

# Fluidistic description of astrophysical and space plasmas

## - Part 3 -

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# Small scales: two-fluid equations

→ The dimensionless version, for a length scale  $L_0$ , density  $n_0$  and Alfvén speed  $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\begin{aligned} \frac{d\vec{U}_i}{dt} &= \frac{1}{\varepsilon} (\vec{E} + \vec{U}_i \times \vec{B}) - \frac{\beta}{n} \vec{\nabla} p_i - \frac{\eta}{\varepsilon n} \vec{J} \\ \frac{m_e}{m_i} \frac{d\vec{U}_e}{dt} &= -\frac{1}{\varepsilon} (\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n} \vec{\nabla} p_e + \frac{\eta}{\varepsilon n} \vec{J} \end{aligned} \quad \text{where} \quad \vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon} (\vec{U}_i - \vec{U}_e)$$

→ We define the Hall parameter  $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma *beta*  $\beta = \frac{p_0}{m_i n_0 v_A^2}$  and the electric resistivity  $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

→ Adding these two equations yields:

$$\frac{d\vec{U}}{dt} = (\vec{\nabla} \times \vec{B}) \times (\vec{B} + \varepsilon_e^2 \vec{\nabla} \times \vec{J}) - \vec{\nabla} p$$

where  $\vec{U} = \frac{m_i \vec{U}_i + m_e \vec{U}_e}{m_i + m_e}$

and  $p = p_i + p_e$

$$\varepsilon_e = \sqrt{\frac{m_e}{m_i}} \varepsilon = \frac{c}{\omega_{pe} L_0}$$

# Retaining electron inertia

➡ In the equation for electrons (assuming incompressibility)

$$\frac{m_e}{m_i} \frac{d\vec{U}_e}{dt} = -\frac{1}{\epsilon} (\vec{E} + \vec{U}_e \times \vec{B}) - \beta_e \vec{\nabla} p_e + \frac{\eta}{\epsilon} \vec{J} \quad \vec{J} = \vec{\nabla} \times \vec{B} = \frac{1}{\epsilon} (\vec{U}_i - \vec{U}_e)$$

we replace  $\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi$  and  $\vec{B} = \vec{\nabla} \times \vec{A}$

to obtain the following generalized induction equation ([Andrés et al. 2014ab](#), PoP)

$$\frac{\partial}{\partial t} \vec{B}' = \vec{\nabla} \times \left[ (\vec{U} - \epsilon \vec{J}) \times \vec{B}' \right] + \eta \nabla^2 \vec{B} \quad , \quad \vec{B}' = \vec{B} - \epsilon_e^2 \nabla^2 \vec{B} - \frac{\epsilon_e^2}{\epsilon} \vec{\omega}$$

➡ See also [Abdelhamid, Lingam & Mahajan 2016](#) for extended MHD.

➡ Electron inertia is quantified by the dimensionless parameter  $\epsilon_e = \sqrt{\frac{m_e}{m_i}} \epsilon = \frac{c}{\omega_{pe} L_0}$

➡ Just as the Hall effect introduces the new spatial scale  $k_H = \frac{1}{\epsilon}$  (the ion skin depth), electron inertia introduces the electron skin depth  $k_e = \frac{1}{\epsilon_e}$  which satisfies  $k_e = \sqrt{\frac{m_i}{m_e}} k_H \gg k_H$

# Normal modes in EIH MHD

- ➔ If we linearize our equations around an equilibrium characterized by a uniform magnetic field, we obtain the following dispersion relation:

$$\left( \frac{\omega}{\vec{k} \cdot \vec{B}_0} \right)^2 \pm \frac{k\epsilon}{1 + \epsilon_e^2 k^2} \left( \frac{\omega}{\vec{k} \cdot \vec{B}_0} \right) - \frac{1}{1 + \epsilon_e^2 k^2} = 0$$

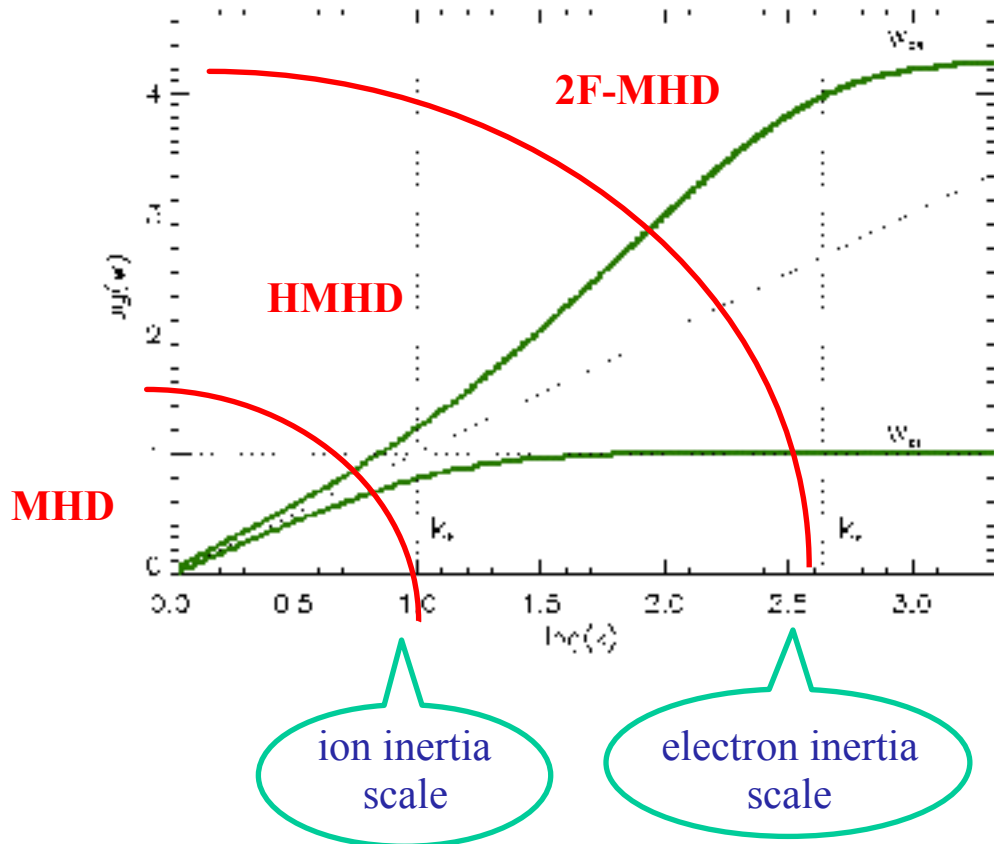
- ➔ Asymptotically, at very large  $k$ , we have two branches

$$\omega \xrightarrow{k \rightarrow \infty} \omega_{ce} \cos \theta$$

$$\omega \xrightarrow{k \rightarrow \infty} \omega_{ci} \cos \theta$$

while for very small  $k$ , both branches simply become Alfvén modes, i.e.

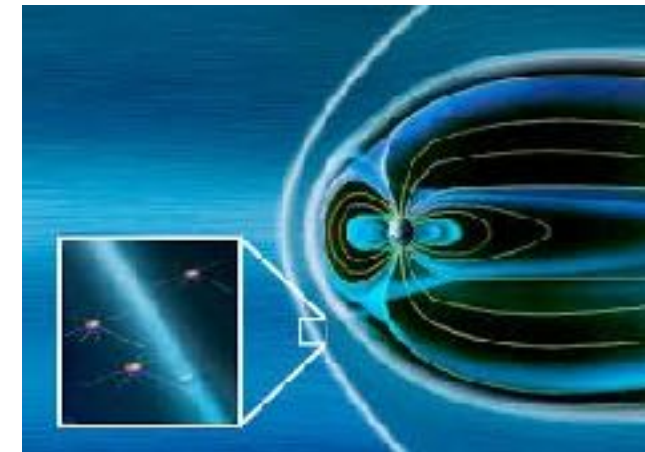
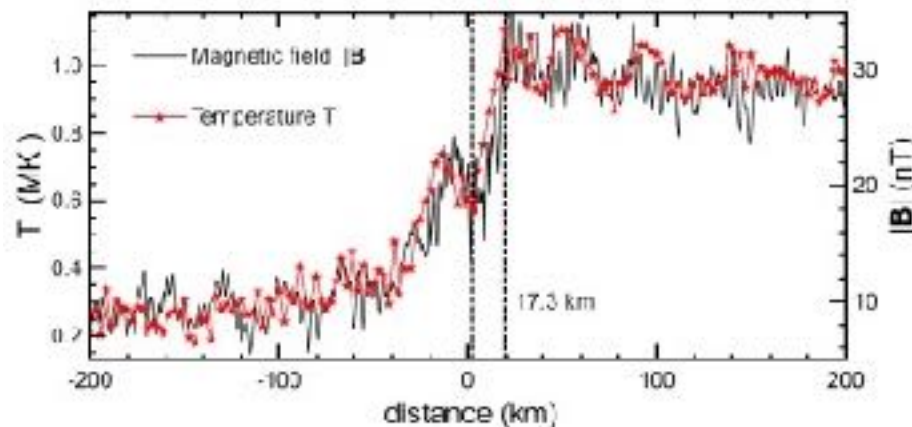
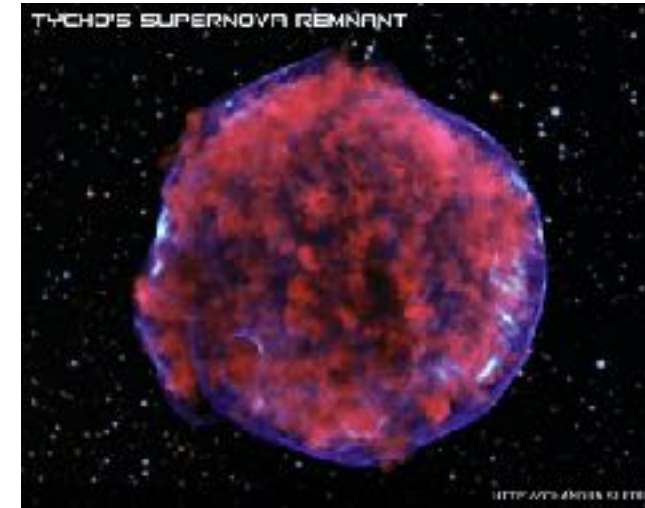
$$\omega \xrightarrow{k \rightarrow 0} k \cos \theta$$



- ➔ Different approximations, just as one-fluid MHD, Hall-MHD and electron-inertia MHD can clearly be identified in this diagram.

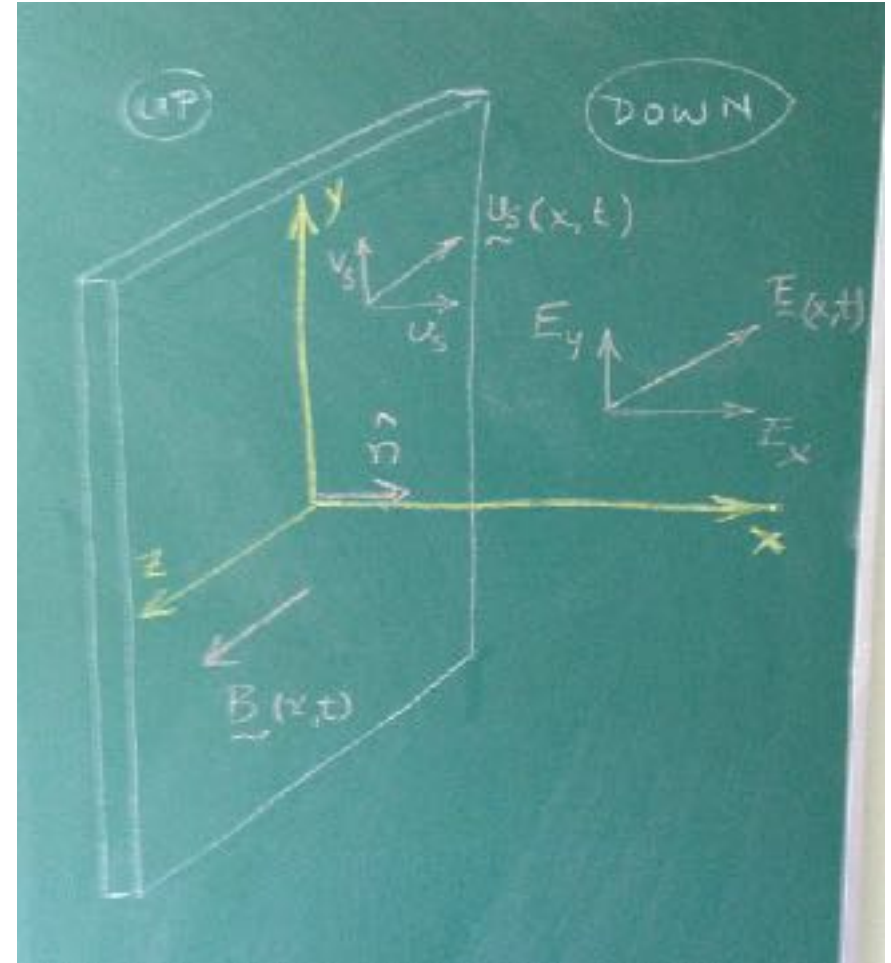
# Shocks in Space & in Astrophysics

- ➔ Shocks are an important source of heating and compression in space and astrophysical plasmas. Also, shocks are an important source of particle acceleration.
- ➔ Because of the very low collisionality of most space and astrophysical plasmas, the shock thickness is determined by plasma processes.
- ➔ For instance, shocks are formed during supernova explosions, when the stellar material propagates supersonically in the interstellar medium. Or when the solar wind impacts on each planet of the solar system.
- ➔ Even though the upstream and downstream regions can often be described using one-fluid MHD, the internal structure of the shock involves much smaller scales and therefore a fluidistic description like MHD cannot describe it properly.
- ➔ The image below shows the transverse structure of the Earth bow shock as observed by Cluster. The thickness is only a few electron inertial lengths ([Schwartz et al 2011](#)).



# One dimensional two-fluid model

- ➡ We decided to use a **two-fluid** description to study the generation and propagation of perpendicular shocks.
- ➡ Perpendicular shocks correspond to the particular case where the magnetic field is tangential to the shock.
- ➡ We adopted a 1D version of the equations. All fields only vary in the direction across the shock (i.e. “x”).
- ➡ **Tidman & Krall 1971** derive stationary solutions (solitons) for these same equations.
- ➡ Our two species are ions and electrons.
- ➡ Compressibility is essential for shock formation and dissipation is neglected.



# Multi-species plasmas

➡ A multi-species plasma description incorporates new physics along with new spatial and temporal scales.

➡ For each species  $s$  we have (Goldston & Rutherford 1995):

- ♦ Mass conservation 
$$\frac{\partial n_s}{\partial t} + \vec{\nabla} \cdot (n_s \vec{U}_s) = 0$$
- ♦ Equation of motion 
$$m_s n_s \frac{d\vec{U}_s}{dt} = q_s n_s (\vec{E} + \frac{1}{c} \vec{U}_s \times \vec{B}) - \vec{\nabla} p_s + \vec{\nabla} \cdot \vec{\sigma}_s + \sum_{s'} \vec{R}_{ss'}$$
- ♦ Momentum exchange rate 
$$\vec{R}_{ss'} = -m_s n_s \nu_{ss'} (\vec{U}_s - \vec{U}_{s'})$$

➡ The moving electric charges are sources of electric and magnetic fields:

- ♦ Charge density 
$$\rho_c = \sum_s q_s n_s \approx 0$$
- ♦ Current density 
$$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = \sum_s q_s n_s \vec{U}_s$$

➡ In the incompressible limit:

$$n_s = 0 \quad \vec{\nabla} \cdot \vec{U}_s = 0$$

➡ In the simple case of a **two-fluid** plasma (ions and electrons), this description introduces two new spatial scales (Andrés, Dmitruk & Gómez 2014a, PoP)

$$\lambda_i = \frac{c}{w_{pi}} = \sqrt{\frac{m_i c^2}{4\pi e^2 n}}$$

ion inertial length

$$\lambda_e = \frac{c}{w_{pe}} = \sqrt{\frac{m_e c^2}{4\pi e^2 n}}$$

electron inertial length



# 1D equations

- ➡ Note that  $U_i = U_e = U$ .
- ➡ As a result, the electrostatic potential can be obtained from the // Euler eqs.
- ➡ The perpendicular Euler equations predict that  $m_e V_e + m_i V_i = \text{const}$ , and we choose it equal to zero.
- ➡ The only linear mode propagating in a homogeneous background are fast magnetosonic waves.
- ➡ We integrate this set of 1D eqs. using a pseudo spectral code with periodic boundary conditions and RK2.
- ➡ The initial profile is a finite amplitude train of fast magnetosonic waves.
- ➡ Dissipation in these simulations is set to zero.

Two-fluid 1D eqs

Cont.  $\partial_t n + \partial_x (n u_s) = 0$   $n_e = n_i$

Euler //  $m_s n (\partial_t u_s + u_s \partial_x u_s) = \frac{q_s n}{E} (-\partial_x \phi + v_s B) - \beta_s \partial_x n^{\gamma}$

Euler  $\perp$   $m_s n (\partial_t v_s + u_s \partial_x v_s) = \frac{q_s n}{E} (-\partial_t A - u_s B)$

Ampère //  $0 = n (u_i - u_e)$

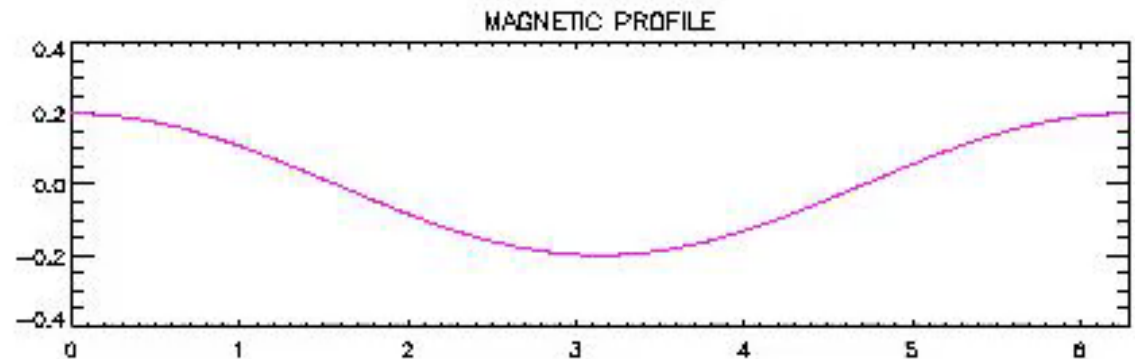
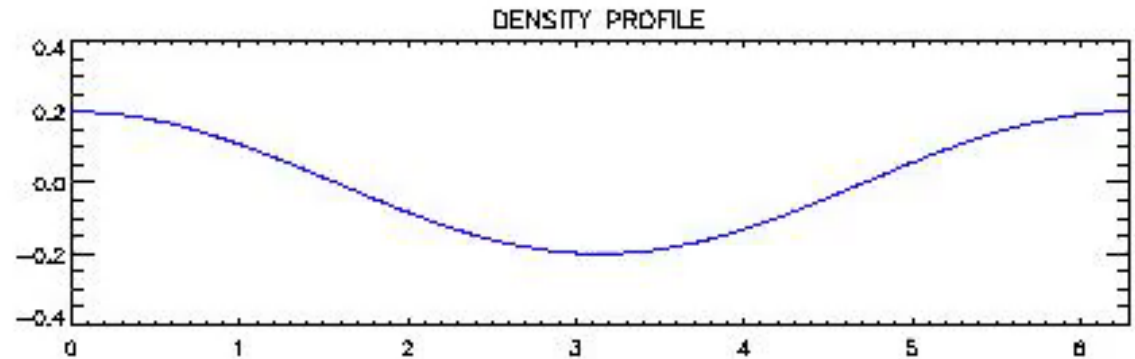
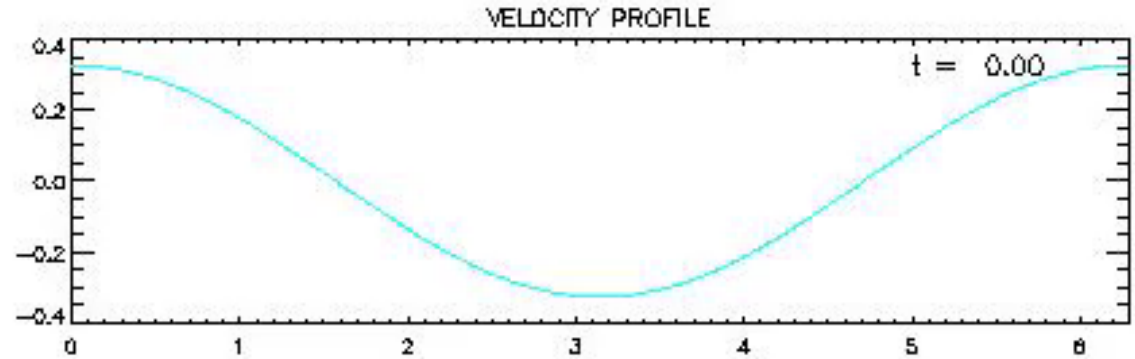
Ampère  $\perp$   $-\epsilon \partial_x B = n (v_i - v_e)$

$\underline{E} = -\partial_x \phi \hat{x} - \partial_t A \hat{y}$



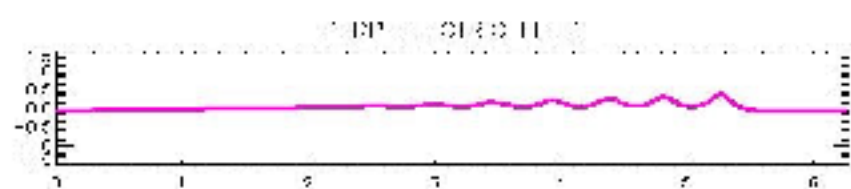
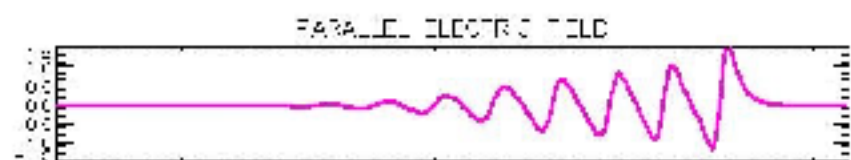
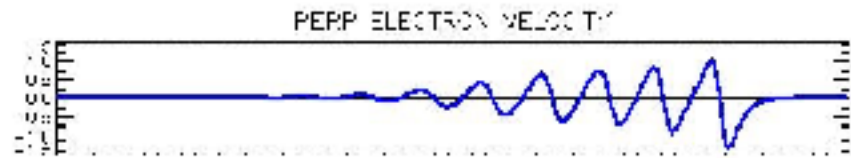
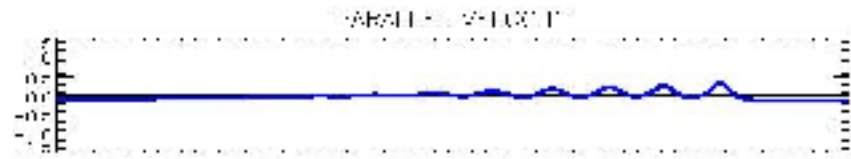
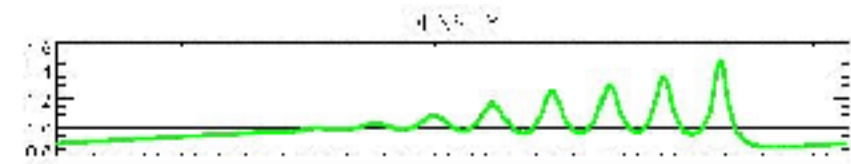
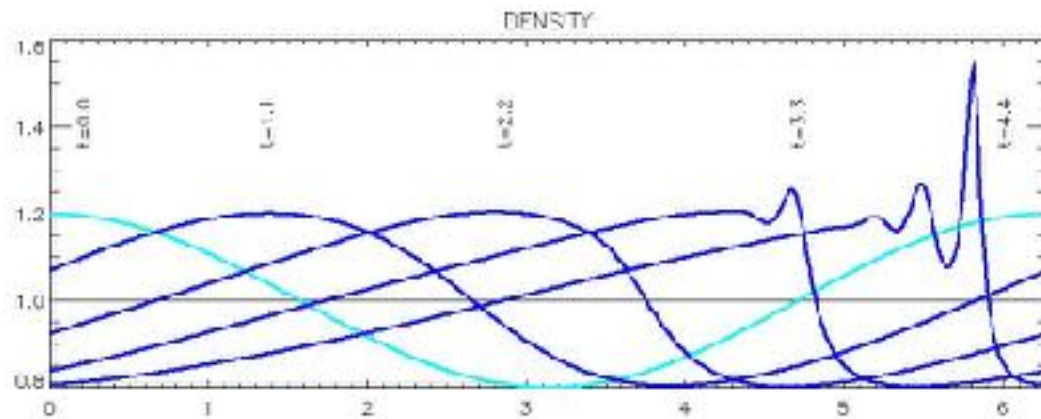
# Shock formation

- ➡ We numerically integrate the 1D **two-fluid** equations to study the generation of the shock, internal structure and propagation properties (Gomez et al 2018, ApJ, in prep.).
- ➡ Our initial condition is a finite amplitude fast magnetosonic wave.
- ➡ The movie shows the evolutions of various profiles (parallel velocity, particle density and perpendicular magnetic field).
- ➡ Once the shock is formed, it propagates without distortion.



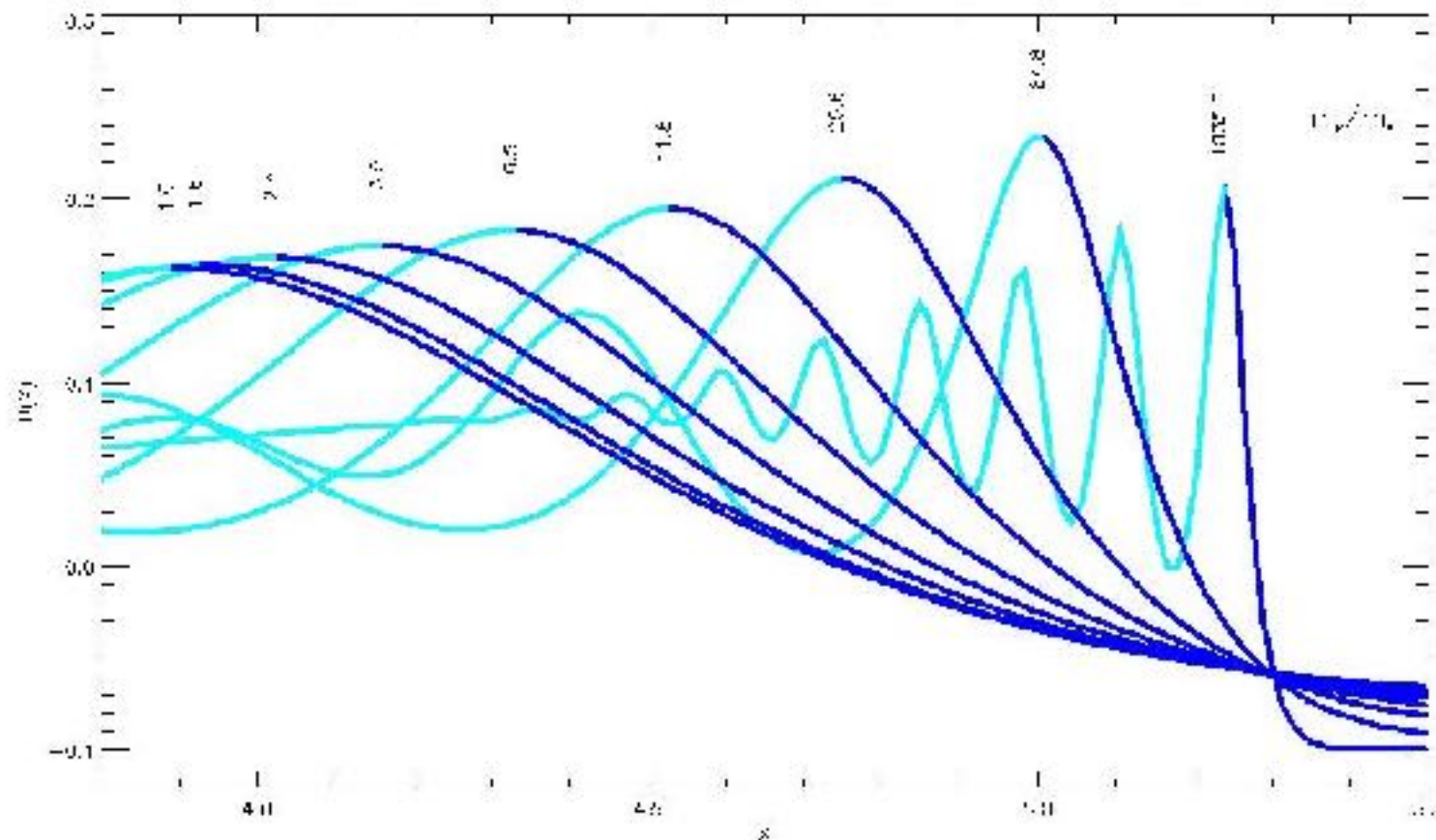
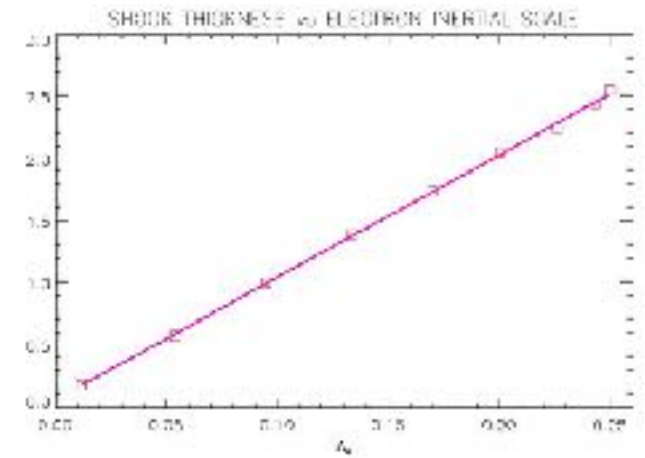
# Shock profiles

- Once the shock is formed, we can study the transverse structure of the various relevant physical quantities.
- We can see the propagation of trailing waves in the downstream region, with wavelengths of a few electron inertial lengths.
- Below we see the temporal evolution of the particle density profile. The initial profile is shown in light blue.



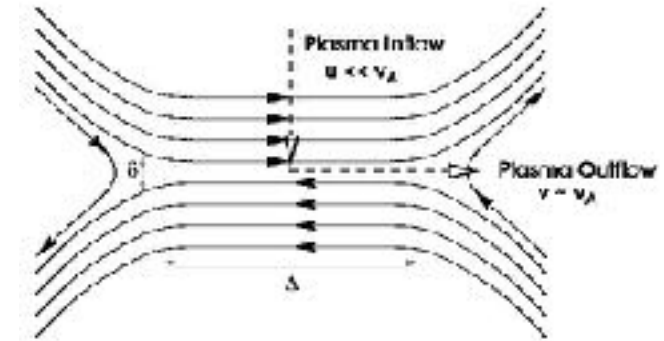
# Ramp thickness

- We overlap the profiles  $U(x)$  for various runs with different mass ratios. In dark blue we show the ramp portion of each profile.
- We estimate the ramp thickness for each run and plot it against the corresponding electron inertial length. We find a perfect correlation, with ramp thicknesses of about ten electron inertial lengths.



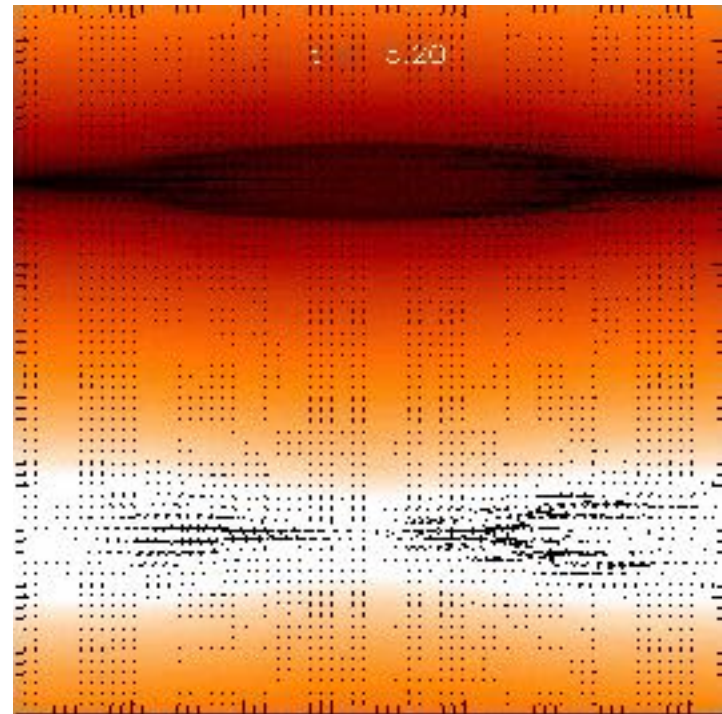
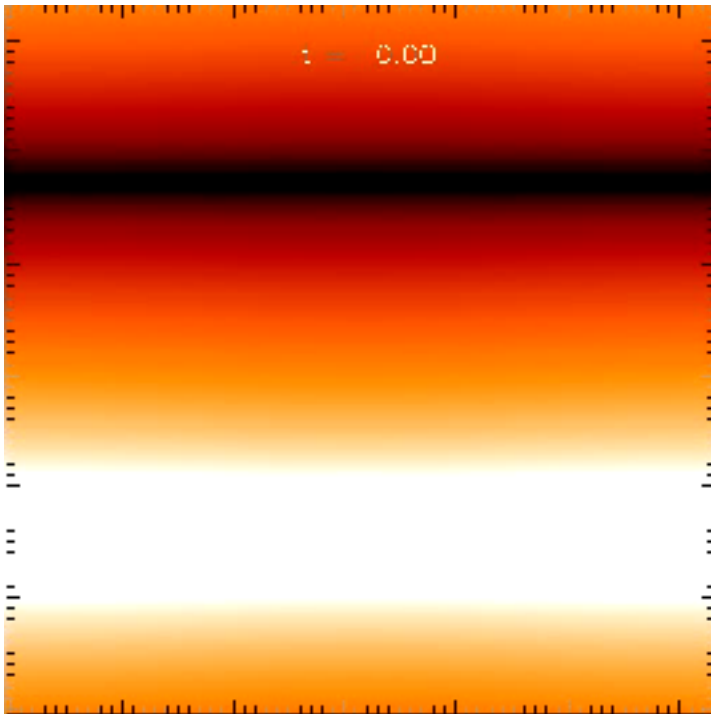
# Magnetic reconnection

- ➡ The standard theoretical model for two-dimensional stationary reconnection is the so-called **Sweet-Parker model** (Parker 1958)
- ➡ It corresponds to a stationary solution of the MHD equations. The plasma inflow (from above and below) takes place over a wide region of linear size  $\Delta$  and is much slower than the Alfvén speed (i.e.  $U_{in} \ll V_A$ ).
- ➡ The outflow occurs at a much thinner region (of linear size  $\delta \ll \Delta$ ) at speed  $U_{out} \sim V_A$ .
- ➡ The efficiency of the reconnection process is measured by the so-called reconnection rate, which is the magnetic flux reconnected per unit time.
- ➡ The dimensionless reconnection rate is
 
$$M = \frac{U_{in}}{U_{out}} \approx S^{-1/2}$$
 where  $S = \frac{\Delta v_A}{\eta}$  is the Lundquist number.
- ➡ Since for most astrophysical and space plasmas is  $S \gg 1$ , the reconnection rate is exceedingly low.

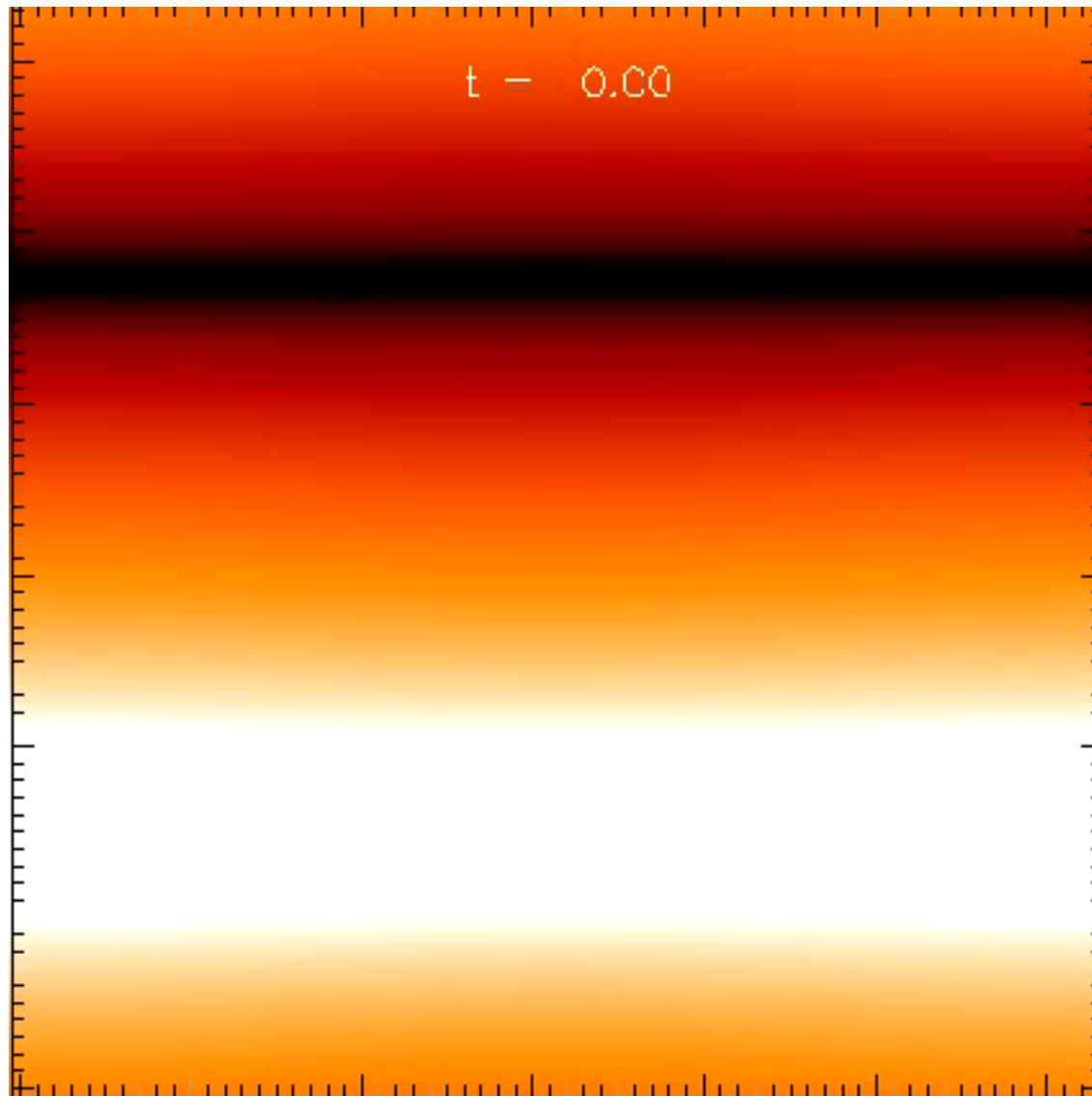


# Two-fluid simulations

- ➡ We perform simulations of the EIHMH equations in 2.5D geometry to study magnetic reconnection. We force an external field with a double hyperbolic tangent profile to drive reconnection at two X points ([Andres et al. 2014a, PoP](#)).
- ➡ We also study the turbulent regime of the EIHMH description, to look for changes at the electron skin-depth scale ([Andres et al. 2014b, PoP](#)).

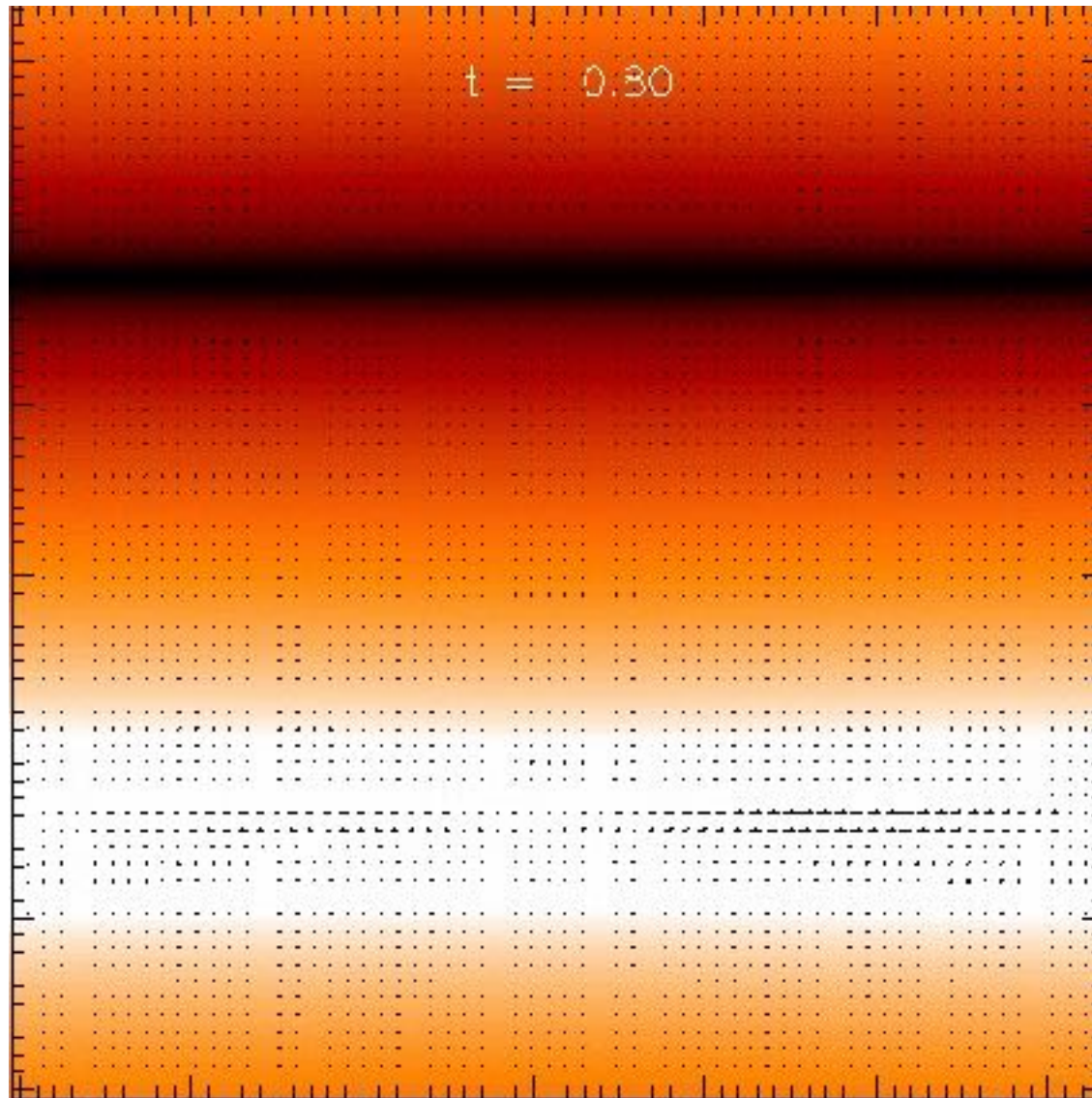


# Two-fluid reconnection



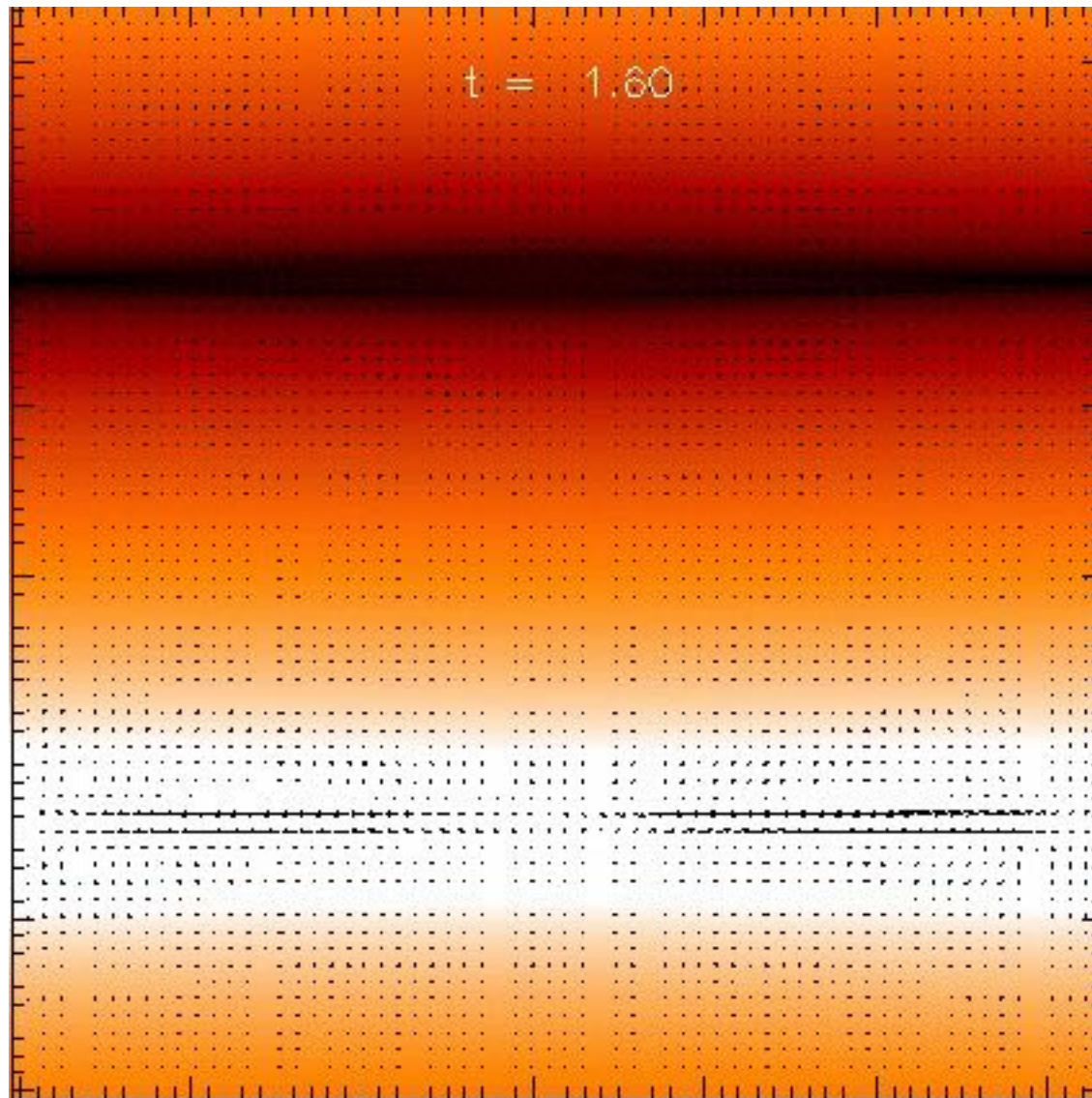


# Two-fluid reconnection

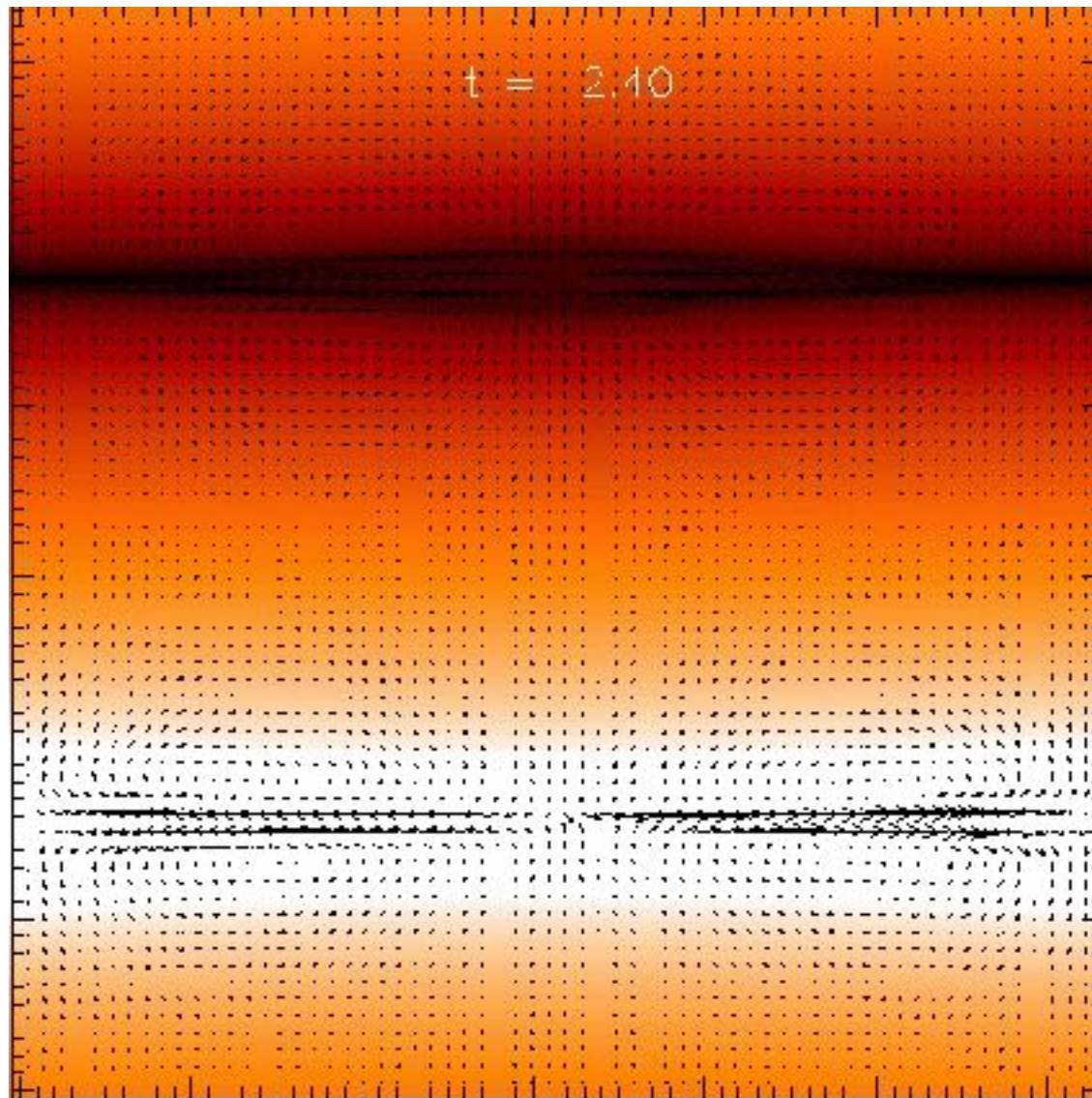




# Two-fluid reconnection

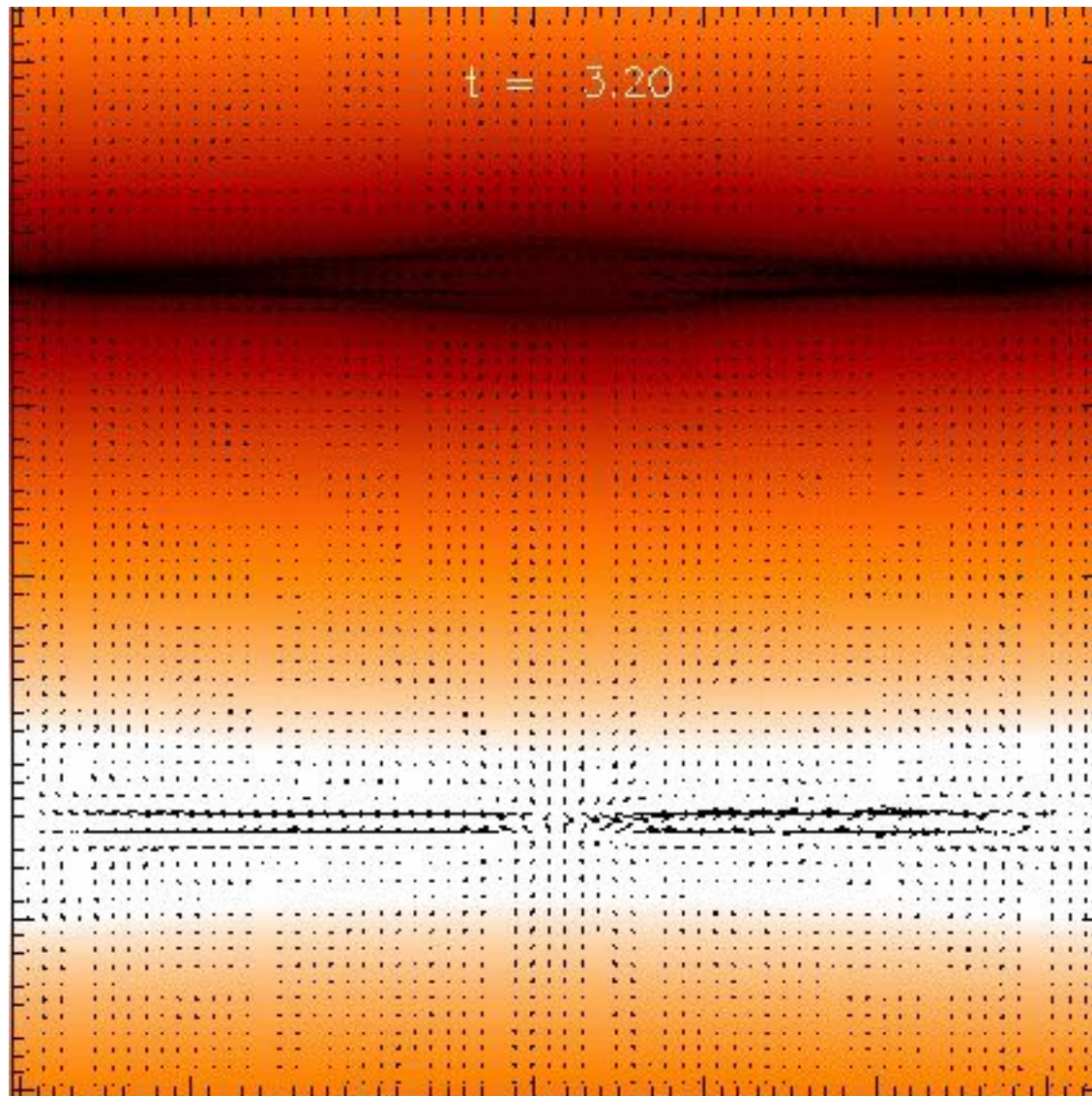


# Two-fluid reconnection

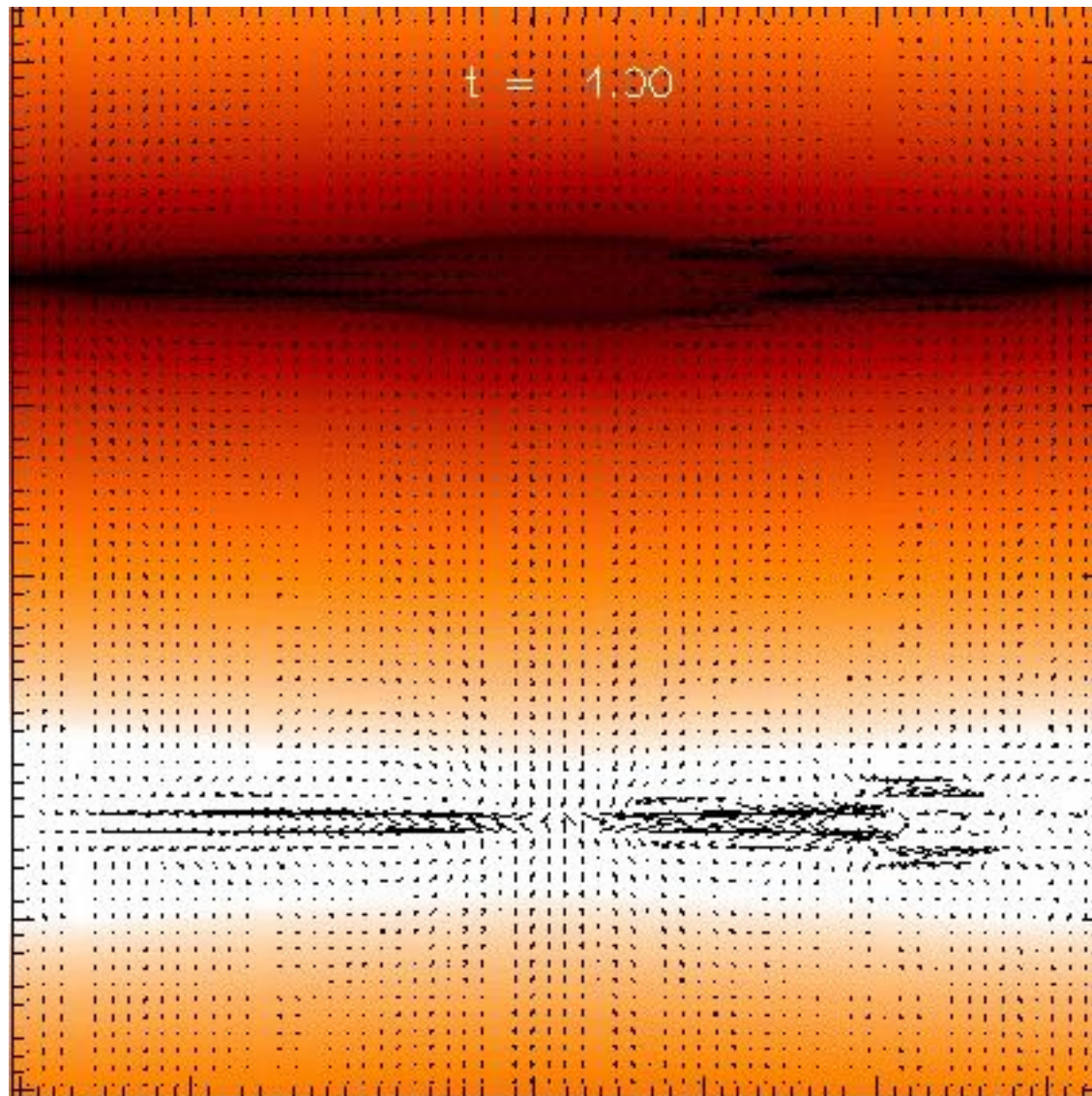




# Two-fluid reconnection

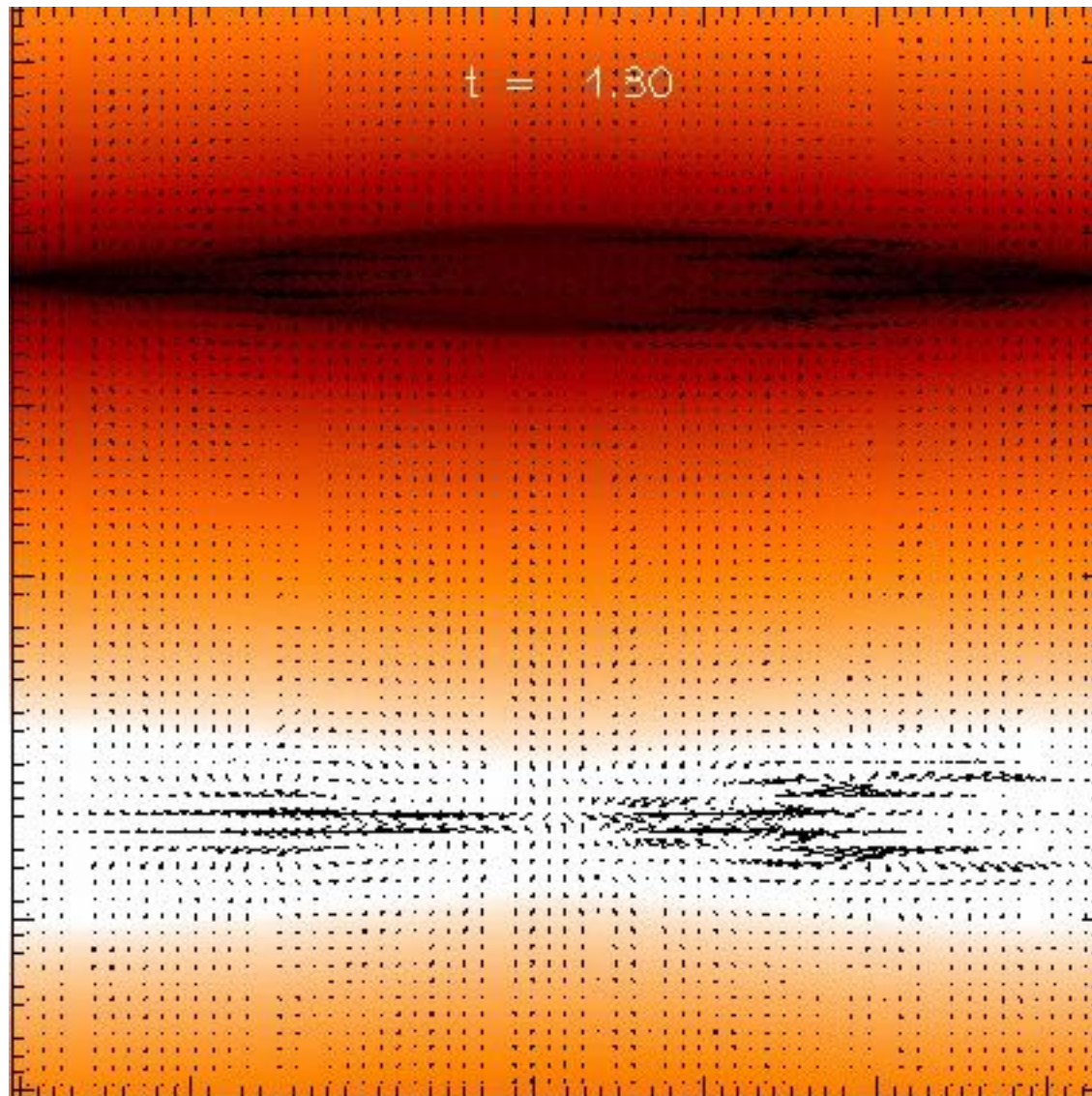


# Two-fluid reconnection

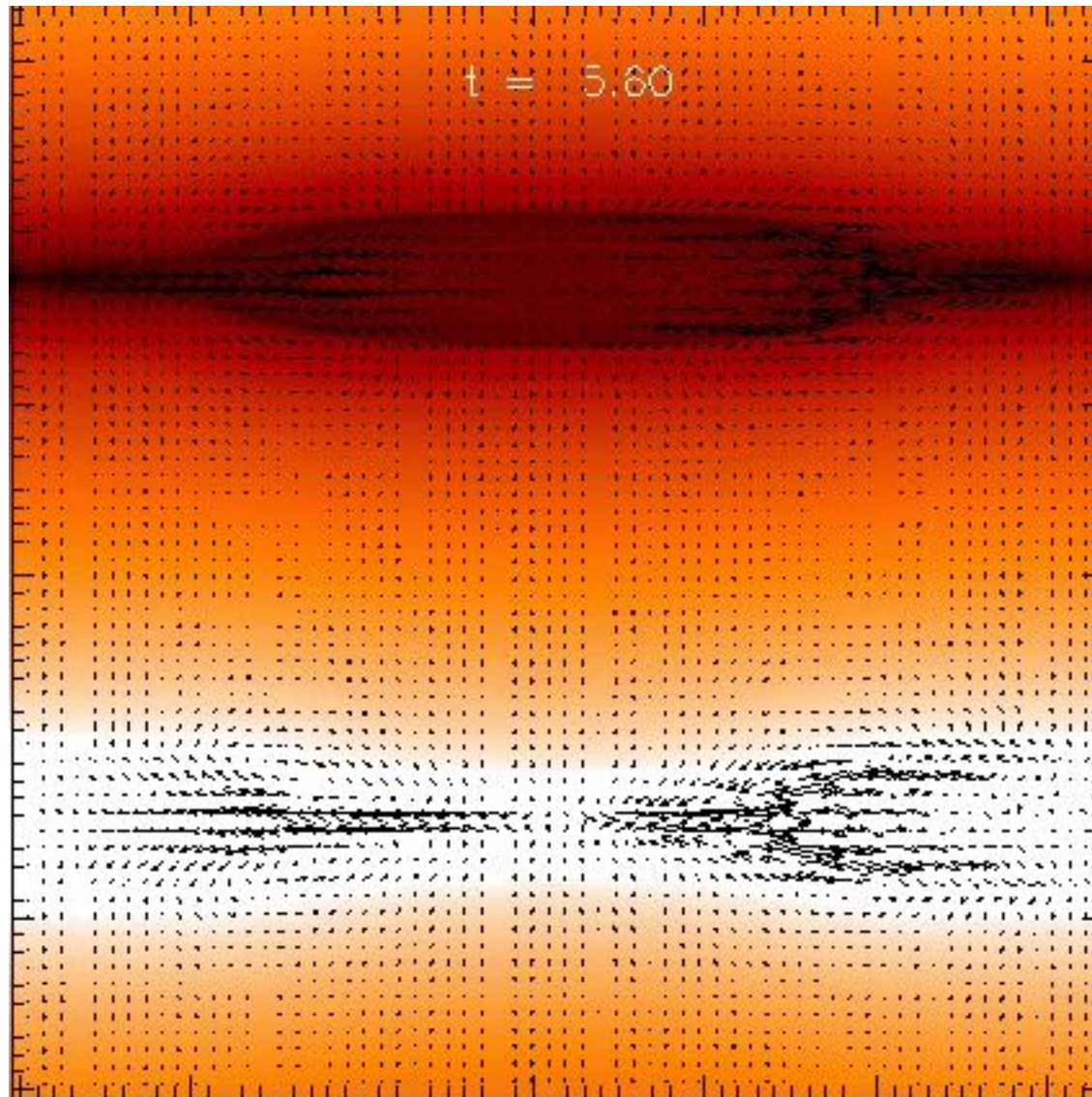




# Two-fluid reconnection

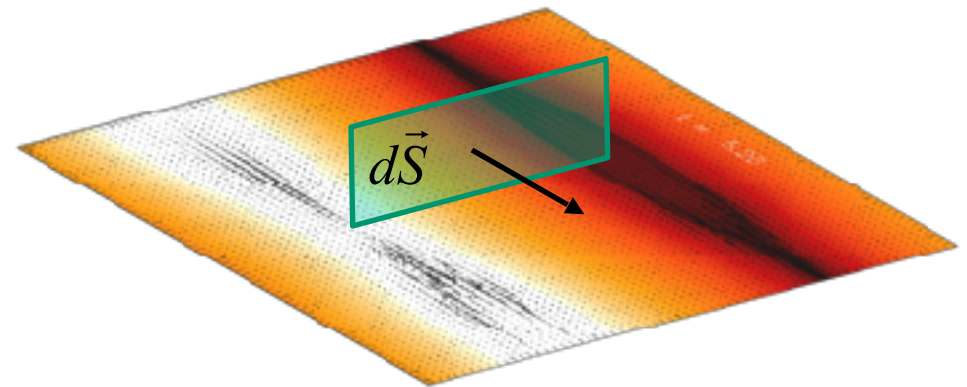
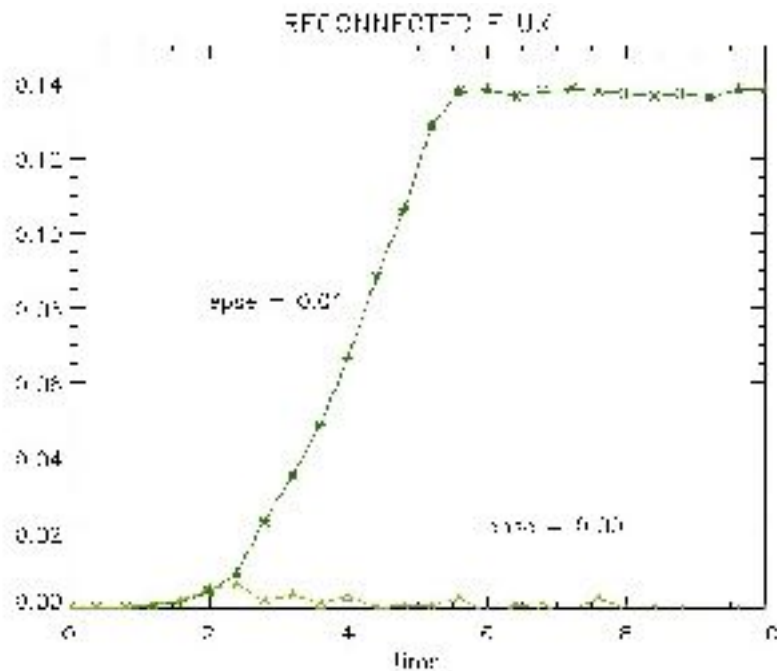


# Two-fluid reconnection



# Reconnected flux in 2F-MHD

- ➡ The total reconnected flux at the X-point is the magnetic flux through the perpendicular surface that extends from the O-point to the X-point.
- ➡ We compare the total reconnected flux between a run that includes electron inertia and another one that does not.



- ➡ The reconnection rate is the time derivative of these two curves.
- ➡ The apparent saturation is just a spurious effect stemming from the dynamical destruction of the X-point.



# Reconnection rate in 2F-MHD

➔ For the 2D configuration and assuming incompressibility, we run several simulations with different values of the Hall parameter, which is the dimensionless ion inertial length.

➔ We compare the corresponding reconnected flux (above) and the reconnection rate (below) vs. time.

➔ The reconnection rate is  $E_z$  at the X-point. From the equation for electrons, under stationary conditions

$$E_z = -\frac{m_e}{e} \hat{z} \cdot \vec{u}_e \times \vec{\nabla}_e$$

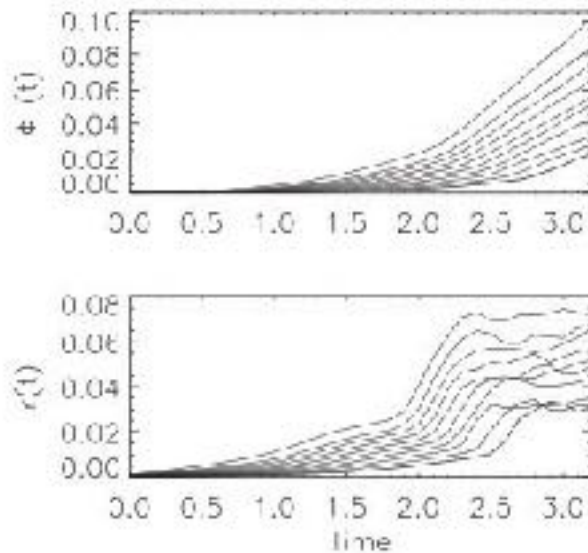


FIG. 3. Reconnected flux  $\Phi$  (upper panel) and reconnection rate  $\eta$  (lower panel) as a function of time for  $\lambda = 0.17, \dots, 0.16$  (from bottom to top). For all runs, the electron to ion mass ratio is  $m_e/m_i = 0.015$ .

➔ At electron scales

$$\vec{u}_e \sim -\frac{1}{en} \vec{j}$$

from where we obtain the following estimate for the dimensionless reconnection rate

$$R = \frac{c E_z}{B_0 v_A} \sim \frac{c}{w_{pi} L_0}$$

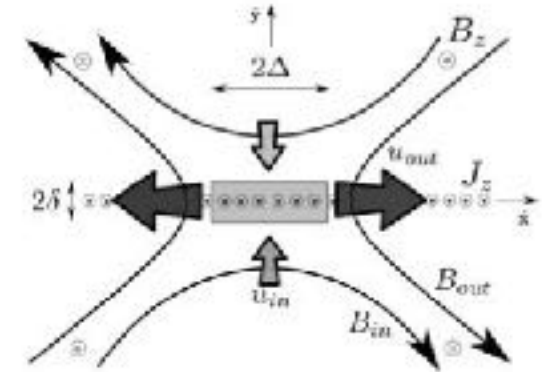


FIG. 4. Schematic of 2D reconnection region

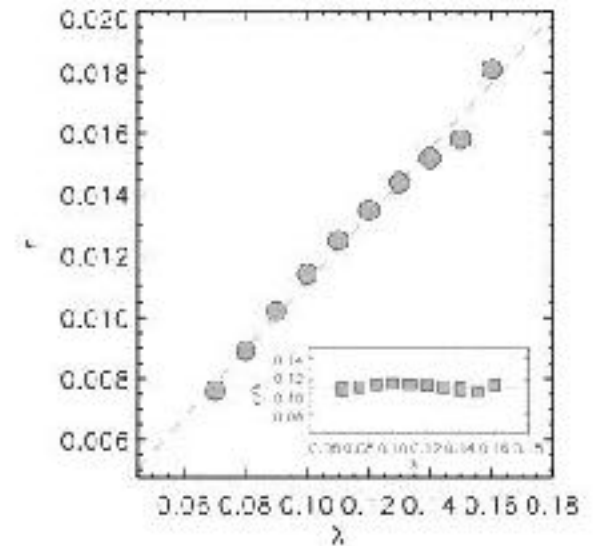
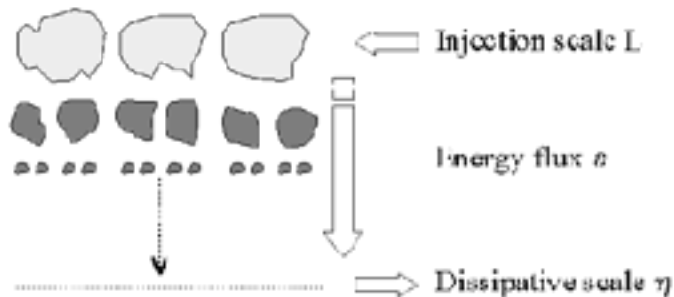


FIG. 6. Quasi-stationary reconnection rate  $\eta$  (gray circles) as a function of the Hall parameter  $\lambda$ . The best linear fit for  $\log \eta = \log c + x$  shown in gray-dashed line. Inset: Ratio between quasi-stationary reconnection rates and the Hall parameter (gray squares) as a function of the Hall parameter.

# Two-fluid turbulence



## → Energy cascade

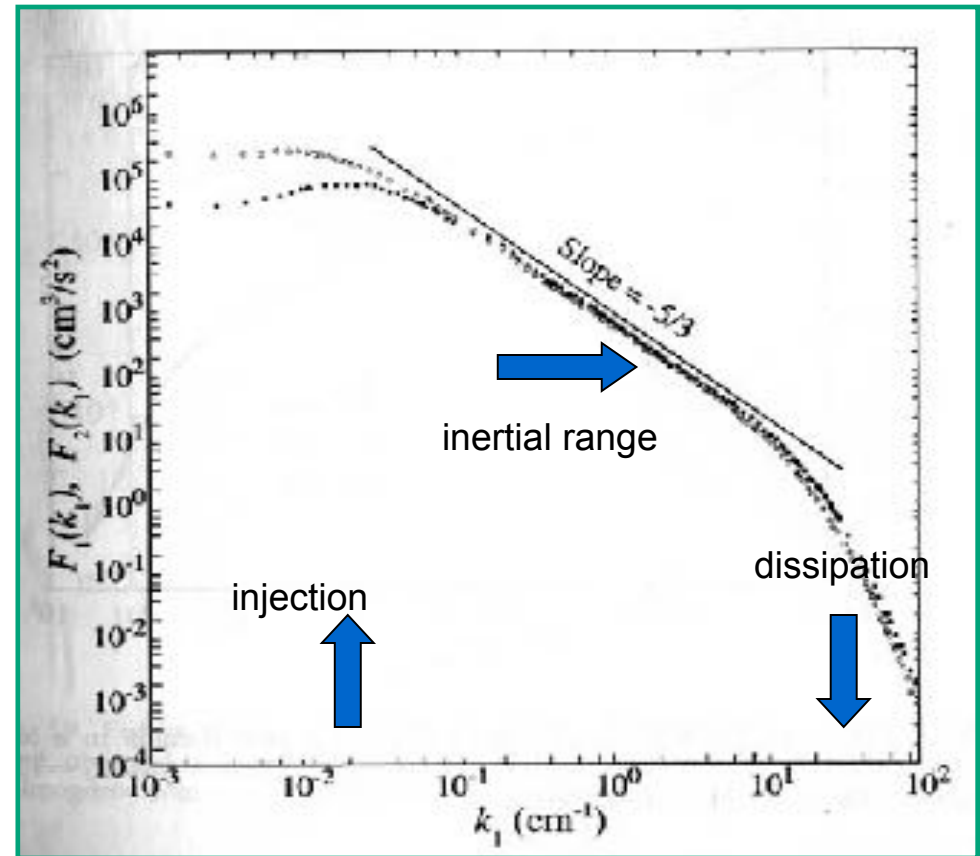
- energy flux toward high  $k$  →  $\epsilon_k \approx \frac{u_k^2}{\tau_k}$
- vortex breakdown

$$\tau_k \approx \frac{1}{ku_k}, \quad \epsilon_k \approx \frac{u_k^2}{\tau_k} = \text{const.}$$

## → Scale invariance

- energy flux in  $k$ :
- energy power spectrum: →  $E_k \approx \frac{u_k^2}{k}$

## → Therefore →



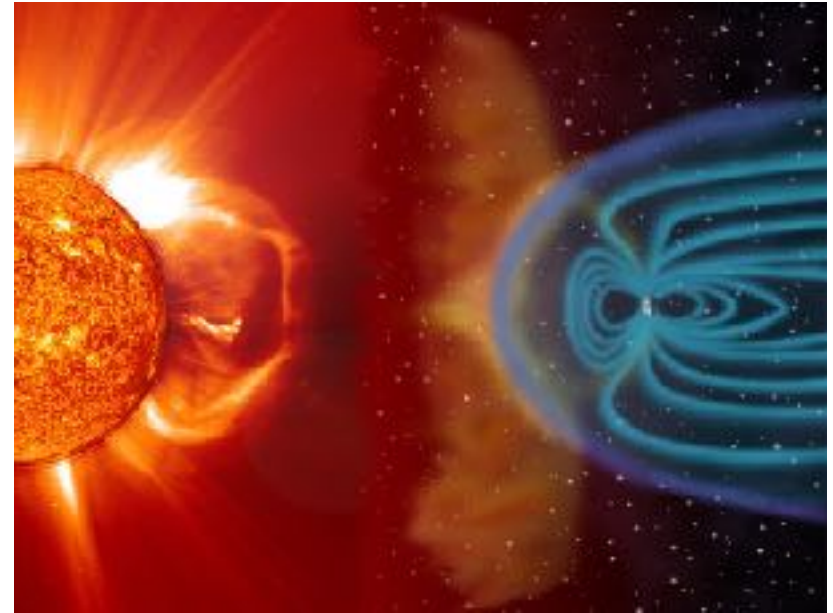
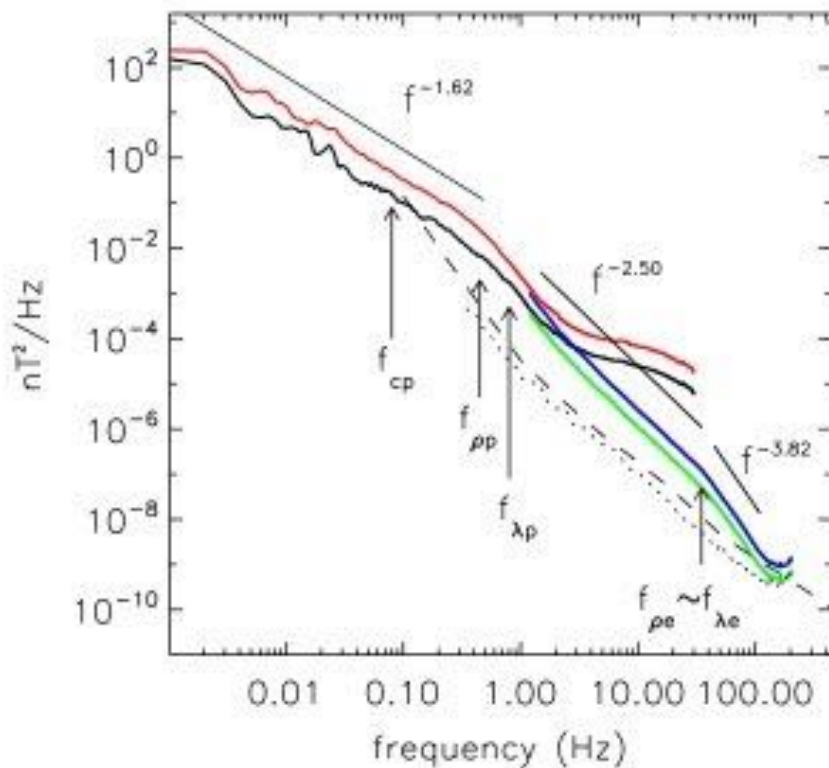
Kolmogorov spectrum (K41)

$$E_k \approx \frac{u_k^2}{k} = \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

# Turbulence in the Solar Wind

➡ The **solar wind** is a stream of plasma released from the upper atmosphere of the Sun, which impacts and affects the planetary magnetospheres.

➡ [Sahraoui et al. 2009](#) used magnetograms from the Cluster mission to derive power spectra of magnetic energy.



➡ They combine low-cadence data from FGM (**parallel** and **perpendicular** components) with high-cadence from STAFF-SC (also **parallel** and **perpendicular**).

➡ At the largest scales, they obtain a K41 power spectrum ( $k^{-1.62}$ ).

➡ As they go to smaller scales, they identify two breakups. An intermediate range with a power law  $k^{-2.50}$ , and an even steeper range at the smallest scales ( $k^{-3.82}$ ).

# Turbulence in 2F-MHD simulations

- ➔ These breakups are a manifestation of physical effects beyond MHD.
- ➔ We performed incompressible 3072x3072 simulations of the full two-fluid equations. We excited a ring of large-scale Fourier modes and let the system relax while the turbulent energy cascade takes place (Andres et al. 2014b, PoP).
- ➔ The magnetic energy power spectrum shows two breakups at the approximate locations of the proton ( $k_p$ ) and electron ( $k_e$ ) scales.

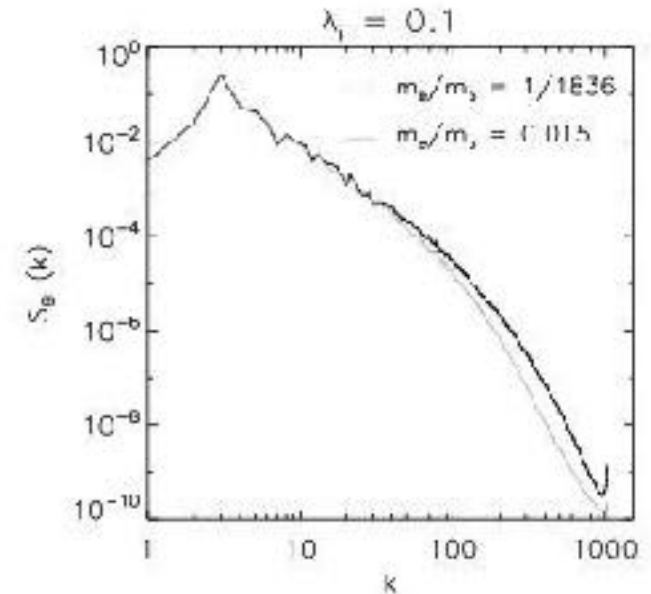
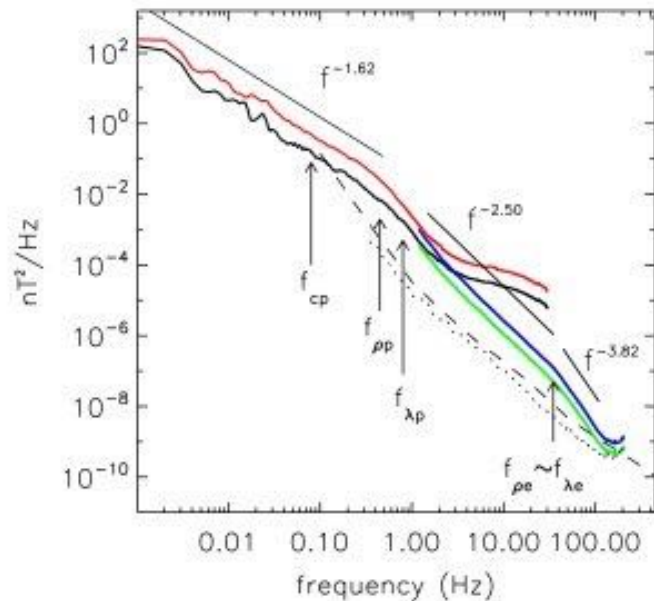


FIG. 1. Magnetic energy spectra for HHMHD cases with  $\nu = 1/10$  and  $m_e/m_i = 1/1836$  (black) and  $m_e/m_i = 0.015$  (gray).

- ➔ The spectrum is K41 (i.e.  $k^{-5/3}$ ) at  $k \ll k_p$ .
- ➔ At intermediate scales ( $k_p \ll k \ll k_e$ ) is  $k^{-7/3}$ .
- ➔ Beyond the electron scale ( $k_e \ll k$ ) a new range takes place  $k^{-11/3}$ .
- ➔ All these inertial ranges can be obtained using Kolmogorov-like arguments on the energy transfer rate given by

$$F_k \simeq k(u_k^3 + u_k B_k B'_k + (1 - \delta) \lambda J_k B_k B'_k + (1 - \delta) \delta \lambda^2 \partial_t J_k B_k).$$

# Turbulence in 2F-MHD simulations

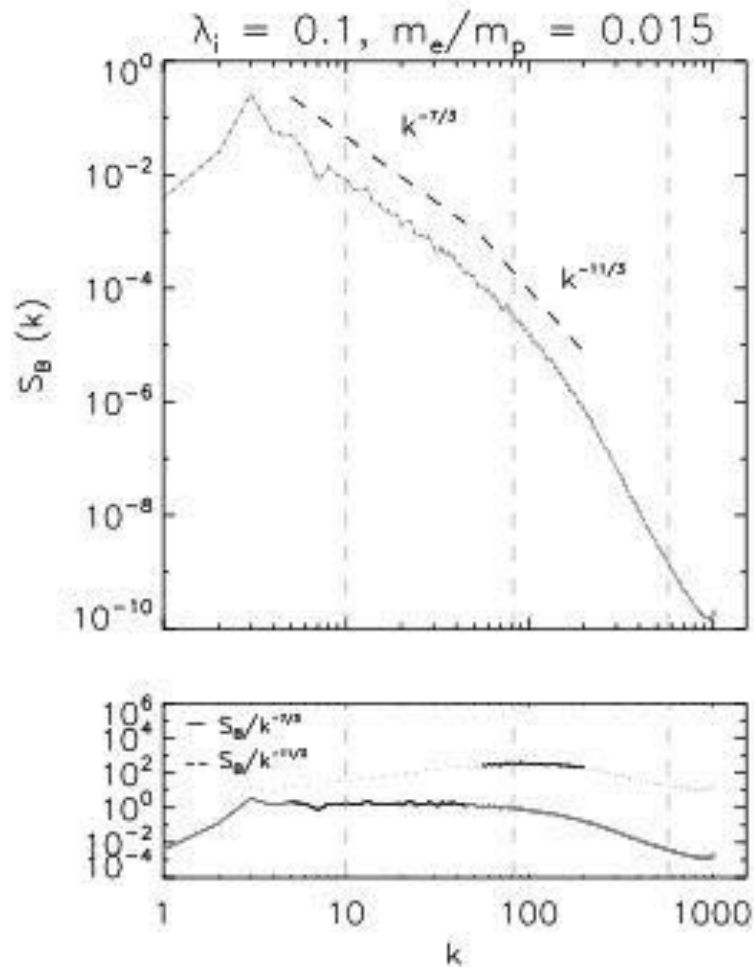


FIG. 2. Magnetic energy spectra for  $m_e/m_p=0.015$ . Vertical dashed gray lines correspond to  $k_x \sim 10$ ,  $k_y \sim 82$ , and  $k_z \sim 550$ . The compensated spectrum for the HMHD (solid line) and EHMHD (dashed line) regions are shown in the lower panel.

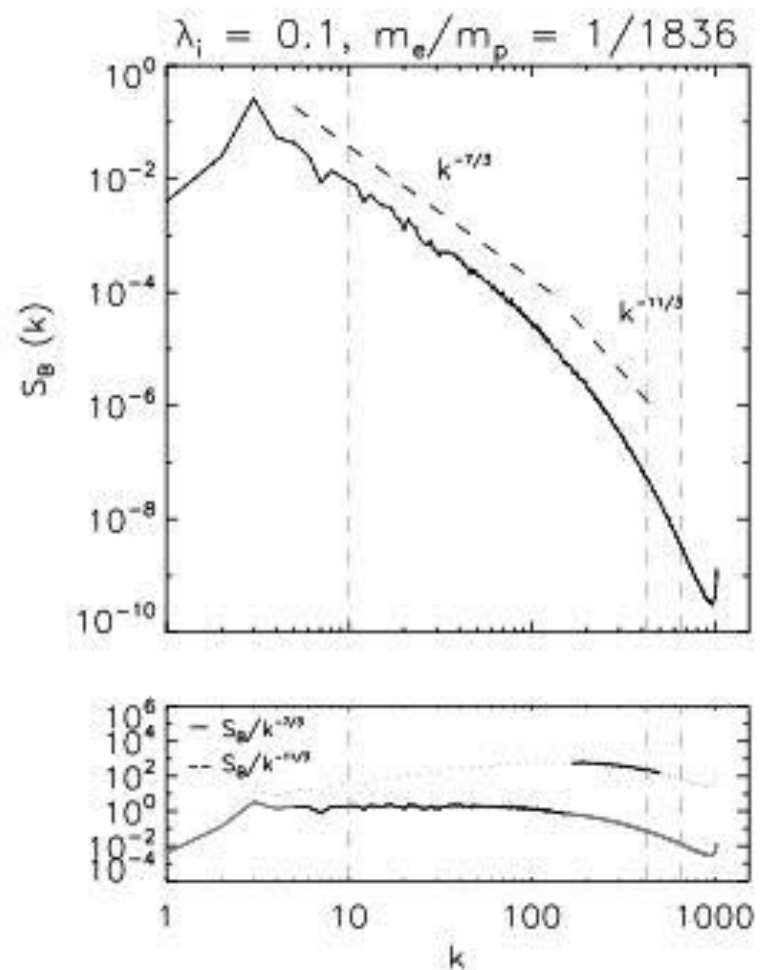


FIG. 3. Magnetic energy spectra for  $m_e/m_p = 1/1836$ . Vertical dashed gray lines correspond to  $k_x \sim 10$ ,  $k_y \sim 430$ , and  $k_z \sim 650$ . The compensated spectrum for the HMHD (gray line) and EHMHD (green line) regions in the same format as Figure 2.



# Conclusions

- **One-fluid MHD** is a reasonable theoretical framework to describe the large-scale dynamics of plasmas.
- **Two-fluid MHD** introduces new physics (Hall, electron pressure, electron inertia) and also new spatial scales, such as the proton and electron inertial lengths. We studied the role of these plasma effects on three relevant phenomena for astrophysical and space plasmas: shocks, reconnection and turbulence.
- Shocks:  
1D simulations show that a train of fast magnetosonic waves decay into a train of shocks. Thickness is a few times the electron inertial scale.
- Reconnection:  
Our 2F-MHD simulations show fast magnetic reconnection without energy dissipation (Andres et al. 2014a, PoP). The reconnection rate scales like the ion inertial scale and is independent from the electron mass.
- Turbulence:  
Externally driven 2F-MHD runs show turbulent regimes. The magnetic energy spectrum displays breakups at the ion and electron inertial scales (Andres et al. 2014b, PoP). Spectral slopes can also be obtained by dimensional analysis.