_{cut-off} = \frac{\omega_p^2}{f}\left(1 + \frac{\Omega_c^2}{f\omega_p^2\gamma_+^2\gamma_-^2}\right)$$

if $\gamma_+\gamma_- \gg 1 \Longrightarrow c^2k^2 = \omega^2 - \frac{\omega_p^2}{f}$ approaches to a light wave in a plasma!

For high–amplitude, the plasma wave behaves as if the plasma were unmagnetized. Simiarly, the Alfven mode frequency decreases

Estimations

For a pair plasma with $n_0 \approx 10^8\text{cm}^{-3}$, and then $\omega_p \approx 3 \times 10^8\text{s}^{-1}$, in a magnetosphere in a pulsar with magnetic field $B_0 \approx 10^{10}\text{G}$, then $\Omega_c \approx 2 \times 10^{17}\text{s}^{-1}$. For high temperatures, $f \approx 4k_B T/(mc^2)$. For $T \sim 10^{11}K$, then $f \approx 100$.
The cut–off

$$\omega_{cut-off} \approx \frac{\omega_p}{\sqrt{f}}\sqrt{1 + \frac{10^{16}}{\gamma_+^2\gamma_-^2}}$$

Then if $\gamma_+ \sim \gamma_- \sim 10^5$, the wave behaves as a light wave with $\omega_{cut-off} \approx \omega_p/\sqrt{f}$.

That’s all (for now).
Thanks!