Symplectic Reduction: The pre-Frölicher Space Case

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AIMS-SENEGAL, Mbour, 24 June 2018

Outline

1. Differential spaces, Frölicher and pre-Frölicher spaces
2. Hamiltonian action and momentum mapping
3. Symplectic quotient of a pre-Frölicher space
4. Non-equivariant case
5. Hereditary properties on the quotient

Differential spaces
Why differential spaces?

For a smooth structure without charts.

Definition

A differential structure on a topological space (M, τ_M) is a collection $\mathcal{D}$ of real-valued functions satisfying the following conditions:

(1) The topology τ is induced by the family $\{f^{-1}(I)|f \in \mathcal{D} \text{ and } \mathrm{I} \in \tau_{\mathbb{R}}\}$.

(2) (Smooth composition) If $f_1, ..., f_n \in \mathcal{D}$, and $\omega \in C^\infty(\mathbb{R}^n)$ for any positive integer n, then $\omega \circ (f_1, ..., f_n) \in \mathcal{D}$.

(2) (Localization) If $f : M \longrightarrow \mathbb{R}$ is such that, for every $x \in M$, there exist an open neighborhood U of x and a function $f_x \in \mathcal{D}_0$ satisfying $f_x|U = f|U$, then $f \in \mathcal{D}$.

The structure can be generated by a finite set $\mathcal{D}_0$ of functions $\alpha_1, ..., \alpha_k$ $(k < n)$, which yield $\mathcal{D}$ following the three conditions above, and are then called *generators* of the structure. See contribution to the area in:

- Aronszajn, N., Notices of the AMS, **14**, III (1967)
- Spallek, K., Math. Annals, **180**, 269-296 (1969)
- Sikorski, R., Introduction to Differential Geoometry (in Polish), Scientifiic Publishers (1972).
- Sasin, W., Demonstratio Mathematica, **19**(4),1063-1075 (1986).
- Marshall, D., Journal of Differential Geometry, **10**, 551-574 (1975)
- Mostov, M.A., Journal of Differential Geometry, **14**, 255-293 (1979)
- Sniatycki, J., Differential Geometry of Singular Spaces and Reduction of Symmetry, Cambridge Press (2013).

Canonical examples

- The Euclidean space $\mathbb{R}^n$, with the standard topology and $\mathcal{D}_{\mathbb{R}^n} = C^\infty(\mathbb{R}^n)$, the usual smooth functions in calculus.
- A smooth manifold M. The differential structure is given by the ring of smooth functions on M.
- A set M together with a trivial topology. The differential structure is the set of constant functions. For the discrete topology, all functions form the differential structure on M.

Definition

$F : M \longrightarrow N$ is said to be smooth map of the differential space $(M, \mathcal{D}_M)$ into the differential space $(N, \mathcal{D}_N)$ if, for any $f \in \mathcal{D}_N$, $f \circ F \in \mathcal{D}_M$.

The category of differential spaces proved to fail the Cartesian closedness property (See Cherenack P., 2000).

Frölicher spaces

Question/ Why Frölicher spaces? Answer/To extend, in a canonical way, the smooth structure from smooth spaces to the set of smooth mappings between them.

Let $\mathbb{R}^M$ and $M^{\mathbb{R}}$ the sets of scalar functions (outputs) on a set M and the set of contours (curves, inputs) into M, respectively. Let $\mathcal{P}(\mathbb{R}^M)$ and $\mathcal{P}(M^{\mathbb{R}})$ be their power sets ordered by inclusion. Denote them by $\mathbf{C_f}$ and $\mathbf{C_c}$ respectively.
Now, let $\Gamma : \mathbf{C_f} \longrightarrow \mathbf{C_c}$ and $\Phi : \mathbf{C_c} \longrightarrow \mathbf{C_f}$ be given by

$$\Gamma(\mathcal{F}) = \{c : \mathbb{R} \longrightarrow M; f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all f} \in \mathcal{F}\},$$

$$\Phi(\mathcal{C}) = \{f : M \longrightarrow \mathbb{R}; f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all c} \in \mathcal{C}\},$$

That is, any set $\mathcal{F}$ of functions on M determines a set $\Gamma\mathcal{F}$ of curves. Similarly, any set $\mathcal{C}$ of curves determines a set $\Phi\mathcal{C}$ of functions.

Definition-Examples

Definition

A Frölicher space is a triple $(M, \mathcal{C}_M, \mathcal{F}_M)$, where $\mathcal{C}_M \in \mathcal{P}(M^{\mathbb{R}})$ and $\mathcal{F}_M \in \mathcal{P}(\mathbb{R}^M)$ satisfy the Frölicher condition

$$\Gamma\mathcal{F}_M = \mathcal{C}_M, \qquad \Phi\mathcal{C}_M = \mathcal{F}_M. \qquad \textbf{(FC)}$$

A set mapping φ between Frölicher spaces $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(N, \mathcal{C}_N, \mathcal{F}_N)$ is smooth if $\varphi_*\mathcal{C}_M \subseteq \mathcal{C}_N$, or equivalently $\varphi^*\mathcal{F}_N \subseteq \mathcal{F}_M$. It is important to notice that the Frölicher (smooth) struture is a pair of mapping sets, contrary to most smooth structures in today's comparative smootheology. The Frölicher condition is a maximality condition as $\Gamma\Phi\Gamma = \Gamma$ and $\Phi\Gamma\Phi = \Phi$ which can be easily verified.

As for differential spaces,

- the Euclidean space $\mathbb{R}^n$ is a Frölicher space. (A. Frölicher, 1965, J. Boman, 1967).
- smooth manifolds are Frölicher spaces through their atlases.
- it is easy to prove that the composition of smooth mappings is a smooth mapping, so that Frölicher spaces and smooth mappings between them form the so-called category of Frölicher spaces.
- the set denoted by $\underline{\mathcal{M}}(Y, Z)$ of smooth mappings between Frölicher spaces is endowed with a Frölicher structure in a canonical way. That, the set of curves is

$$\mathcal{C}_{\mathcal{M}(Y,Z)} := \{c : \mathbb{R} \to \underline{\mathcal{M}}(Y, Z);\ \tilde{c} : \mathbb{R}\Pi Y \to Z \text{ is a smooth map}\},$$

where $\tilde{c}(s, t) := c(s)(t)$. Then $\Phi\mathcal{C}_{\mathcal{M}(Y,Z)}$ yields the structure functions on $\mathcal{M}(Y, Z)$. Thus, $(\Gamma\Phi\mathcal{C}_{\mathcal{M}(Y,Z)}, \Phi\mathcal{C}_{\mathcal{M}(Y,Z)})$ is the smooth structure on the mapping space.

Definition

Let $(M, \mathcal{C}, \mathcal{F})$ be a Frölicher space. An operational tangent vector at $p \in M$ is a derivation $v : \mathcal{F} \longrightarrow \mathbb{R}$. The set of all tangent vectors on M at x is the tangent space denoted by T_pM. It is clearly a linear space. Now, let $a \in \mathbb{R}$ and $c \in \Gamma\mathcal{F}$ such that $c(a) = p$. A kinematic tangent vector at p is defined by

$$v_c(f) = \lim_{t \to a} \frac{f \circ c(t) - f \circ c(a)}{t - a}, \quad f \in \mathcal{F}.$$

We denote by T_pCM the set of all kinematic tangent vectors on M at p. Clearly, T_pCM is a cone and may not be a linear space. The disjoint union of tangent spaces defines the tangent bundle TM or TCM on M, the sections of which are vector fields.The exterior operators are defined in the usual way and induce the Cartan's exterior calculus similar to that of smooth manifolds.

Initial and final objects in the cateory of Frölicher spaces

1. **Product of Frölicher spaces**: Let $(M_i, \mathcal{C}_i, \mathcal{F}_i)_{i\in I}$ be a collection of Frölicher spaces. Then the Cartesian product $P = \Pi_{i\in I}$ has a smooth structure generated by the collection $\mathcal{F}_{P^*} = \cup_{i\in I}\{f_i \circ \pi_i;\ f_i \in \mathcal{F}_i,\ i \in I\}$, in which the set of structure is easily seen to be $\mathcal{C}_P = \{c : \mathbb{R} \longrightarrow P;\ \mathrm{if\ c(t) = (c_i(t))_{i\in I},\ c_i} \in \mathcal{C}_\mathrm{i}\}$, and $\mathcal{F}_P = \Phi\Gamma\mathcal{F}^*_P$.
2. **Frölicher Subspaces**: Let $(M, \mathcal{C}, \mathcal{F})$ be a Frölicher space, and $A \subset M$. Then $(A, \mathcal{C}_A, \mathcal{F}_A)$ is a Frölicher subspace of $(M, \mathcal{C}, \mathcal{F})$, where

$$\begin{aligned} \mathcal{C}_A &= \{c : \mathbb{R} \longrightarrow A;\ i_A \circ c \in \mathcal{C}_M\}, \\ \mathcal{F}_A &= \Phi\Gamma\{f \circ i_A;\ f \in \mathcal{F}_M\}. \end{aligned}$$

The smooth structure on TM comes from that $TM \subset M \times \mathcal{D}_M$, where $\mathcal{D}_M = \{D : \mathcal{F}_M \longrightarrow \mathbb{R}\}$, D a

1. **Coproduct of Frölicher spaces.** The coproduct of Frölicher spaces has as structure functions the restrictions $f_{|M_i}$ of the structure functions on the copies $(M_i, \mathcal{C}_i, \mathcal{F}_i)$.
2. **Quotient by an equivalence relation.** The quotient Frölicher space on $M/\mathcal{R}$ by an equivalence relation has structure functions the mappings $\tilde{f} : M/\mathcal{R} \longrightarrow \mathbb{R}$ such that $\tilde{f} \circ \pi \in \mathcal{F}_M$.

It follows form the structure that the curves on $\mathbb{Q}$ are constant functions, and that the tangent cone spaces reduce to $\{0\}$. Similarly, the tangent spaces are trivial since the constant curves induce a structure where all functions from $\mathbb{Q}$ must be smooth. Nevertheless, $\mathbb{Q}$ as a differential subspace of $\mathbb{R}$ has structure functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ which are, locally in the usual topology, the restrictions of local smooth functions on $\mathbb{R}$. Thus, the tangent space to a point $q \in \mathbb{Q}$ is the same as the tangent space when q is regarded as a point in $\mathbb{R}$, and is one-dimensional.

Pre-Frölicher spaces

Q/ Why pre-Frölicher spaces?

A/They form a reflexive subcategory of differential spaces that is Cartesian closed and whose associated Frölicher spaces have non-trivial geometry.

Lemma

Let $(M, \mathcal{C}, \mathcal{F})$ be a Frölicher space. Then, $(M, \mathcal{F})$ is a differential space. (P. Cherenack, 2000)

Lemma

Let M be a nonempty set. Let $\mathcal{D}_0 \subseteq \mathbb{R}^M$. Then $\mathcal{D}_0 \subseteq \Phi\Gamma\mathcal{D}_0$.

If $\mathcal{D}$ is a Sikorski differential structure on M, then $\mathcal{D} \subseteq \Phi\Gamma\mathcal{D}$ always occurs for the associated Frölicher structure, and the tangent bundles and differential geometry on $(M, \mathcal{D})$ and $(M, \Gamma\mathcal{D}, \Phi\Gamma\mathcal{D})$ do not always coincide, on the rationals $\mathbb{Q}$ for instance.

Therefore, one needs a class of smooth spaces which are Sikorski differential spaces and can be made Frölicher in such a way that they preserve the Sikorski differential structure; that is, that $\mathcal{D}$ is $\Phi\Gamma$- invariant. This gives smooth spaces (1) with the relative topology on subspaces, (2) holding same differential geometry in becoming a Frölicher space, (3) with a Frölicher smooth structure extending to sets of smooth mappings between them, (4) where differential forms and vector fields can be defined in a way similar to smooth manifolds.

Definition

Let M be a nonempty set. A pre-Frölicher (**pF**-) space is a triple $(M, \mathcal{D}, \tau)$, where M is a nonempty set, $\mathcal{D}$ a family of real-valued functions on M inducing the weakest topology in which they are continuous, and satisfying the following conditions:

(1) If $\alpha_1, ..., \alpha_k \in \mathcal{D}$, and $\omega \in C^\infty(\mathbb{R}^k)$ for any positive integer k, then $\omega(\alpha_1, ..., \alpha_k) \in \mathcal{D}$.

(2) If $f : M \longrightarrow \mathbb{R}$ is such that, for every $x \in M$, there exist an open neighborhood U of x and a function $f_x \in \mathcal{D}$ satisfying $f_x|U = f|U$, then $f \in \mathcal{D}$.

(3) $\Phi\Gamma\mathcal{D} = \mathcal{D}$,
where $\Gamma\mathcal{D} = \{c : \mathbb{R} \longrightarrow M;\ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\ \forall f \in \mathcal{D}\}$,
$\Phi\Gamma\mathcal{D} = \{f : M \longrightarrow \mathbb{R};\ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\ \forall c \in \Gamma\mathcal{D}\}$.

Consequently, $\mathbb{Q}$ is not a pre-Frölicher space although it is both Frölicher and Sikorski space. Condition (3) is not satisfied.

Theorem

Let $(M, \mathcal{D})$ be a set and N be a **pF***-space. Let $(\mathcal{C}, \mathcal{D})$ be the Frölicher structure induced on M by means of maps $f_i : M \longrightarrow N$, $i \in I$ so that $\varphi : M \longrightarrow N^I$; $\varphi(x) = (f_i(x))_I$, 1-1. Then φ is a diffeomorphism onto $\varphi(M)$ of N^I [see Batubenge, 2004, Batubenge-Tshilombo 2016].*

Definition

A pre-Frölicher space is locally diffeomorphic to $\mathbb{R}^n$ (or locally Euclidean), for some positive integer n, if at each point $p \in M$ there is an open neighborhood U_p of p in M and a diffeomorphism $\varphi : U_p \longrightarrow \varphi(U_p) \subseteq \mathbb{R}^n$.

Here the openness of $\varphi(U_p)$ is not guaranteed, and n is not fixed.

Example

Let $X = [0, \pi)$. Consider the map given on X by setting $\varphi(x) = (-cos\ x, k(x))$, where $k(x) = -1$ for all $x \in X$. The function $-cos\ x$ is point-separating in X so that the map φ is an $\mathbb{F}$-diffeomorphism of $[0, \pi)$ onto the interval $J = \varphi[X] \subset \mathbb{R}^2$, which is neither open, nor closed.
More generally, one can consider $Y = [0, \infty)$ and the map defined on Y by setting $\varphi(x) = (id(x), \theta(x))$ for all $x \in Y$, where $\theta(x) = 0$.

Theorem

Let M be a pre-Frölicher space that is locally diffeomorphic to $\mathbb{R}^n$. Then there is an atlas of local charts on M and a partition of unity subordinated to this atlas [see Batubenge-Tshilombo, 2016].

The **F**-chart mappings are Frölicher diffeomorphisms, i.e. $\{\varphi := (f_i)_i\}$, where $f_i \in \mathcal{F}_M$, with one of them separating points in M, and the maximum number of them being the dimension of the tangent space at the point under consideration.

Smooth action by a Frölicher Lie group.

In what follows, we shall call smooth space a pre-Frölicher space wherever there is no fear of confusion, and smooth group for Frölicher Lie group in the following sense.

A **Frölicher Lie group** is an abstract group G which is also a smooth space such that the group multiplication $G \times G \to G; \quad (x, y) \mapsto xy$, and inversion $G \to G; \quad x \mapsto x^{-1}$, are smooth in the category of Frölicher spaces).
The vector space $\mathfrak{g} = T_e G$ is called the Lie algebra of the corresponding Lie group G, where $e \in G$ is the identity element.
Note that for each $g \in G$, $\Phi_g : X \to X$ defined by $\Phi_g(x) = \Phi(g, x)$ is a diffeomorphism. In the category of pre-Frölicher spaces, the Lie bracket is defined in the usual sense.

Let M be a smooth space. A symplectic structure on M is a 2-form $\omega \in \Omega^2(M)$ which satisfies the following two conditions:

(i) ω is closed. That is, $d\omega = 0$.

(ii) ω is nondegenerate. That is, on each tangent space T_mM, $m \in M$, if $\omega_m(X, Y) = 0$ for all $Y \in T_mM$, then $X = 0$.

A smooth space M is called a symplectic if there is defined on M a closed 2-form ω which is nondegenerate. The pair (M, ω) will then be called a symplectic space.

Let $\Phi : G \times M \to M$, $(g, m) \mapsto \Phi_g(m) = g \cdot m$ be an action of a Lie group G on a symplectic space (M, ω). Then the action Φ is called symplectic if for each $g \in G$, the diffeomorphism $\Phi_g : M \to M$, $m \mapsto \Phi_g(m)$ is such that $\Phi_g^* \omega = \omega$.

Let G be a smooth group and let Φ be a smooth action of G on a smooth space M. Let $\mathfrak{g}$ be the Lie algebra of G. We define the infinitesimal generator of the action Φ corresponding to $X \in \mathfrak{g}$ to be $X_M(m) = \frac{d}{dt} \Phi_{\exp tX}(m) \mid_{t=0}$ where $\exp : \mathfrak{g} \to G$ is the exponential map.

Definition

Let $\Phi : G \times M \to M$, be a symplectic action of a smooth group G on a symplectic space (M, ω), and let X_M be the infinitesimal generator of the action corresponding to $X \in \mathfrak{g}$. Then the map $\mu : M \to \mathfrak{g}^*$ is called the **momentum mapping** for the action if for every $X \in \mathfrak{g}$ there is a function $\hat{\mu}_X : M \to \mathbb{R}$ such that the relation $\hat{\mu}_X(m) = \mu(m) \cdot X$ holds, and where $d\hat{\mu}_X = i_{X_M}\omega$. Such an action is said to operate on M in *Hamiltonian fashion* and the space (M, ω, Φ, μ) is called a Hamiltonian G-space.

A momentum mapping μ is said Ad^*-**equivariant** provided $\mu(\Phi_g(x)) = Ad^*_{g^{-1}}\mu(x)$, for every $g \in G$.

In the case of non-equivariance, the following techniques can be used to make it equivariant and perform the symplectic reduction.

Let G be a smooth group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual of its Lie algebra. The function $\sigma : G \to \mathfrak{g}^*$ defined by $\sigma(g) = \mu(\Phi_g(m)) - Ad^*_{g^{-1}}\mu(m)$ for all $m \in M$ is called a **co-adjoint cocycle** on G or simply one-cocycle as σ satisfies the cocycle identity $\sigma(gh) = \sigma(g) + Ad^*_{g^{-1}}\sigma(h)$ for all $g, h \in G$.

Notice that if $\sigma = 0$ then the momentum mapping is Ad^*-equivariant.

Theorem

Let Φ be a symplectic action of a smooth group G on a symplectic smooth space (M, ω) which admits a momentum mapping μ. Let σ be a one-cocycle. Define a map $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ by $\Psi(g, \alpha) = Ad_g^* \alpha + \sigma(g)$. Then the map Ψ is an action and the momentum mapping is equivariant with respect to this action.

Proof

First we need to check that the conditions of an action are satisfied. From the definition $\sigma(g) = \mu(\Phi_g(m)) - Ad_g^* \mu(m)$ we have $\sigma(e) = \mu(\Phi_e(m)) - Ad_e^* \mu(m) = \mu(m) - \mu(m) = 0$ since Ad_e^* is the identity map. Thus $\Psi(e, \alpha) = Ad_e^* \alpha + \sigma(e) = \alpha$, i.e $\Psi(e, \alpha) = \alpha$.

Using the cocycle identity we have
$\Psi(gh,\alpha) = Ad^*_{gh}\alpha + \sigma(gh) = Ad^*_g(Ad^*_h\alpha) + \sigma(g) + Ad^*_g\sigma(h)$
$= Ad^*_g(Ad^*_h\alpha + \sigma(h)) + \sigma(g) = Ad^*_g(\Psi(h,\alpha)) + \sigma(g)$
$= \Psi(g, \Psi(h,\sigma))$.
Hence Ψ is an action. To see that the momentum mapping is equivariant with respect to this action, we have
$\mu(\Phi_g(m)) - \Psi(g,\mu(m)) = \mu(\Phi_g(m)) - (Ad^*_g\mu(m) + \sigma(g))$
$= (\mu(\Phi_g(m)) - Ad^*_g\mu(m)) - \sigma(g) = \sigma(g) - \sigma(g) = 0.$
Thus, $\mu(\Phi_g(m)) = \Psi(g,\mu(m))$. This concludes the proof of the proposition.
We call $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ an **affine action.**

Theorem

Let $\Phi : G \times M \to M$ be a symplectic action of G on M which admits a momentum mapping $\mu : M \to \mathfrak{g}^*$ and let $\sigma : G \to \mathfrak{g}^*$ be the cocycle of μ. Let the function $\hat{\sigma_\eta} : G \to \mathbb{R}$ be defined by $\hat{\sigma_\eta}(g) = \sigma(g) \cdot \eta$.
Define also a function $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ by $\Sigma(\xi, \eta) = d\hat{\sigma_\eta}(e) \cdot \xi$ for all $\xi, \eta \in \mathfrak{g}$. Then Σ is skew symmetric bilinear form on $\mathfrak{g}$ and satisfies the Jacobi's identity
$$0 = \Sigma(\xi, [\eta, \zeta]) + \Sigma(\eta, [\zeta, \xi]) + \Sigma(\zeta, [\xi, \eta])$$

Theorem

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual of its Lie algebra. Let $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Psi(g, \alpha) = Ad_g^* \alpha + \sigma(g)$ be the affine action of G on $\mathfrak{g}^*$. Then the orbit $G \cdot \beta = \{\Psi(g, \beta) : g \in G\}$ is a symplectic smooth space with the 2-form given by

$$\omega_\beta(\xi_{\mathfrak{g}^*}(v), \eta_{\mathfrak{g}^*}(v)) = -\beta[\xi, \eta] + \Sigma(\eta, \xi)$$

[see Batubenge-Haziyu, to appear, 2018]

Let (M, ω) be a symplectic ringed **pF**-space, hence, a Hausdorff topological space. Let $G \times M \longrightarrow \mathbb{R}$ be a Hamiltonian action of a connected smooth group G on M. Assume that the action is free and proper, with an Ad^*-equivariant momentum map $\mu : M \longrightarrow \mathcal{G}^*$, where $\mathcal{G}$ is the Lie algebra of the Lie group G. Let G_θ denote the isotropy subgroup of the regular value θ of μ such that the action of G_θ on the level set $\mu^{-1}(\theta)$ is free and proper. Then under these conditions it turns out that the **pF**-subspace $M_\theta = \mu^{-1}(\theta)/G_\theta$ is a symplectic ringed **pF**- space provided with the symplectic form

$$\pi_\theta^* \omega_\theta = i_\theta^* \omega,$$

where $\pi_\theta : \mu^{-1}(\theta) \longrightarrow M_\theta$ is the projection to the quotient space and $i_\theta : \mu^{-1}(\theta) \longrightarrow M$ is the inclusion. By Sard's lemma it is known that the pre-image of such a μ at a regular point is a closed subspace of M.

Hereditary properties

Theorem

If M is Hausdorff paracompact, then the subspace $K = \mu^{-1}(\theta) \subset M$ is Hausdorff paracompact. [Tshilombo, 2016]

Theorem

The symplectic quotient of a Hausdorff paracompact smooth space is a Hausdorff paracompact space. [Tshilombo, 2016]

Theorem

Let $(M, \mathcal{D}, \omega)$ be a symplectic pre-Frölicher space, and $(M, \Gamma\mathcal{D}, \mathcal{D}, \omega)$ the associated Frölicher space. If $\mathcal{D}$ has a point-separating function and is locally Euclidean, then the symplectic quotient of M is Hausdorff and locally Euclidean.

Proof highlight:

- Since $\mathcal{D}$ has a point-separating function, then the topology of M is Hausdorff. We say that the smooth space is a ringed space in the sense of R. Palais.
- A function $\tilde{f}$ is in the quotient structure if, and only if $\tilde{f} \circ \pi \in \mathcal{D}$, where π is the canonical projection. That is, there is a point-separating structure function on the quotient. Thus, the quotient space will be Hausdorff and locally Euclidean.

Open questions:

- On smooth manifolds, the momentum mapping exists provided the operating Lie group is semi-simple. What about pre-Frölicher spaces or their corresponding Frölicher spaces?
- If the action is not proper/not free, does the quotient exist a symplectic pre-Frölicher space?

References

1. Abraham R. and Marsden J., *Foundations of Mechanics*, Second Edition, Addison-Wesley, Reading, 1978.
2. A. Batubenge and H. Tshilombo, Topologies and Smooth Maps on Initial and Final Objects in the Category of Frölicher spaces, *in Demonstratio Mathematica*, **42**, **3**, 2009.
3. T.A. Batubenge and M.H. Tshilombo: Notes on Pre-Frölicher Spaces, Quaest. Math., **39** (8), 2016.
4. Boman, J., Differentiability of a function and its composition with functions of one variable, *Math. Scand.* **20** (1967), 249-268.
5. P. Cherenack, Frölicher versus Differential Spaces: A Prelude to Cosmology, G. Brümmer and C. Gilmour (eds), 2000 Kluwer Academic Publishers, The Netherlands (1998), 63-91.

Thank you for your attention