Nuclear Structure
(I) Single-particle models

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Overview of nuclear models

Ab initio methods: Description of nuclei starting from the bare nn & nnn interactions.

Nuclear shell model: Nuclear average potential + (residual) interaction between nucleons.

Mean-field methods: Nuclear average potential with global parameterization (+ correlations).

Phenomenological models: Specific nuclei or properties with local parameterization.

Independent-particle shell model

Independent motion of individual neutrons and protons in a mean-field potential.

Existence of shell structure with 'magic numbers' 2, 8, 20, 28, 50, 82, 126 of increased stability.

Crucial ingredient: spin-orbit interaction (Fermi).

Nobel prize in 1963:

Mayer & Jensen: "…for their discoveries concerning shell structure."

Wigner: "…for his contributions to the theory of the atomic nucleus and the elementary particles…"

Nuclear shell model

Ingredients:

Mean-field potential.

Residual interaction between (some of) the nucleons.

Difficulties:

Nucleonic interactions from QCD (EFT).

Large-matrix diagonalization.

Issues of current interest:

Changing shell structure and three-body forces in exotic nuclei.

Continuum effects (nucleus = open quantum system).

Words of warning

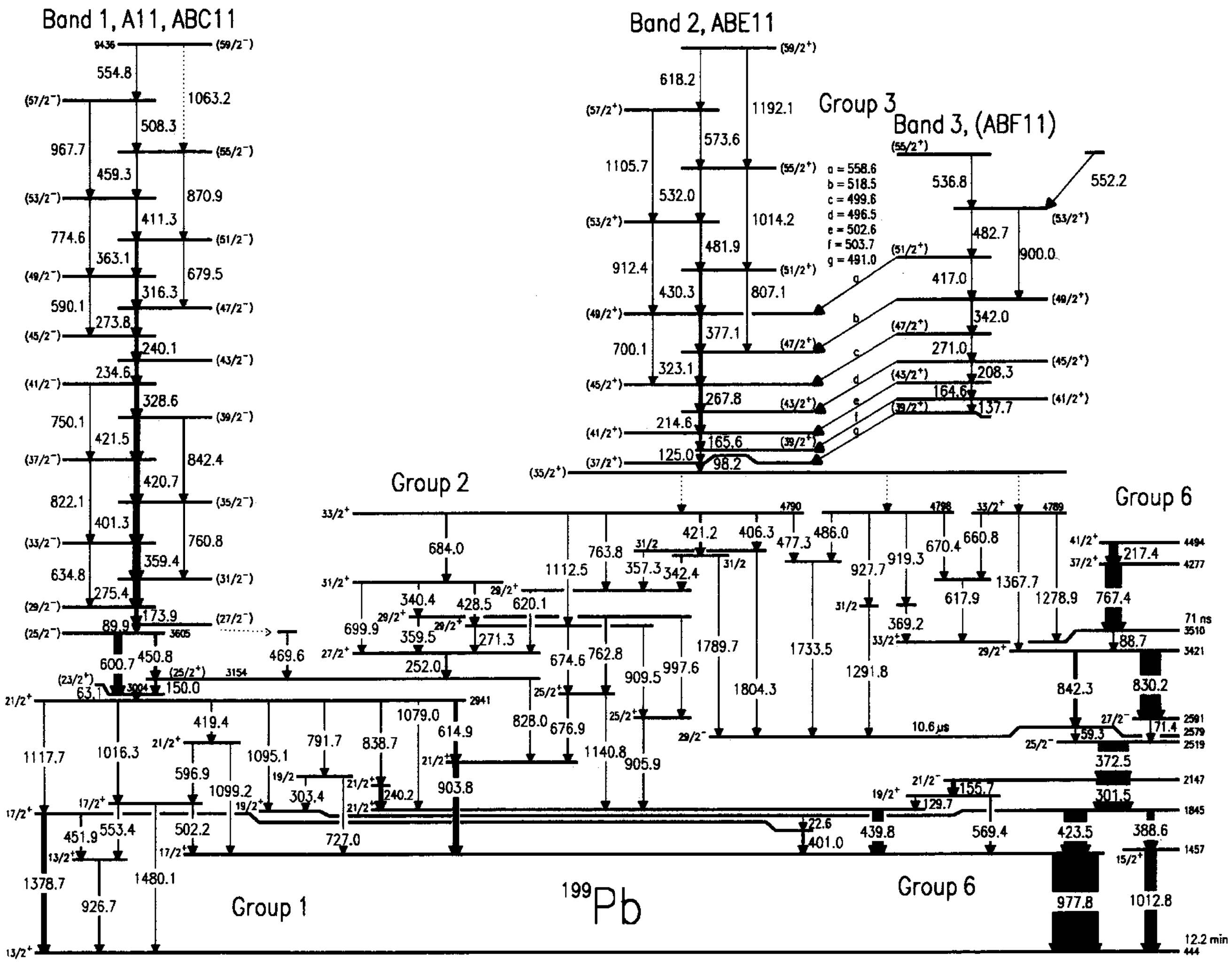
Bethe:

The complexity of the nuclear many-body problem is such that the shell-model wave functions cannot be the true eigenfunctions of the nuclear hamiltonian.

Wigner:

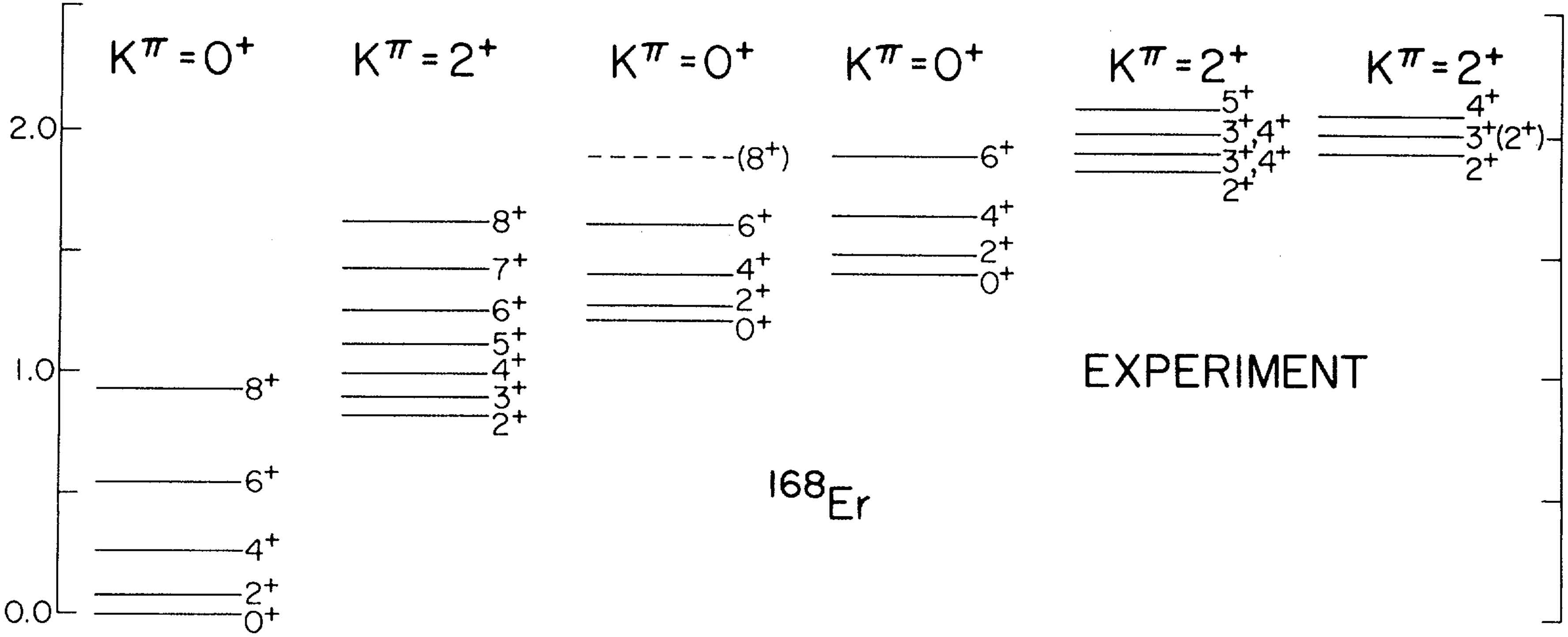
It is nice to know that the computer understands the problem. But I would like to understand it too.

Example: ^{199}Pb (N=117, Z=82)



G. Baldsiefen *et al.*, Nucl. Phys. A **574** (1994) 521

Example: ^{168}Er (N=100, Z=68)



D. D. Warner *et al.*, Phys. Rev. Lett. **45** (1980) 1761

Nuclear shell model

Many-body quantum mechanical problem:

$$\hat{H} = \sum_{k=1}^{A} \frac{p_k^2}{2m_k} + \sum_{k<l}^{A} \hat{V}_2(\boldsymbol{r}_k, \boldsymbol{r}_l)$$

$$= \underbrace{\sum_{k=1}^{A} \left[\frac{p_k^2}{2m_k} + \hat{V}(\boldsymbol{r}_k) \right]}_{\text{mean field}} + \underbrace{\left[\sum_{k<l}^{A} \hat{V}_2(\boldsymbol{r}_k, \boldsymbol{r}_l) - \sum_{k=1}^{A} \hat{V}(\boldsymbol{r}_k) \right]}_{\text{residual interaction}}$$

Independent-particle assumption. Choose *V* and neglect residual interaction:

$$\hat{H} \approx \hat{H}_{\mathrm{IP}} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m_k} + \hat{V}(\boldsymbol{r}_k) \right]$$

Independent-particle shell model

Solution for one particle:

$$\left[\frac{p^2}{2m} + \hat{V}(\boldsymbol{r})\right]\phi_i(\boldsymbol{r}) = E_i \phi_i(\boldsymbol{r})$$

Solution for many particles:

$$\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) = \prod_{k=1}^{A} \phi_{i_k}(\boldsymbol{r}_k)$$

$$\hat{H}_{\mathrm{IP}} \Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) = \left(\sum_{k=1}^{A} E_{i_k}\right) \Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A)$$

Independent-particle shell model

Anti-symmetric solution for many particles (Slater determinant):

$$\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{i_1}(\boldsymbol{r}_1) & \phi_{i_1}(\boldsymbol{r}_2) & \cdots & \phi_{i_1}(\boldsymbol{r}_A) \\ \phi_{i_2}(\boldsymbol{r}_1) & \phi_{i_2}(\boldsymbol{r}_2) & \cdots & \phi_{i_2}(\boldsymbol{r}_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i_A}(\boldsymbol{r}_1) & \phi_{i_A}(\boldsymbol{r}_2) & \cdots & \phi_{i_A}(\boldsymbol{r}_A) \end{vmatrix}$$

Example for $A=2$ particles:

$$\Psi_{i_1 i_2}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}\left[\phi_{i_1}(\boldsymbol{r}_1)\phi_{i_2}(\boldsymbol{r}_2) - \phi_{i_1}(\boldsymbol{r}_2)\phi_{i_2}(\boldsymbol{r}_1)\right]$$

Hartree-Fock approximation

Vary ϕ_i (*ie V*) to minize the expectation value of *H* in a Slater determinant:

$$\delta \frac{\int \Psi^*_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) \hat{H} \Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_A}{\int \Psi^*_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) \Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_A} = 0$$

Application requires choice of *H*. Many global parameterizations (Skyrme, Gogny,…) have been developed.

Poor man's Hartree-Fock

Choose a simple, analytically solvable V that approximates the microscopic HF potential:

$$\hat{H}_{\mathrm{IP}} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m} + \frac{m\omega^2}{2} r_k^2 - \zeta \, \boldsymbol{l}_k \cdot \boldsymbol{s}_k - \kappa \, l_k^2 \right]$$

Contains

Harmonic oscillator potential with constant ω.

Spin-orbit term with strength ζ.

Orbit-orbit term with strength κ.

Adjust ω, ζ and κ to best reproduce HF.

R. D. Lawson, *Theory of the Nuclear Shell Model*

Harmonic oscillator solution

Energy eigenvalues of the harmonic oscillator:

$$E_{nlj} = \left(N + \tfrac{3}{2}\right)\hbar\omega - \kappa\hbar^2 l(l+1) + \zeta\hbar^2 \begin{cases} -\tfrac{1}{2}l & j = l + \tfrac{1}{2} \\ \tfrac{1}{2}(l+1) & j = l - \tfrac{1}{2} \end{cases}$$

$N = 2n + l = 0,1,2,\ldots :$ oscillator quantum number

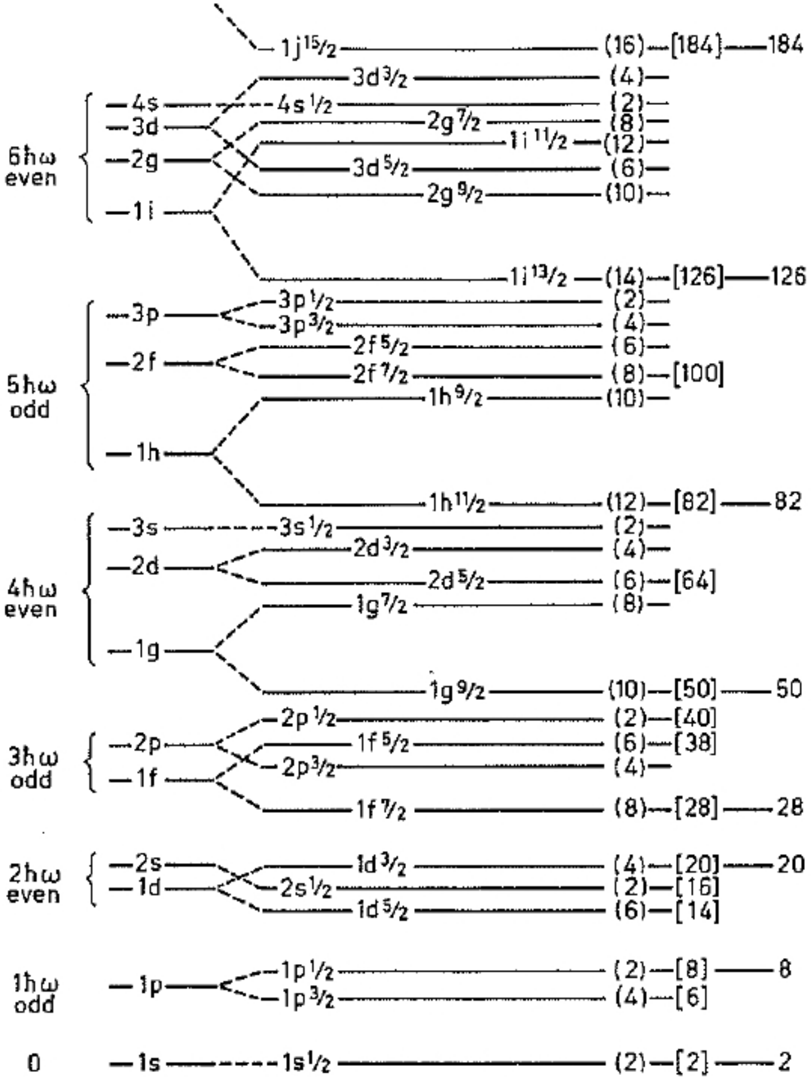
$n = 0,1,2,\ldots :$ radial quantum number

$l = N, N-2, \ldots, 1 \text{ or } 0 :$ orbital angular momentum

$j = l \pm \tfrac{1}{2} :$ total angular momentum

$m_j = -j, -j+1, \ldots, +j :$ z projection of j

Energy levels of harmonic oscillator



Typical parameter values:

$$\hbar\omega \approx 41\ A^{-1/3}\ \mathrm{MeV}$$

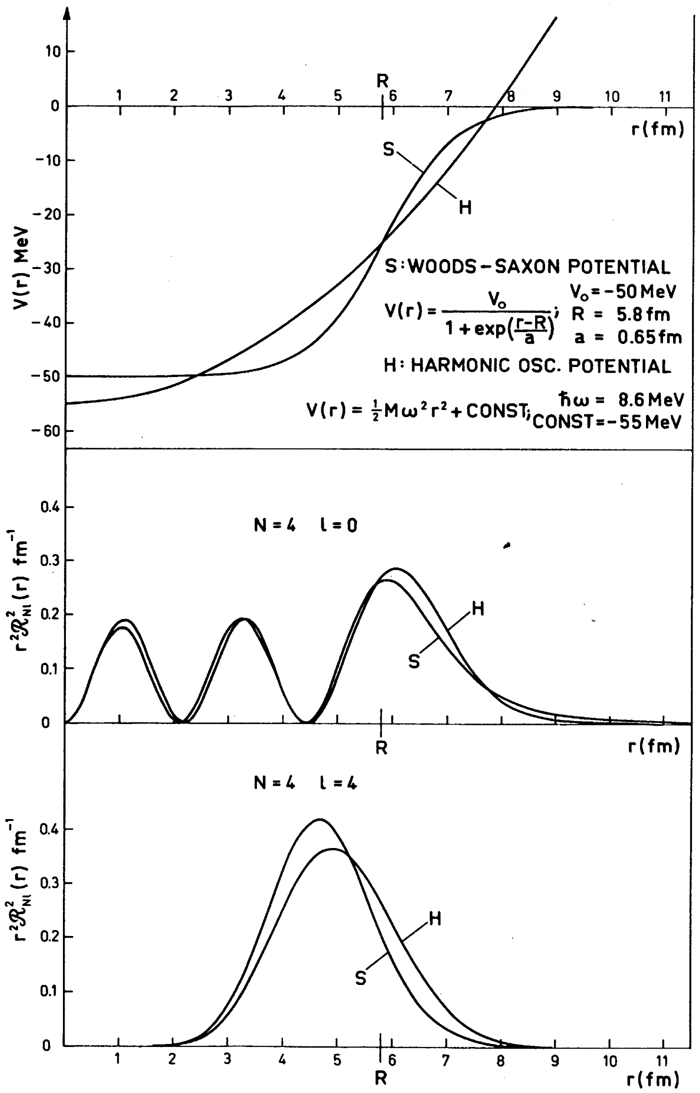
$$\zeta\,\hbar^2 \approx 20\ A^{-2/3}\ \mathrm{MeV}$$

$$\kappa\,\hbar^2 \approx 0.1\,\mathrm{MeV}$$

$$\therefore\ b \approx 1.0\ A^{1/6}\ \mathrm{fm}$$

'Magic' numbers at 2, 8, 20, 28, 50, 82, 126, 184,…

Why an orbit-orbit term?



Nuclear mean field is close to Woods-Saxon:

$$\hat{V}_{\mathrm{WS}}(r) = \frac{V_0}{1 + \exp\dfrac{r - R_0}{a}}$$

2n+l=N degeneracy is lifted $\Rightarrow E_l < E_{l-2} < \ldots$

Why a spin-orbit term?

Relativistic origin (*ie* Dirac equation).

From general invariance principles:

$$\hat{V}_{\mathrm{SO}} = \zeta(r)\boldsymbol{l}\cdot\boldsymbol{s}, \quad \zeta(r) = \frac{r_0^2}{r}\frac{\partial V}{\partial r}\left[= \zeta \text{ in HO}\right]$$

Spin-orbit term is surface peaked ⇒ diminishes for diffuse potentials.

Evidence for shell structure

Evidence for nuclear shell structure from

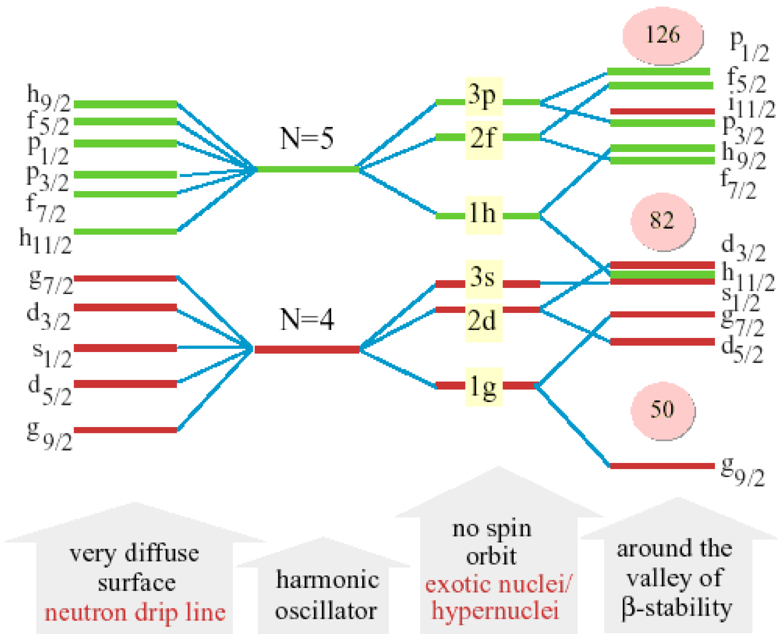
2^+ in even-even nuclei [E_x, $B(E2)$].

Nucleon-separation energies & nuclear masses.

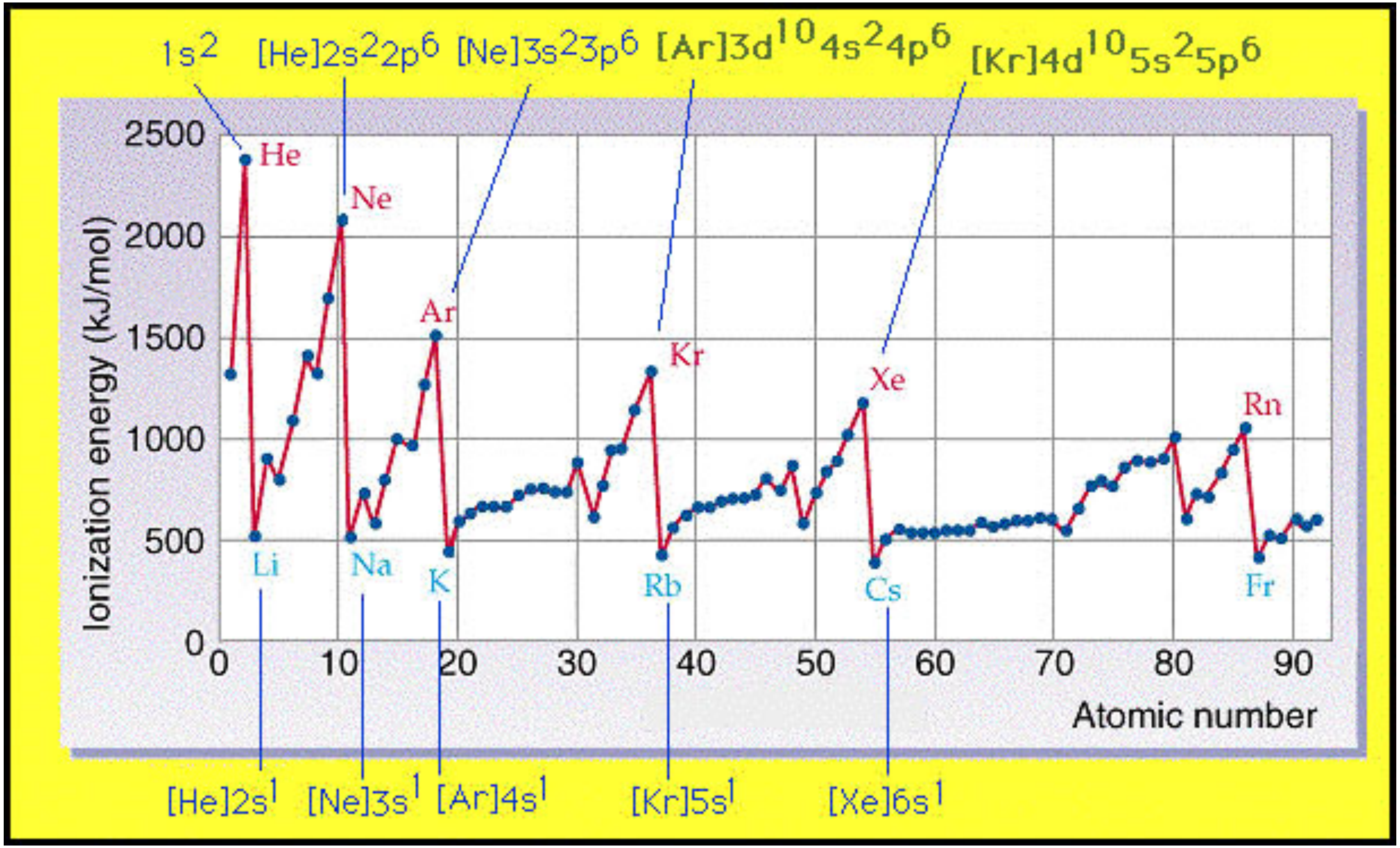
Nuclear level densities.

Reaction cross sections.

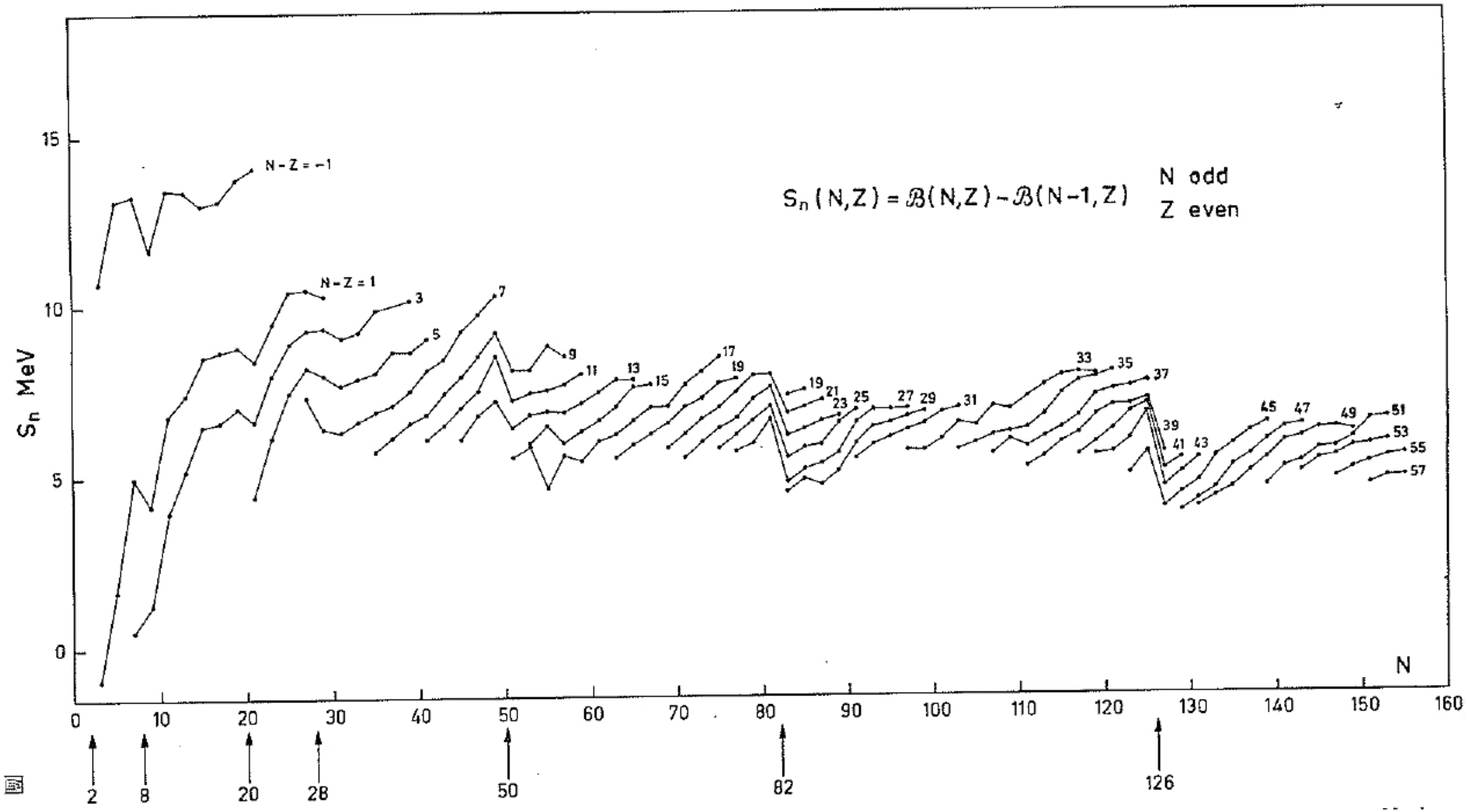
Is nuclear shell structure modified away from the line of stability?



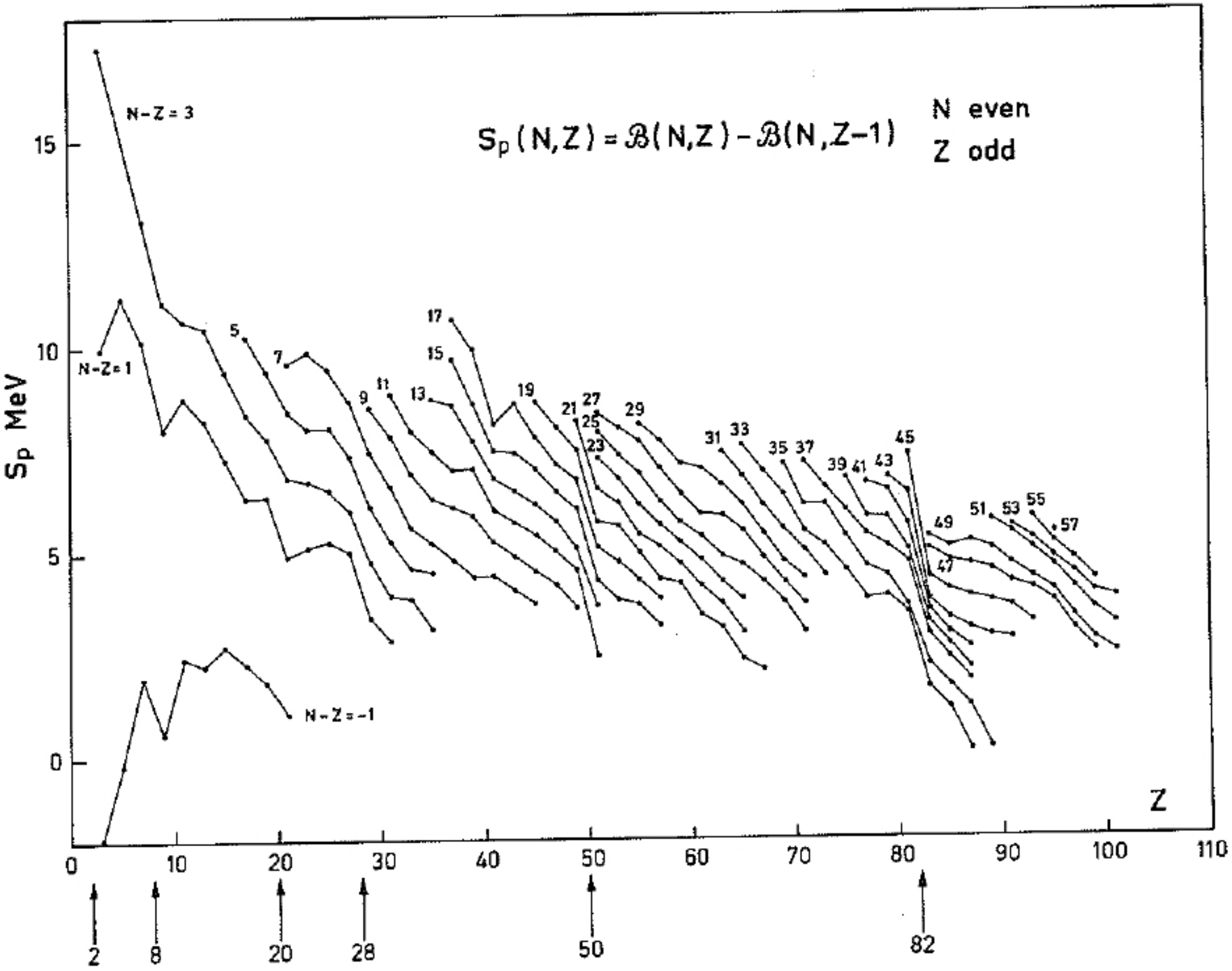
Ionization energy of atoms



Neutron separation energies



Proton separation energies



Liquid-drop mass formula

Binding energy of an atomic nucleus:

$$B(N,Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_s' \frac{(N-Z)^2}{A} + a_p \frac{\Delta(N,Z)}{A^{1/3}}$$

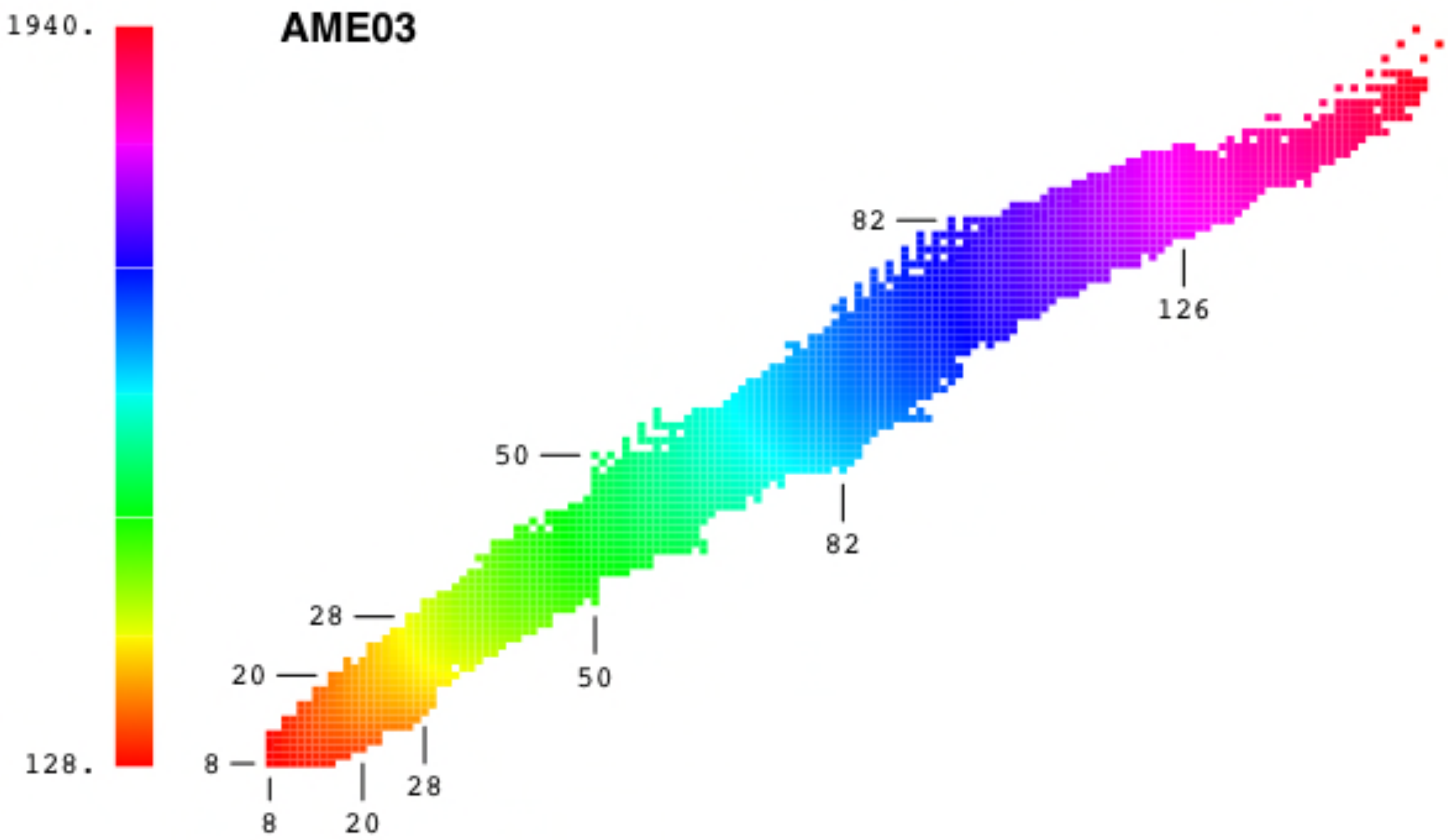
For 2149 nuclei ($N,Z \geq 8$) in AME03:

$a_v \approx 16$, $a_s \approx 18$, $a_c \approx 0.71$, $a_s' \approx 23$, $a_p \approx 6$

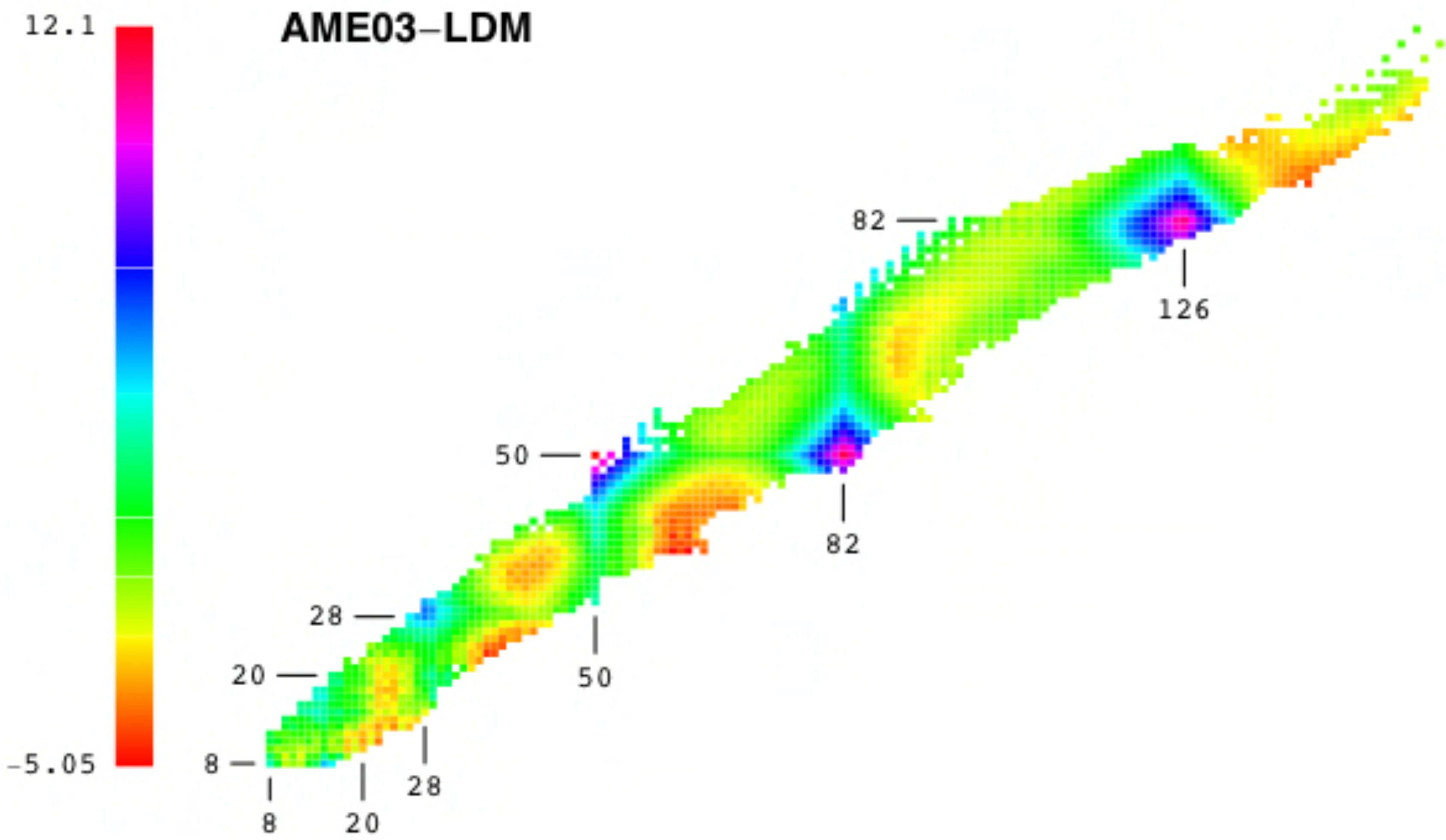
$\Rightarrow$ $\sigma_{rms} \approx 2.93$ MeV.

C.F. von Weizsäcker, Z. Phys. **96** (1935) 431
H.A. Bethe & R.F. Bacher, Rev. Mod. Phys. **8** (1936) 82

The nuclear mass surface



‘Unfolding’ of the mass surface



Liquid-drop mass formula

Binding energy of an atomic nucleus:

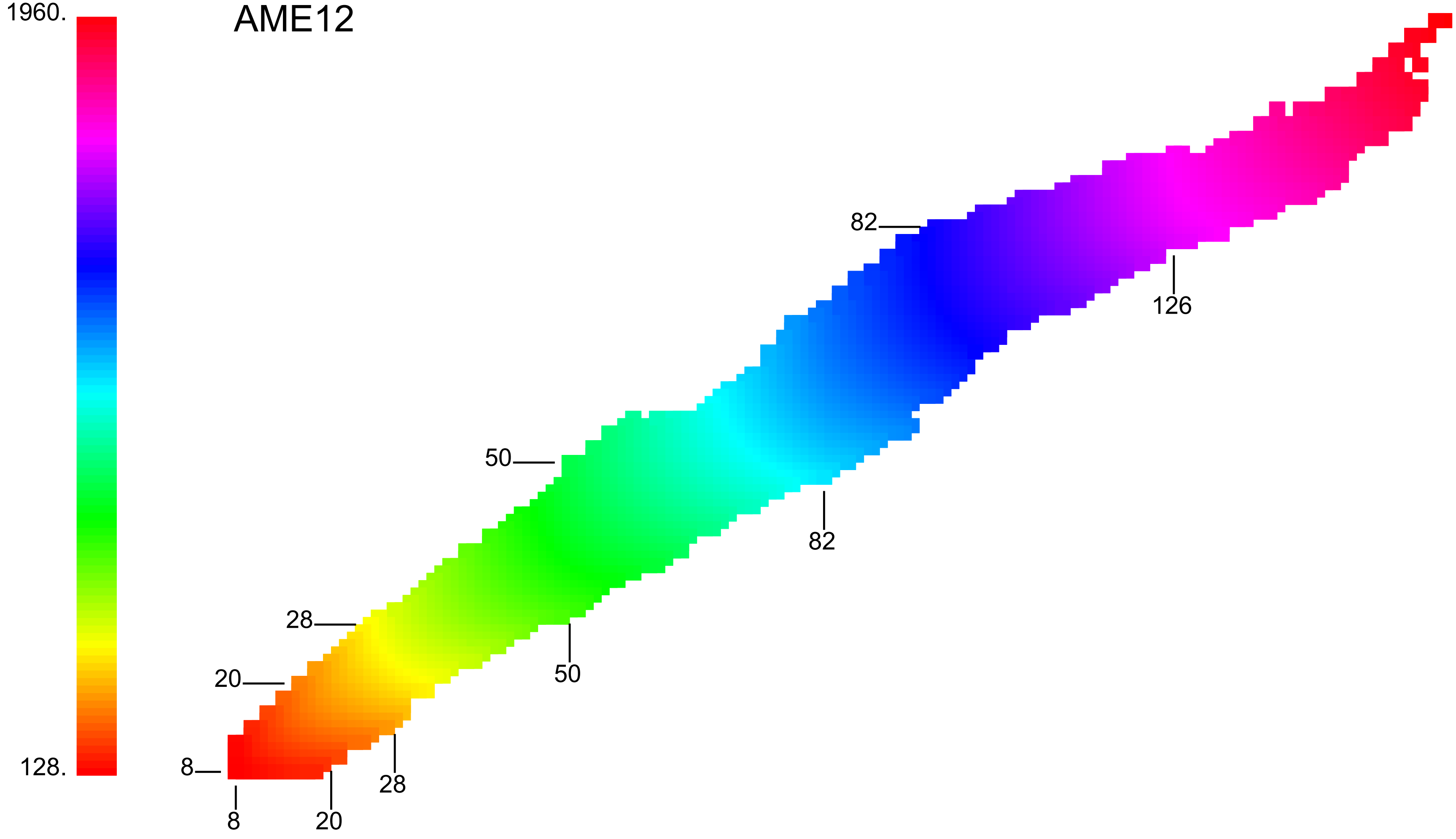
$$B(N,Z) = a_{\mathrm{v}} A - a_{\mathrm{s}} A^{2/3} - a_{\mathrm{c}} \frac{Z(Z-1)}{A^{1/3}} - a'_{\mathrm{s}} \frac{(N-Z)^2}{A} + a_{\mathrm{p}} \frac{\Delta(N,Z)}{A^{1/3}}$$

For 2353 nuclei ($N,Z \geq 8$) in AME12:

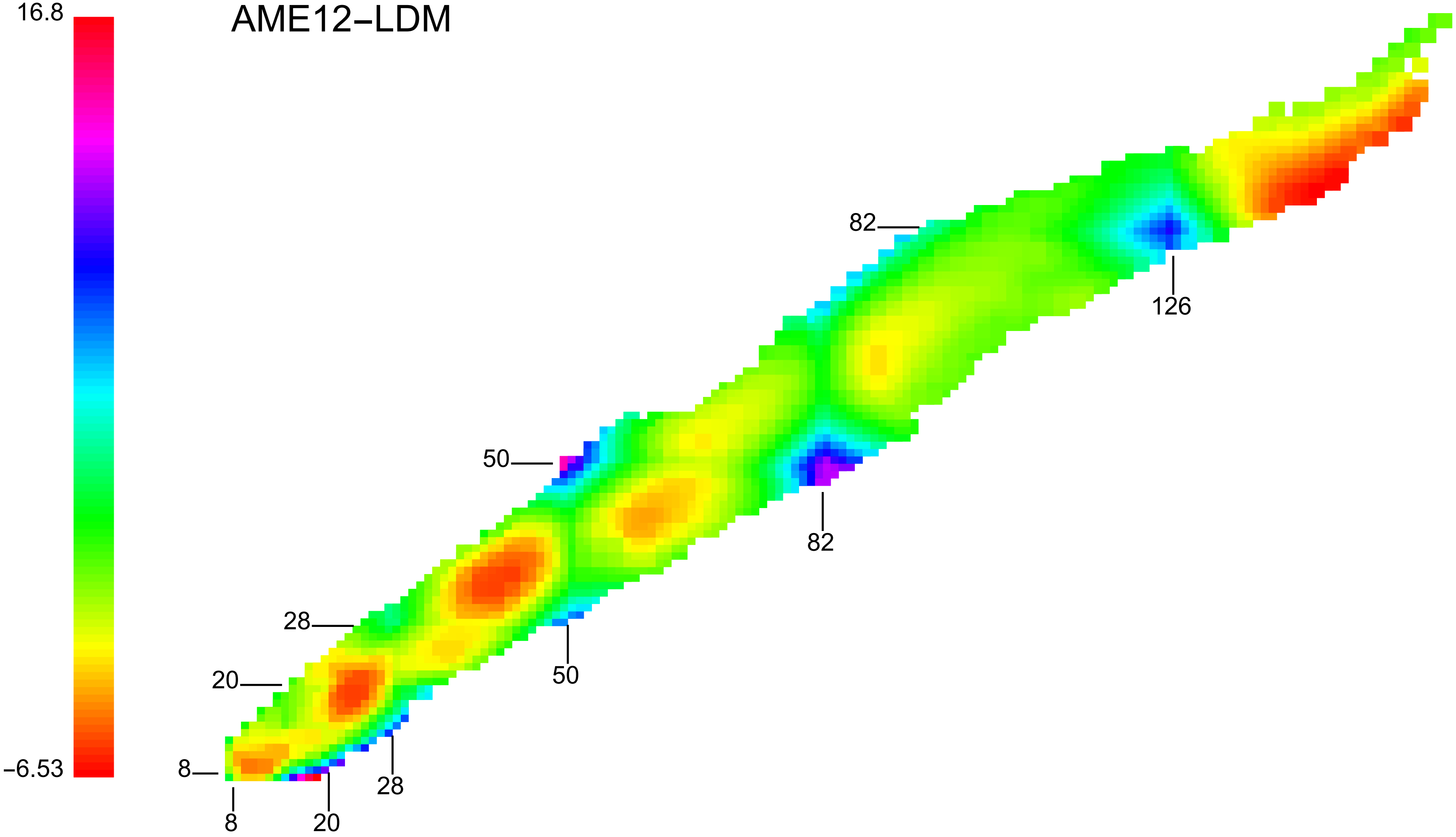
$a_{\mathrm{v}} \approx 15.7$, $a_{\mathrm{s}} \approx 17.9$, $a_{\mathrm{c}} \approx 0.713$, $a'_{\mathrm{s}} \approx 23.2$, $a_{\mathrm{p}} \approx 4.69$

$\Rightarrow \sigma_{\mathrm{rms}} \approx 3.10$ MeV.

AME12



AME12-LDM



Modified liquid-drop formula

Add surface, Wigner and 'shell' corrections:

$$B(N,Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} + a_p \frac{\Delta(N,Z)}{A^{1/3}}$$

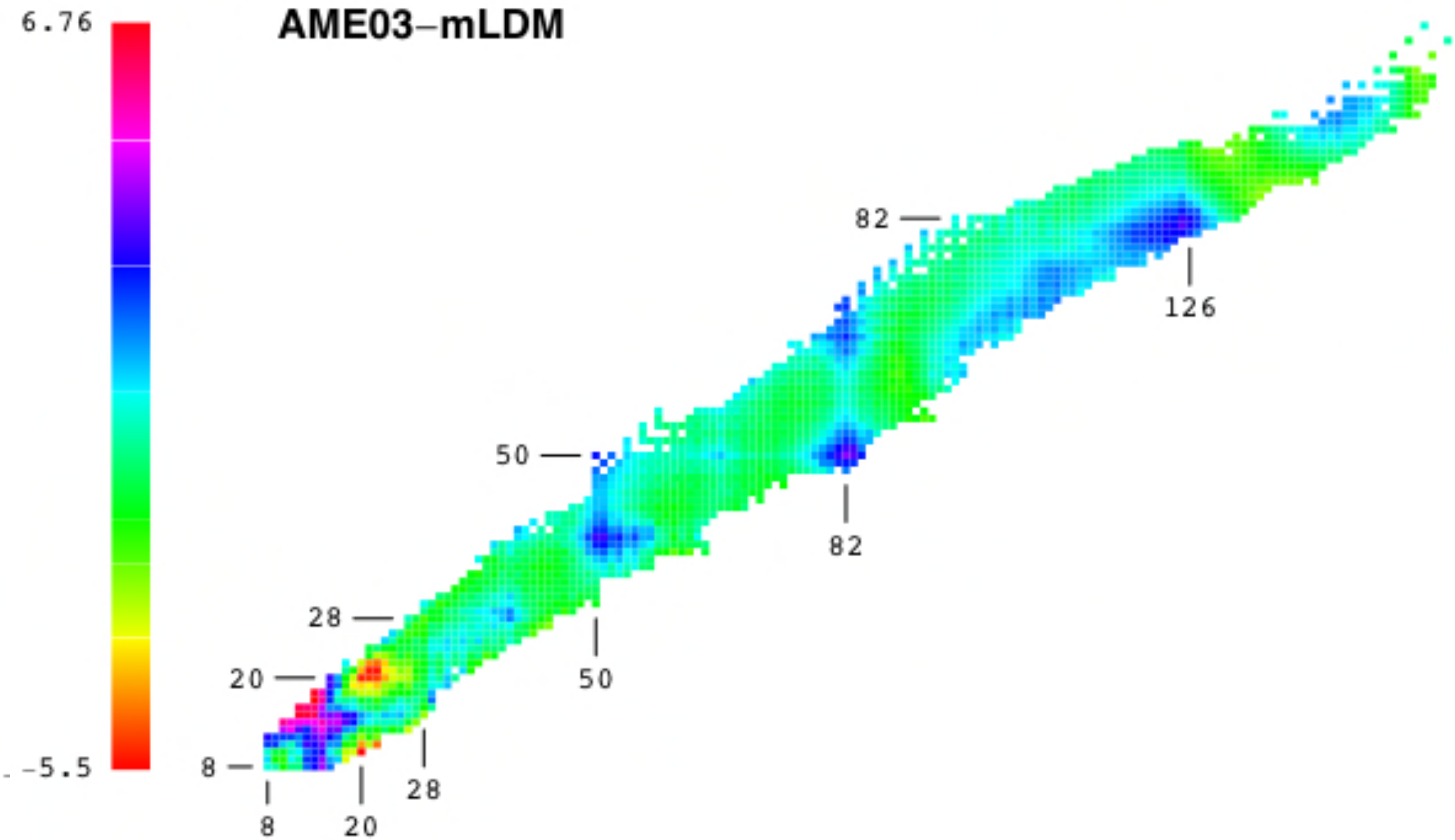
$$- \frac{S_v}{1 + y_s A^{-1/3}} \frac{4T(T+1)}{A} - a_f (n_\nu + n_\pi) + a_{ff} (n_\nu + n_\pi)^2$$

For 2149 nuclei ($N,Z \geq 8$) in AME03:

$a_v \approx 16$, $a_s \approx 18$, $a_c \approx 0.71$, $S_v \approx 35$, $y_s \approx 2.9$, $a_p \approx 5.5$, $a_f \approx 0.85$, $a_{ff} \approx 0.016$

$\Rightarrow \sigma_{rms} \approx 1.16$ MeV.

AME03-mLDM



Modified liquid-drop formula

Add surface, Wigner and 'shell' corrections:

$$B(N,Z) = a_{\mathrm{v}} A - a_{\mathrm{s}} A^{2/3} - a_{\mathrm{c}} \frac{Z(Z-1)}{A^{1/3}} + a_{\mathrm{p}} \frac{\Delta(N,Z)}{A^{1/3}}$$

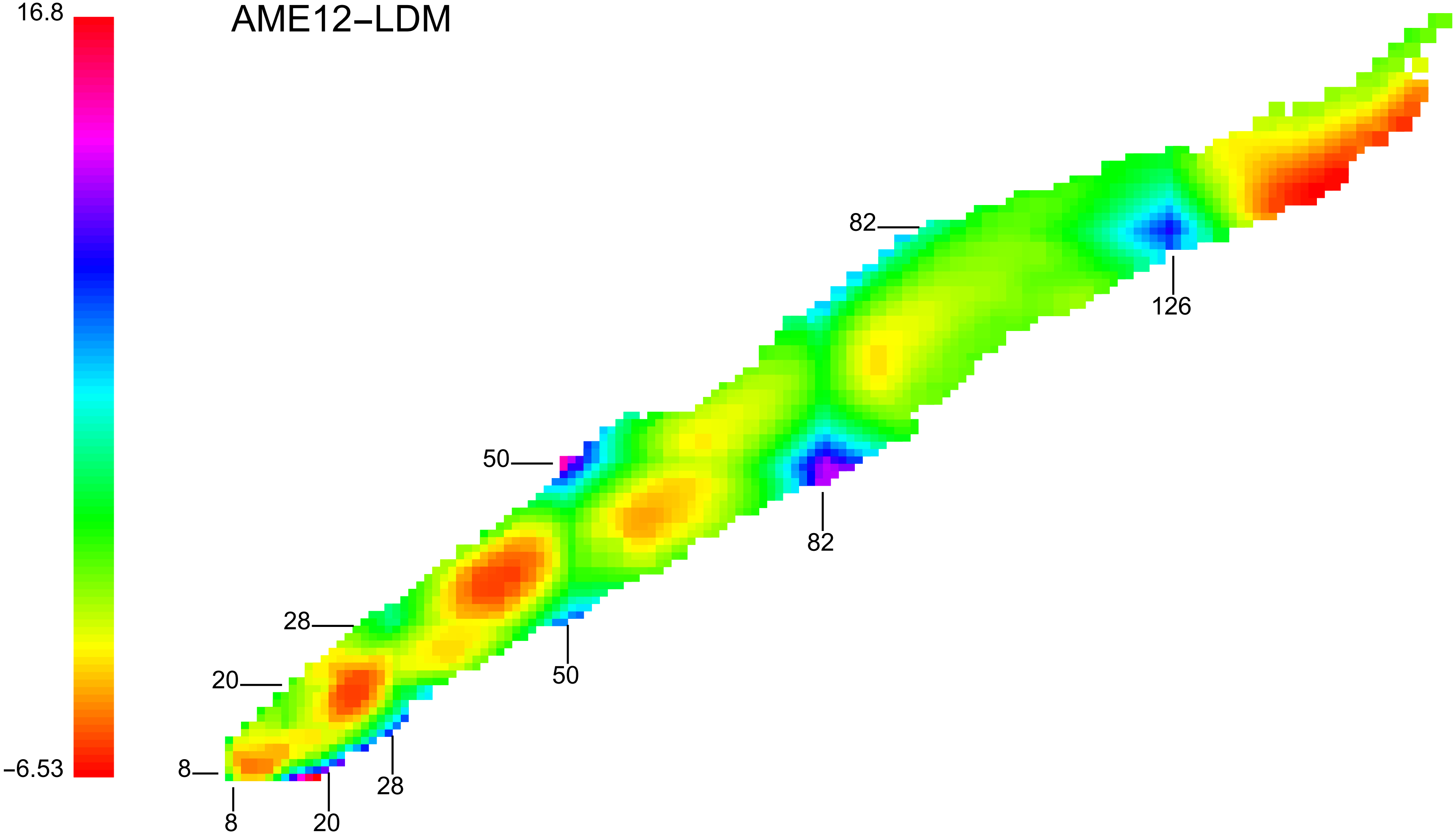
$$- \frac{S_{\mathrm{v}}}{1 + y_{\mathrm{s}} A^{-1/3}} \frac{4T(T+1)}{A} - a_{\mathrm{f}}(n_\nu + n_\pi) + a_{\mathrm{ff}}(n_\nu + n_\pi)^2$$

For 2353 nuclei ($N,Z \geq 8$) in AME12:

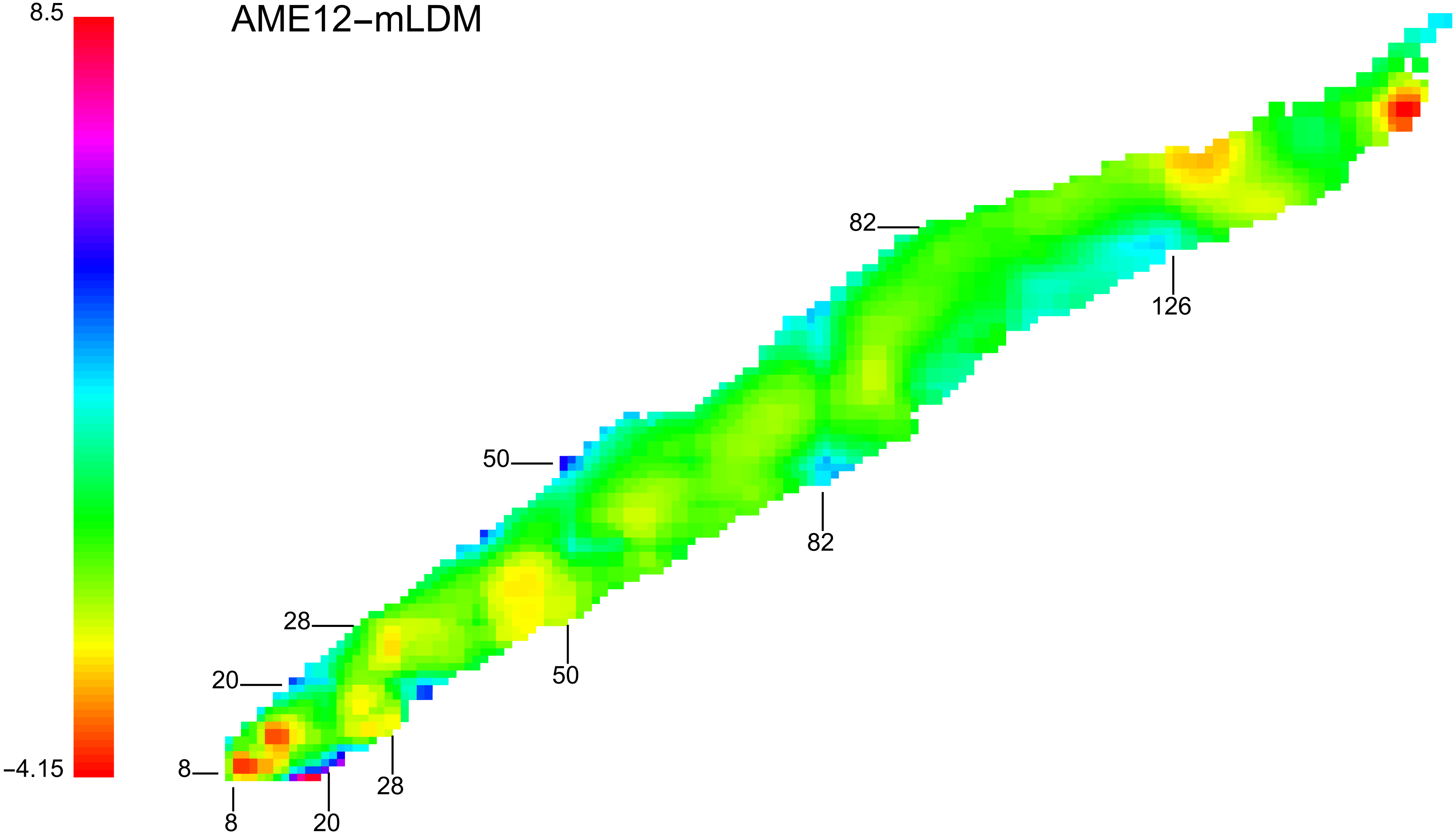
$a_v \approx 15.7$, $a_s \approx 17.6$, $a_c \approx 0.710$, $S_v \approx 33.7$, $y_s \approx 2.75$,
$a_p \approx 5.25$, $a_f \approx 0.86$, $a_{ff} \approx 0.016$

$\Rightarrow \sigma_{rms} \approx 1.19$ MeV.

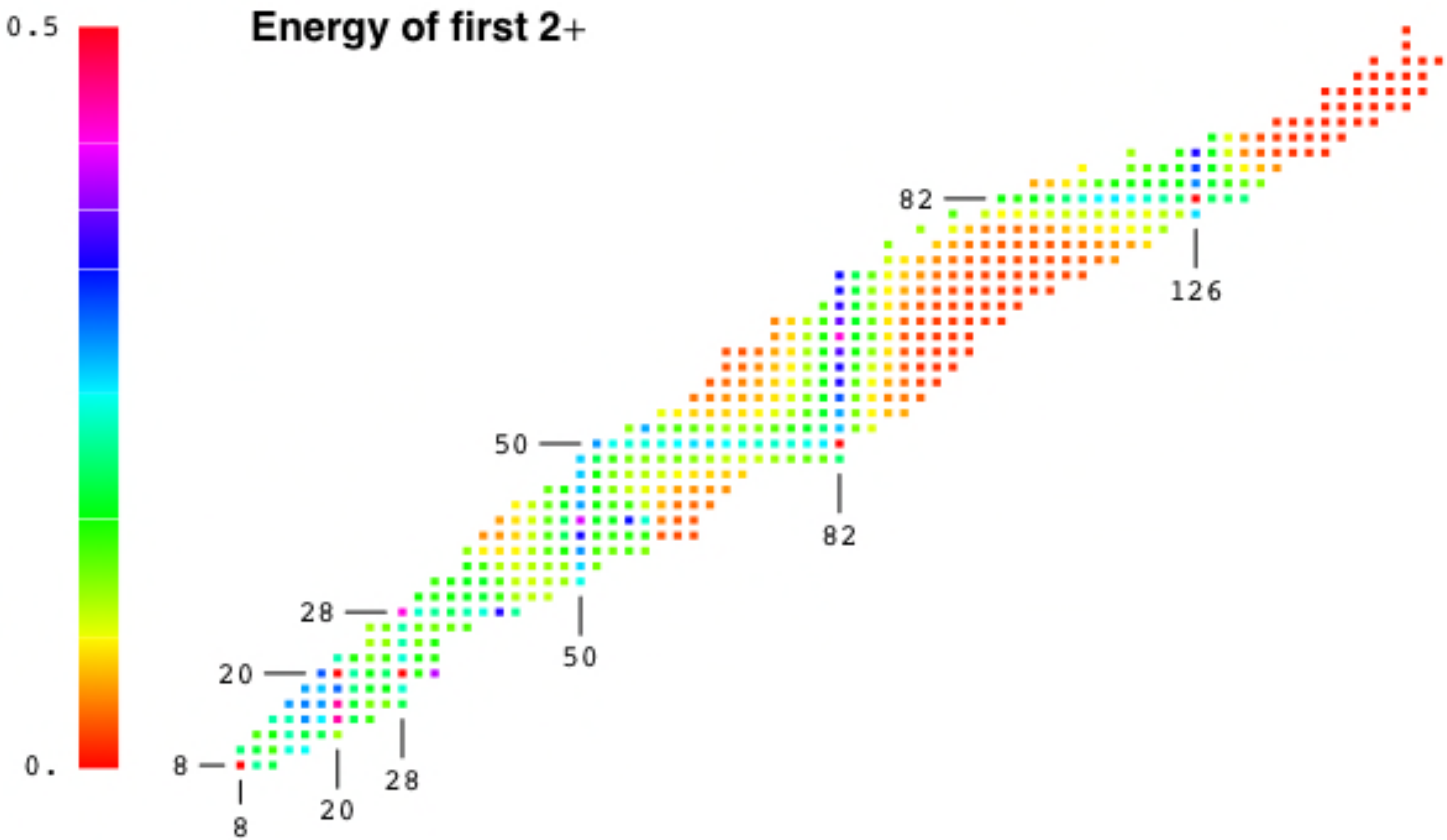
AME12-LDM



AME12–mLDM



Shell structure from $E_x(2_1)$



Nuclear shell model

The full shell-model hamiltonian:

$$\hat{H} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m} + \hat{V}(\boldsymbol{r}_k) \right] + \sum_{k<l}^{A} \hat{V}_{\mathrm{RI}}(\boldsymbol{r}_k, \boldsymbol{r}_l)$$

Valence nucleons: Neutrons or protons that are in excess of the last, completely filled shell.

Usual approximation: Consider the residual interaction V_{RI} among valence nucleons only.

Sometimes: Include selected core excitations ('intruder' states).

Residual shell-model interaction

Several approaches:

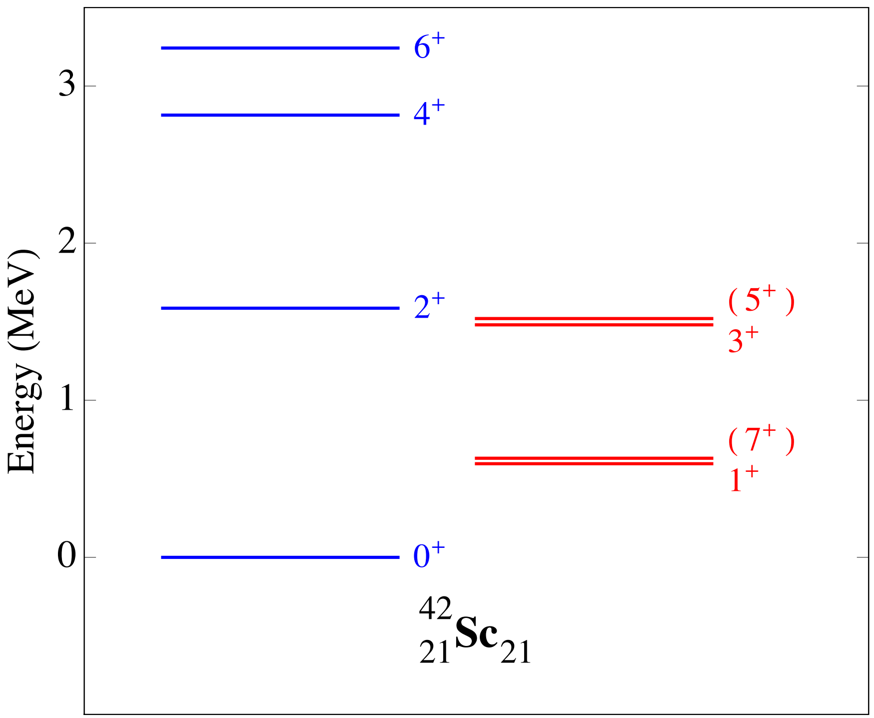
Microscopic: Derive from free nn interaction taking account of the nuclear medium.

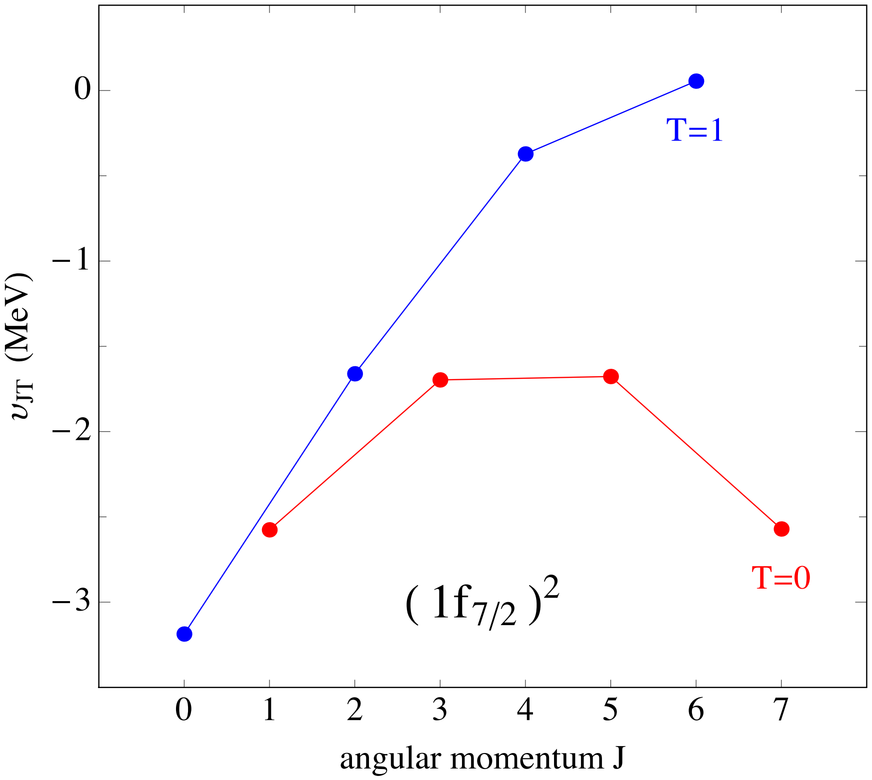
Empirical: Adjust matrix elements of residual interaction to data. Examples: p, sd and pf shells.

Microscopic-empirical: Effective interaction with some adjusted (monopole) matrix elements.

Schematic: Assume a simple spatial form and calculate its matrix elements in a harmonic-oscillator basis. Example: δ interaction.

Empirical nn interaction

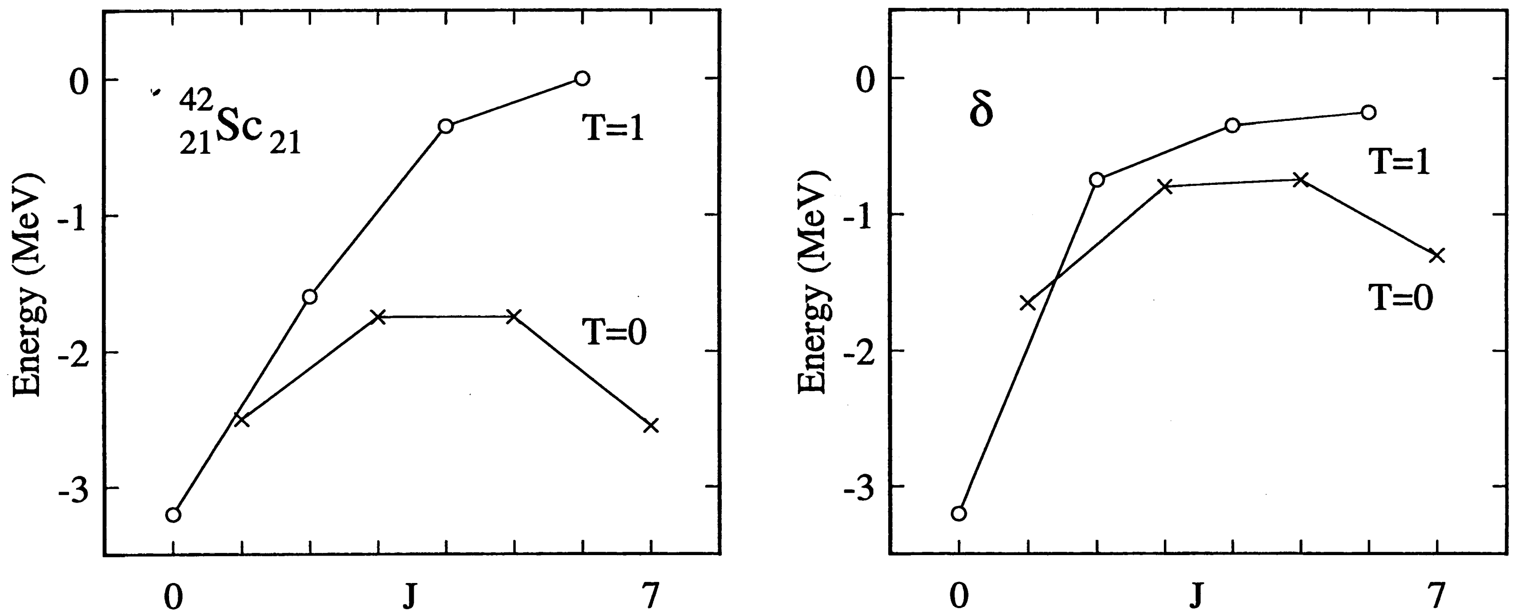




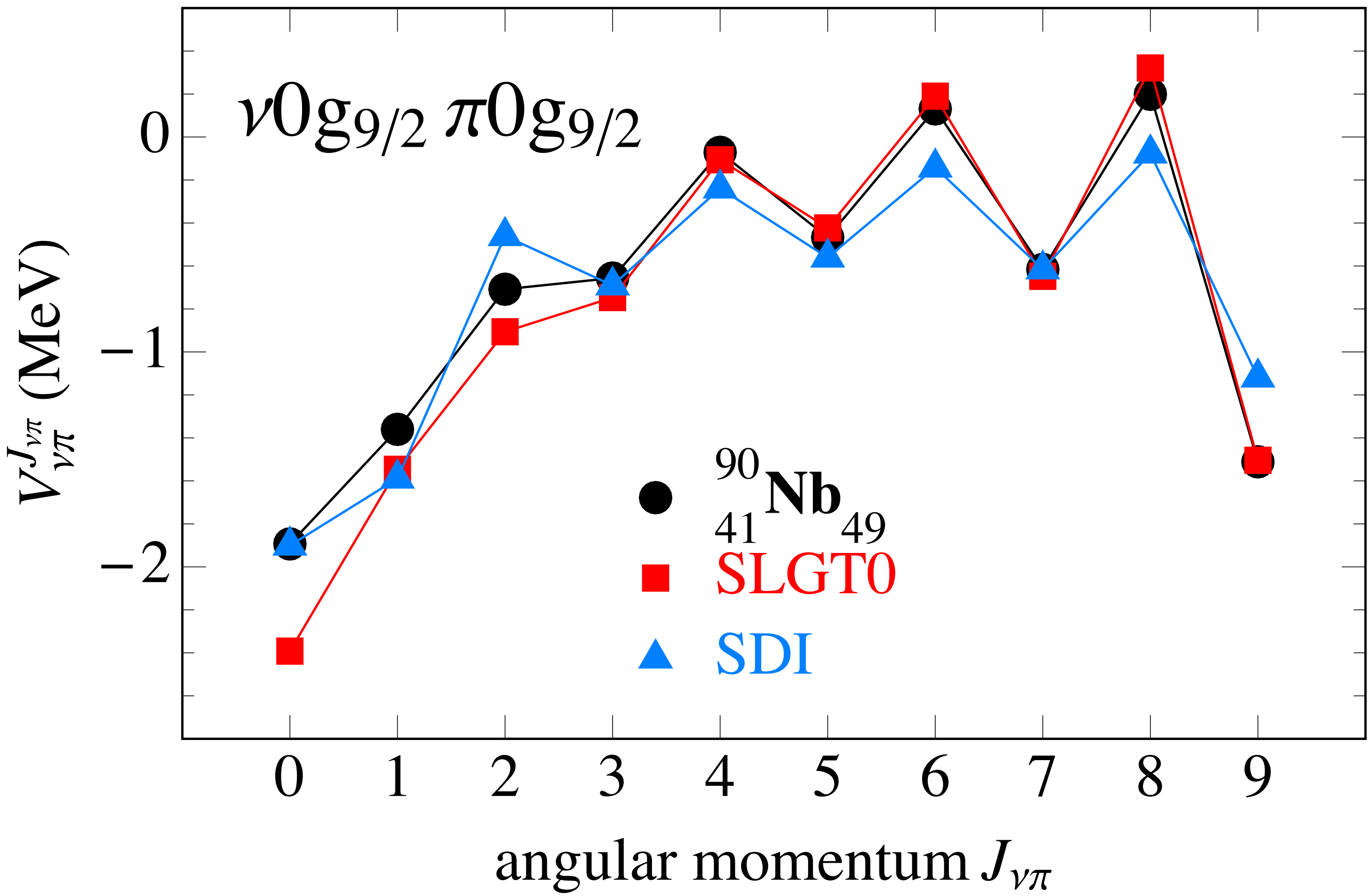
Schematic short-range interaction

Delta interaction in harmonic-oscillator basis:

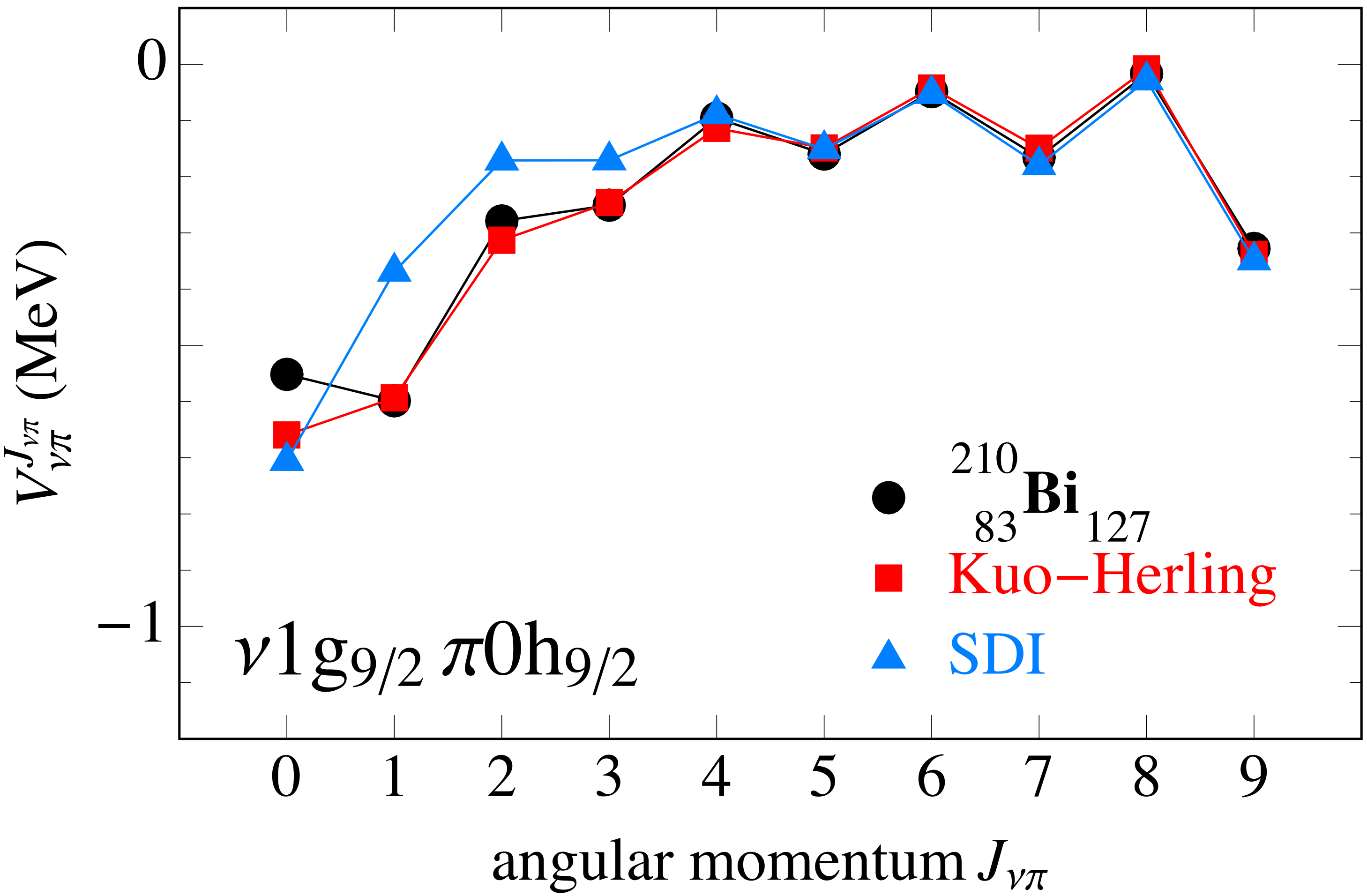
Example of ${}^{42}Sc_{21}$ (1 neutron + 1 proton).



The $0g_{9/2}$-$0g_{9/2}$ interaction



The $\nu 1g_{9/2}$-$\pi 0h_{9/2}$ interaction



Symmetries of the shell model

Three *bench-mark* solutions:

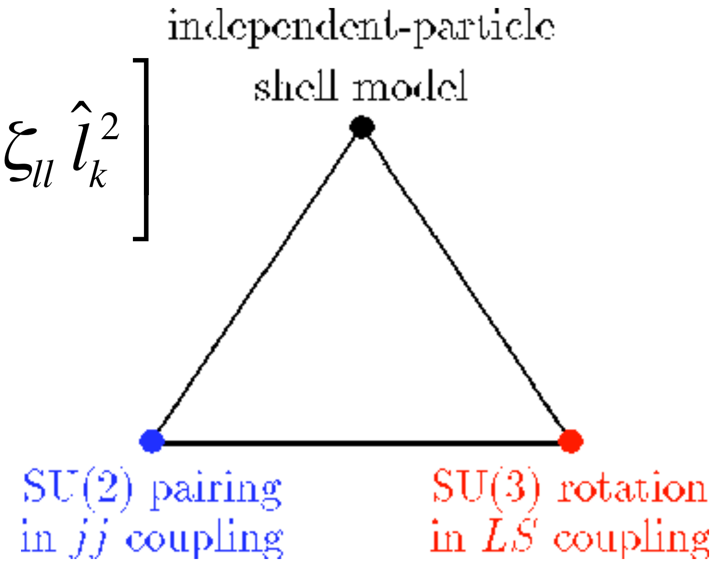
No residual interaction ⇒ IP shell model.

Pairing (in jj coupling) ⇒ Racah's SU(2).

Quadrupole (in LS coupling) ⇒ Elliott's SU(3).

Symmetry triangle:

$$\hat{H} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m} + \frac{1}{2} m\omega^2 r_k^2 - \zeta_{ls} \hat{\boldsymbol{l}}_k \cdot \hat{\boldsymbol{s}}_k - \zeta_{ll} \hat{l}_k^2 \right] + \sum_{1 \le k < l}^{A} \hat{V}_{\mathrm{RI}}(\boldsymbol{r}_k, \boldsymbol{r}_l)$$

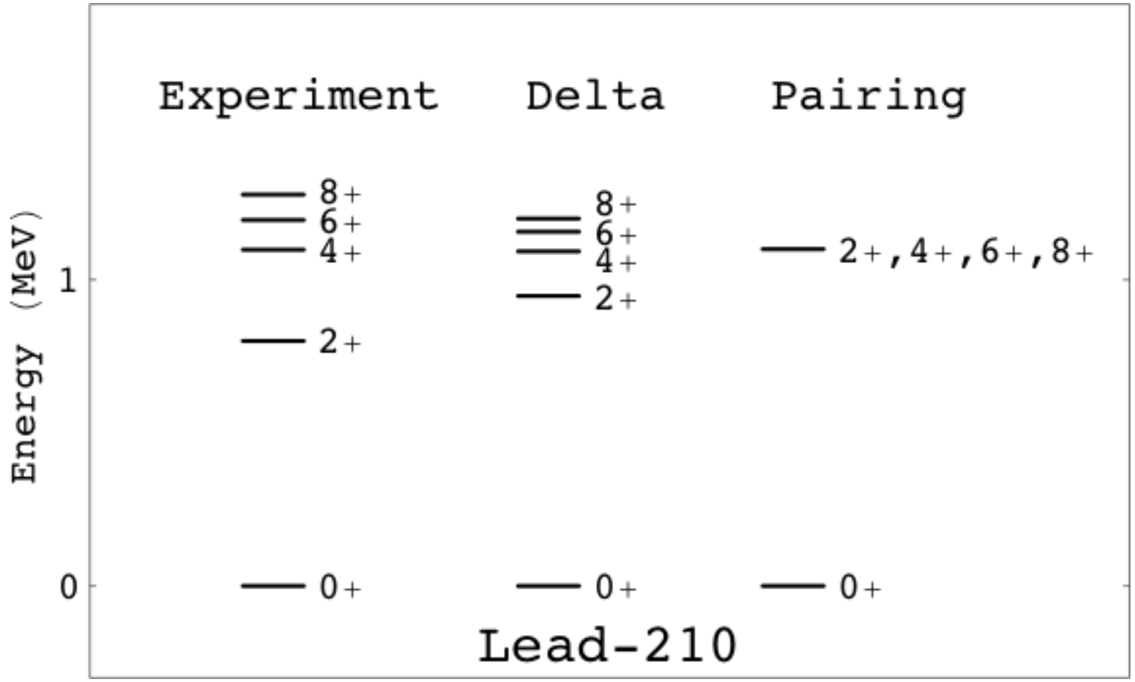


Racah's SU(2) pairing model

Assume pairing interaction in a single-*j* shell:

$$\left\langle j^2 J M_J \left| \hat{V}_{\text{pairing}}(\boldsymbol{r}_1,\boldsymbol{r}_2) \right| j^2 J M_J \right\rangle = \begin{cases} -\frac{1}{2}(2j+1)g_0, & J=0 \\ 0, & J\neq 0 \end{cases}$$

Spectrum ^{210}Pb:



Solution of the pairing hamiltonian

Analytic solution of pairing hamiltonian for identical nucleons in a single-*j* shell:

$$\left\langle j^n \upsilon J \left| \sum_{1 \le k < l}^{n} \hat{V}_{\text{pairing}}(\boldsymbol{r}_k, \boldsymbol{r}_l) \right| j^n \upsilon J \right\rangle = -g_0 \tfrac{1}{4}(n-\upsilon)(2j-n-\upsilon+3)$$

Seniority υ (number of nucleons not in pairs coupled to *J*=0) is a good quantum number.

Correlated ground-state solution (*cf.* BCS).

G. Racah, Phys. Rev. **63** (1943) 367

Pairing gap in semi-magic nuclei

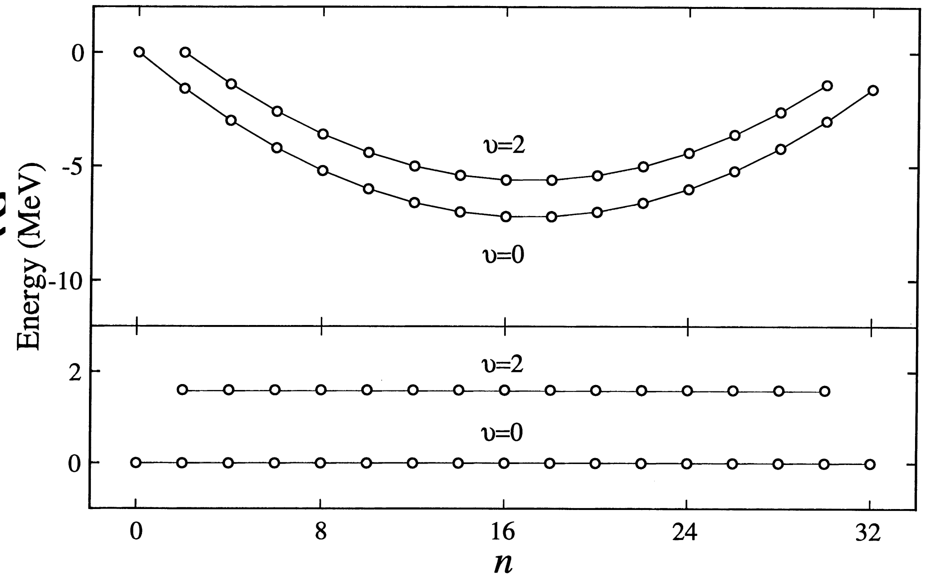
Even-even nuclei:

Ground state: υ=0.

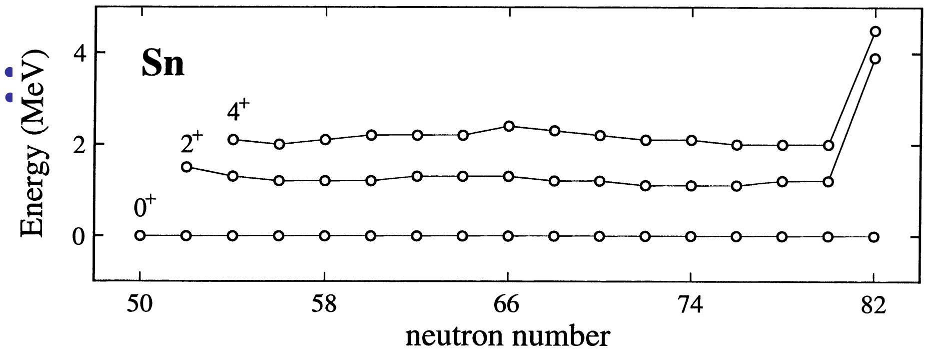
First-excited state: υ=2

Pairing produces constant energy gap:

$$E_x\left(2_1^+\right) = \tfrac{1}{2}(2j+1)G$$



Example of Sn isotopes:

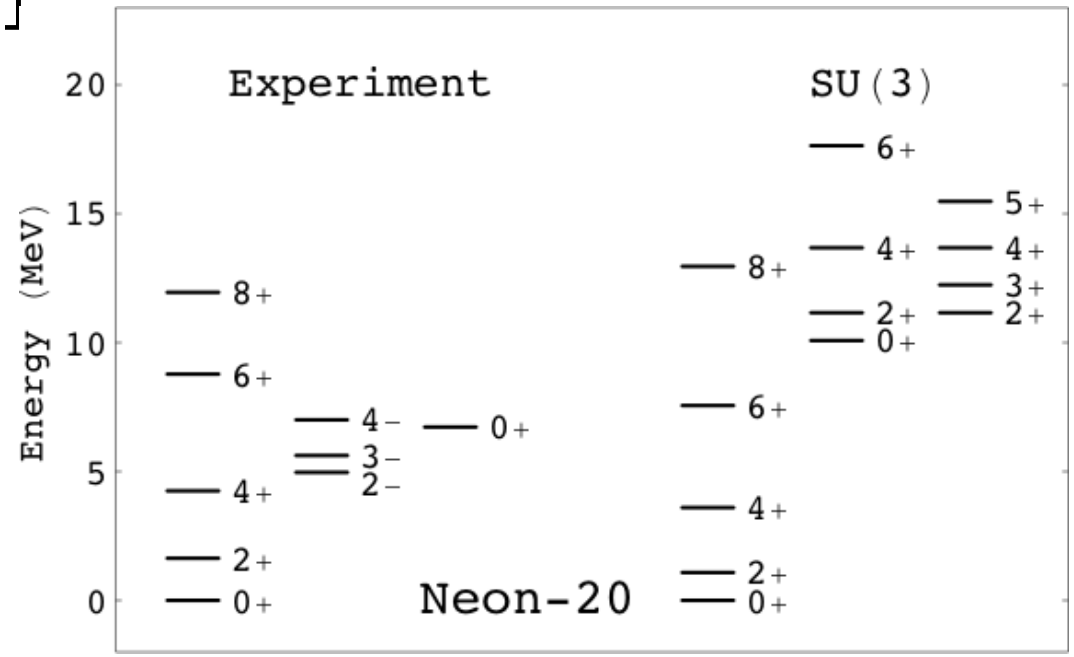


Elliott's SU(3) model of rotation

Harmonic oscillator mean field (*no* spin-orbit) with residual interaction of quadrupole type:

$$\hat{H} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m} + \frac{1}{2} m\omega^2 r_k^2 \right] - g_2 \hat{Q} \cdot \hat{Q},$$

$$\hat{Q}_\mu \propto \sum_{k=1}^{A} r_k^2 Y_{2\mu}(\hat{\boldsymbol{r}}_k) + \sum_{k=1}^{A} p_k^2 Y_{2\mu}(\hat{\boldsymbol{p}}_k)$$



J.P. Elliott, Proc. Roy. Soc. A **245** (1958) 128; 562

The three faces of the shell model