The Angle Function in Contact Geometry

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Introduction

Analysis of the angle function between Reeb vector field and other Riemannian geometric vector fields reveals information ranging from the number of closed characteristics to the Sasakian rigidity of shared K-contact metrics.

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In some contact metric structures, this angle function turns out to be isoparametric whose normalized gradient provides new examples of minimal unit vector fields on noncompact manifolds.

Preliminaries

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The distribution $D_p = \{V \in T_pM : \eta(V) = 0\}$ is called the contact distribution of η. Clearly, D is a symplectic vector bundle with symplectic form $d\eta$.

On a contact manifold (M, η, Z), there is also a non-unique Riemannian metric g and a partial complex operator J adapted to η in the sense that the identities $g(Z, Z) = 1$ and

$$2g(X, JY) = d\eta(X, Y), \quad J^2 X = -X + \eta(X)Z$$

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Assuming that (M, g) is a complete Riemannian manifold, let ψ_t, $t \in \mathbb{R}$ denote the 1-parameter group of diffeomorphisms generated by Z. The contact form η is invariant under the 1-parameter group ψ_t, that is, $\psi_t^*\eta = \eta$.

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If ψ_t is also a 1-parameter group of isometries of g, then the contact metric manifold is called a K-**contact manifold**.

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If the identity

$$(\nabla_X J)Y = g(X, Y)Z - \eta(Y)X$$

is satisfied for any vector fields X and Y on M, then the contact metric structure (M, η, Z, J, g) is called a **Sasakian** structure.

More generally; given a unit Killing vector field Z on a Riemannian manifold (M, g), Z is the Reeb field of a Sasakian structure on M if and only if the following identity is satisfied:

$$\nabla_X(\nabla Z)Y = g(Z, Y)X - g(X, Y)Z$$

for any vector fields X and Y on M.

Condition to be a sphere

Theorem (Obata, 1965)

Let (M, g) be a complete simply connected Riemannian space of dimension n. In order for M to admit a non-trivial solution ρ for the system of differential equations

$$(\nabla^2 d\rho)(X, Y, Z) = -c[2d\rho(X)g(Y, Z) + d\rho(Y)g(X, Z) + d\rho(Z)g(X, Y)],$$

it is necessary and sufficient that M is isometric with a sphere S^n of radius $\frac{1}{\sqrt{c}}$ in the Euclidean $(n+1)$-space E^{n+1}.

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An example of such a ρ is $\rho = div\ W$ where W is an infinitesimal projective transformation.

In this case [**Okumura, 1962**], $d\rho = (n+1)\mu$, where μ is the 1-form determined by

$$(L_W \nabla)(U, V) = \mu(V)U + \mu(U)V$$

or, equivalently:

$$(\nabla_U \nabla W)V = R(U, W)V + \mu(V)U + \mu(U)V.$$

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If (M, g, ξ_1, ξ_2) are two Sasakian structures, then $\rho = g(\xi_1, \xi_2)$ satisfies the above differential equations [Tachibana-Yu, 1970]

Existence of closed characteristics

Theorem (Banyaga-Rukimbira, 1994)

Let $\mathcal{F}'$ be a contact foliation which is a C^1-perturbation of an almost regular contact foliation $\mathcal{F}$ on a compact manifold M. Then $\mathcal{F}'$ has at least two compact leaves.

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Indeed:
Let $\gamma(t)$ be the unique geodesic joining p and $\phi_T(p)$, where T is the period of the almost regular contact flow ϕ_t and define

$$S(p) = \int \alpha(\gamma'(t))dt = \int g(\xi, \gamma'(t))dt.$$

Where g is an adapted Riemannian metric.

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The idea of the proof is to show that for a sufficiently small peturbation, if $\gamma'(0) \neq 0$ or $\gamma'(0) \neq \lambda\xi_p$, then $D_pS \neq 0$.

More specifically: $D_p S(J\gamma'(0)) \neq 0$, where J is the transverse almost contact structure.
As a consequence, any critical point of **S** corresponds thus to a periodic orbit for ξ_α.

Circle invariant contact forms

Theorem

Let M be a $2n+1$-dimensional compact manifold with a circle invariant contact form α. If Z is the infinitesimal generator of the circle action and $\alpha(Z) \neq 0$ everywhere on M, then α admits a least $n+1$ closed characteristics.

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Sketch of the proof:
Z is the Reeb field of $\beta = \frac{1}{\alpha(Z)}\alpha$. The quotient V-manifold Σ carries a symplectic form and since $(d\beta)^n \neq 0$ everywhere, one sees that $cuplength(\Sigma) \geq n$.

It is also known from classical Lusternick-Shnirelman Theory that

$$cat(M) \geq cuplength(M) + 1$$

The circle invariant function $\alpha(Z)$ should have at least $cat(\Sigma) \geq n + 1$ critical circles.

Goldberg conjecture in symplectic geometry

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Essentially still open. With an additional condition,

Theorem (Sekigawa, 1987)

Any compact almost kähler Einstein manifold with nonegative scalar curvature is Kähler.

Problems

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2. Suppose an Einstein compact manifold M admits a compatible Kähler structure and let ω be any other adapted symplectic structure. Is ω necessarily Kähler?

Contact Goldberg conjecture

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Indeed:

Theorem (Boyer-Galicki, 2001)

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Proposition

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Dropping the Einstein condition, we ask:
Suppose a K-contact manifold (M, α, g, X) admits some Sasakian structure (β, g, Z). Is α necessarirly Sasakian?

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Two distint cases:

1. Case I: $[Z, X] \neq 0$.
2. Case II: $[Z, X] = 0$.

Case I:

Let ψ_t be the one parameter group of isometries generated by X. Then $(g, \psi_1^*\alpha, \psi_{1*}Z, \psi_{1*}^{-1}J\psi_{1*})$ are structure tensors of a second Sasakian structure on M.

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i) If $g(Z, \psi_{1*}Z)$ is constant, then M supports a 3-Sasakian steructure. (3 complex structures satisfying quaternionic identities)

ii) If $g(Z, \psi_{1*}Z)$ is nontrivial, then (M, g) is finitely covered by the Euclidean unit sphere.

Case II:

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If $[X, Z] = 0$, *then* $g(X, Z)$ *is a constant function if and only if* $X = \pm Z$.

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Proof: Suppose $X \neq \pm Z$. Let ϕ and J denote the two almost complex structures on the respecttive contact distributions. Then from $\nabla_X Z - \nabla_Z X = 0$, it follows that $-JX + \phi Z = 0$. That is

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$$\phi Z = JX.$$

On the other hand, the gradient of $f = g(X, Z)$ is easily seen to be

$$\nabla f = 2JX = 2\phi Z.$$

Since $Z \neq \pm X$, one has

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The function $f = g(X, Z)$ in this case is actually what is known as **isoparametric**; that is

$$\|\nabla f\| = a(f) \text{ and } \Delta f = b(f)$$

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Defined on the complement of the critical set of f,

$$N = \frac{\nabla f}{\|\nabla f\|}$$

is a geodesic, minimal unit vector field (**Rukimbira, 2015**).

Theorem

Let $(M, g, \alpha, Z, \beta, X)$ be a non-trivial bi-K-ontact structure on a closed manifold M, one of them being Sasakian. Then $[Z, X] \neq 0$.

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Sketch of proof:

Suppose $[Z, X] = 0$. Then the angle function $f = g(Z, X)$ has exactly 2 critical submanifolds Σ_{-1} and Σ_1. Moreover, each of these is totally geodesic.

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Let γ be the minimal geodesic realizing the distance between Σ_{-1} and Σ_1 and $\overline{Z}$ denotes the parallel translate of Z along the geodesic γ.

Since

$$\nabla_N Z = -\frac{1}{\|JX\|} J^2 X = \frac{1}{\|JX\|}(X - \alpha(X)Z),$$

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This vector field $\overline{Z}$ along γ provides a negative second variation for the energy functional along γ; contradicting its minimizing property.

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