Affine Szabó manifolds

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Global Differential Geometry
Summer School and Workshop
14-25 May 2018

25 mai 2018

Introduction

- One of the most important concept in differential geometry is the notion of curvature.
- But the curvature is in general quite difficult to study.
- A problem in differential geometry is to relate algebraic properties of the curvature to the underlining geometric information.
- Due to the fact that the curvature tensor is so difficult to handle, the investigation usually focuses on different objects associated to the curvature.

Introduction

- Let R be the Riemann curvature tensor of a pseudo-Riemannian (M, g) of signature (p, q).
- The Szabó operator [1] S is the self-adjoint linear map which is defined by

$$g(S(x)y, z) = \nabla R(y, x, x, z; x). \tag{1}$$

  It plays an important role in the study of totally isotropic manifolds.
- One says that (M, g) is Szabó if the eigenvalues of $S(x)$ are constant on the pseudo-spheres of unit timelike and spacelike vectors :

$$S^{\pm}(M, g) = \{x \in TM : g(x, x) = \pm 1\}. \tag{2}$$

1. Szabó write the original paper concerning the spectral properties of the this operator in the Riemannian setting

Introduction

- Szabó [2] used techniques from algebraic topology to show in the Riemannian setting that any such metric is locally symmetric.
- Szabó used this observation to give a simple proof that any 2 point homogeneous spaces is either flat or is a rank one symmetric space.
- Gilkey and Stravrov extended his result to show that any Szabó Lorentzian manifold has constant sectional curvature.
- If $p \geq 2$ ad if $q \geq 2$, there exist Szabó pseudo-Riemannian manifolds which are neither locally symmetric, nor locally homogeneous, nor pointwise totally isotropic.
- This motives the study of the Szabó operator in the higher signature setting.

2. Z. I. Szabó, A short topological proof for the symmetry of 2 point homogeneous spaces, Invent. Math., 106 (1991), 61-64.

Outline

1. Preliminaries
2. Affine Szabó manifolds
3. Affine Szabó 2-manifolds
4. Affine Szabó 3-manifolds
5. Applications
6. Conclusion

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Affine connection

Let M be a differentiable manifold of class $\mathcal{C}^\infty$. We shall denote by $\mathcal{F}(M)$ the set of all differentiable function and by $\mathfrak{X}(M)$ the set of all smooth vector fields on M.

Definition

An **affine connection** ∇ on M is a mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$(X, Y) \in \mathfrak{X}(M)^2 \mapsto \nabla_X Y \tag{3}$$

satisfying the following conditions :

- ♣ $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$;
- ♣ $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$;
- ♣ $\nabla_{fX} Y = f \nabla_X Y$;
- ♣ $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

where $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$.

Affine connection

- The affine connections is a classical topic in differential geometry. It was initially developped to solve pure geometrical problems.
- It provides an extremely important tool to study geometrical structures on manifolds and, as such, has been applied with great sources in many different setting.

Affine connection

Let us consider a coordinate system $(x^1, x^2, \cdots, x^n)$ in a neighborhood $\mathcal{U}$ of a point p in M.

- ♣ In $\mathcal{U}$, the connection is given by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x^k} \tag{4}$$

where the system of functions $\Gamma_{ij}^{k}(i, j, k = 1, \cdots, n)$ are called the **Christoffel symbols** for the affine connection relative to the local coordinate system.

Torsion tensor

Given affine connection ∇ on M.

Definition

The **torsion tensor field** T is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]; \tag{5}$$

it is a tensor of type $(1,2)$.

- It satisfy the following property : $T(X,Y) = -T(Y,X)$.
- The components of the torsion tensor T in local coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k. \tag{6}$$

- If $T = 0$, we say that ∇ has **zero torsion** or ∇ is **torsion free**.

Curvature tensor

Given torsion free affine connection ∇ on M.

Definition

The **curvature tensor field** R is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z; \tag{7}$$

it is a tensor of type $(1,3)$.

- The components of the curvature tensor R in local coordinates are

$$R\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)\frac{\partial}{\partial x^j} = \sum_i R^i_{jkl} \frac{\partial}{\partial x^i}. \tag{8}$$

- If $T = 0$ and $R = 0$, we say that ∇ is a **flat affine connection**.
- If $T = 0$ and $\nabla R = 0$, we say that ∇ is a **locally symmetric affine connection**.

Curvature tensor

The curvature tensor field R has the following properties :

- $R(X, Y)Z + R(Y, X)Z = 0$;
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$; (first Bianch identity)
- $(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$ (second Bianchi identity).

Ricci tensor

Given curvature tensor R on M.

Definition

The **Ricci tensor field** Ric is defined by

$$Ric(Y,Z) = \mathrm{trace}\{X \mapsto R(X,Y)Z\}; \tag{9}$$

it is a tensor of type $(0,2)$.

- The components of the Ricc tensor R in local coordinates are

$$Ric_{jk} = Ric\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \sum_i R^i_{jkl}. \tag{10}$$

- If $T = 0$ and $Ric = 0$, we say that ∇ is a **Ricci flat affine connection**.
- If $T = 0$ and $\nabla Ric = 0$, we say that ∇ is a **cyclic parallel affine connection**.

Remark

- In Riemannian geometry, the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is,

$$Ric(X, Y) = Ric(Y, X).$$

- This property is not true for an arbitrary torsion free affine connection. In fact, the property is closely related to the concept of parallel volume element.

Aim

The curvature is a central concepts in differetial geometry. But the curvature is difficult to investigate. We can use the curvature to define several associated operators :

- the affine Jacobi operator ;
- the affine Szabó operator ;
- the affine skew-symmetric operator ; ect...

which are defined in terms of the curvature and its covariane derivative ; and we discuss the spectral properties of these operators.

Affine Szabó operator

Given curvature tensor R on M.

Definition

The **affine Szabó operator** S^∇ is defined by

$$S^\nabla(X)Y := (\nabla_X R)(Y,X)X \tag{11}$$

where $X, Y \in T_pM$.

The covariant derivative of R is defined as

$$\begin{aligned}(\nabla_X R)(Y,X)X &= \nabla_X R(Y,X)X - R(\nabla_X Y,X)X \\ &\quad -R(Y,\nabla_X X)X - R(Y,X)\nabla_X X.\end{aligned} \tag{12}$$

The affine Szabó operator satisfies

- ♣ $S^\nabla(X)X = 0\,;$
- ♣ $S^\nabla(\alpha X)(\cdot) = \alpha^3 S^\nabla(X)(\cdot)$ for $\alpha \in \mathbb{R}^*$.

Riemann extension

For any affine connection ∇ on M, there exist a technique called Riemann extension, which relates affine and pseudo-Riemannian geometries. This technique is very powerful in constructing new examples of pseudo-Riemannian metrics.

Definition

The Riemann extension g_∇ is the pseudo-Riemannian metric on T^*M of neutral signature (n, n) defined in the locally induced coordinates $(x_i, \dots, x_{i'})$ on $\pi^{-1}(U) \subset T^*M$, by

$$g_\nabla = \begin{pmatrix} -2x_{k'}\Gamma_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix}. \tag{13}$$

with respect to $\{\partial_1, \dots, \partial_n, \partial_{1'}, \dots, \partial_{n'}\}$ $(i, j, k = 1, \dots, n; k' = k + n)$, where Γ_{ij}^k are the Christoffel symbols of the torsion free affine connection ∇ with respect to (U, x_i) on M.

Twisted Riemann extension

The twisted Riemann extension which is a generalization of Riemann extension.

Definition

Let ϕ be a symmetric $(0,2)$-tensor field on M. The twisted Riemann extension is the neutral signature metric on T^*M given by

$$g_\nabla = \begin{pmatrix} \phi_{ij}(x) - 2x_{k'}\Gamma_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix}. \tag{14}$$

with respect to $\{\partial_1, \dots, \partial_n, \partial_{1'}, \dots, \partial_{n'}\}$ $(i, j, k = 1, \dots, n; k' = k + n)$, where Γ_{ij}^k are the Christoffel symbols of the torsion free affine connection ∇ with respect to (U, x_i) on M.

Remark

Some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemann extension g_∇. For instance :

- ♣ (M, ∇) is locally symmetric if and only if (T^M, g_∇) is locally symmetric.
- ♣ (M, ∇) is projectively flat if and only if (T^M, g_∇) is locally conformally flat.

Definition

Definition

Let (M, ∇) be a smooth affine manifold and $p \in M$.

1. (M, ∇) is called **affine Szabó** at $p \in M$ if the affine Szabó operator $S^{\nabla}(X)$ has the same characteristic polynomial for every $X \in T_pM$.
2. (M, ∇) is called **pointwise affine Szabó** if the eingenvalues of the affine Szabó operator $S^{\nabla}(X)$ dot not depend on $X \in T_pM$ for every point $p \in M$ (the eigenvalues may vary from point to point).
3. (M, ∇) is called **affine Szabó** if (M, ∇) is affine Szabó at each $p \in M$.

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3. (M, ∇) is called **affine Szabó** if (M, ∇) is affine Szabó at each $p \in M$.

Theorem 1

Theorem

Let (M, ∇) be an n-dimensional affine manifold and $p \in M$. Then (M, ∇) is affine Szabó at $p \in M$ iff the characteristic polynomial of the affine Szabó operator $S^{\nabla}(X)$ is

$$P_\lambda[S^{\nabla}(X)] = \lambda^n \tag{15}$$

for every $X \in T_pM$.

Proof of Theorem 1

- If the characteristic polynomial of the affine Szabó operator at p is given by $P_\lambda(\mathcal{S}^\nabla(X)) = \lambda^n$, then the affine manifold (M, ∇) is obviously affine Szabó.
- Assume that (M, ∇) is affine Szabó,
  - then for $X \in T_pM$, the characteristic polynomial of the affine Szabó operator $\mathcal{S}^\nabla(X)$ is given by

$$P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} - \cdots + (-1)^n\sigma_n. \quad (16)$$

  - Then for $\beta X \in T_pM$ with $\beta \in \mathbb{R}^*$, the characteristic polynomial of the affine Szabó operator $\mathcal{S}^\nabla(\beta X)$ is given by

$$P_\lambda[\mathcal{S}^\nabla(\beta X)] = \lambda^n - \sigma_1\beta^3\lambda^{n-1} + \sigma_2\lambda^{n-2} - \cdots + (-1)^n\beta^{3n}\sigma_n. \quad (17)$$

Hence, since (M, ∇) is affine Szabó, that is $P_\lambda[\mathcal{S}^\nabla(X)] = P_\lambda[\mathcal{S}^\nabla(\beta X)]$, it follows that

$$\sigma_1 = \cdots = \sigma_n = 0$$

which complete the proof.

Theorem 1

Corollary

If (M, ∇) is affine Szabó at $p \in M$, then

- 0 is the only eigenvalue of the affine Szabó operator ;
- the trace of the affine Szabó operator vanishes, that means the Ricci tensor of ∇ is cyclic parallel.

Aim

Classification of 2-dimensional affine Szabó manifolds.

Theorem 2

Theorem

Let (M, ∇) be a two-dimensional smooth affine manifold. Then (M, ∇) is affine Szabó at $p \in M$ iff the Ricci tensor of (M, ∇) is cyclic parallel at $p \in M$.

Proof of Theorem 2

♣ Step 1. The components of the curvature tensor R are given by

$$R(\partial_1, \partial_2)\partial_1 = a\partial_1 + b\partial_2 \tag{18}$$

and

$$R(\partial_1, \partial_2)\partial_2 = c\partial_1 + d\partial_2, \tag{19}$$

where a, b, c and d are given by

$$\begin{aligned}
a &= \partial_1 f_{12}^1 - \partial_2 f_{11}^1 + f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1, \\
b &= \partial_1 f_{12}^2 - \partial_2 f_{11}^2 + f_{11}^2 f_{12}^1 + f_{12}^2 f_{12}^2 - f_{11}^1 f_{12}^2 - f_{11}^2 f_{22}^2, \\
c &= \partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{11}^1 f_{22}^1 + f_{12}^1 f_{22}^2 - f_{12}^1 f_{12}^1 - f_{12}^2 f_{22}^1, \\
d &= \partial_1 f_{22}^2 - \partial_2 f_{12}^2 + f_{11}^2 f_{22}^1 - f_{12}^1 f_{12}^2.
\end{aligned}$$

Proof of Theorem 2

♣ Step 2. Let $X = \alpha_i \partial_i, i = 1, 2$ be a vector on M, then the affine Szabó operator is given by

$$(\nabla_X R)(\partial_1, X)X = A\partial_1 + B\partial_2 \tag{20}$$

and

$$(\nabla_X R)(\partial_2, X)X = C\partial_1 + D\partial_2 \tag{21}$$

where A, B, C and D are given by

Proof of Theorem 2

♣

$$
\begin{aligned}
A &= \alpha_1^2\alpha_2\Big[\partial_1 a - a(f_{11}^1 + f_{12}^2) + bf_{12}^1 - cf_{11}^2\Big] + \alpha_2^3\Big[\partial_2 c - 2cf_{22}^2 + (d-a)f_{22}^1\Big] \\
&\quad + \alpha_1\alpha_2^2\Big[\partial_2 a + \partial_1 c - a(f_{12}^1 + f_{22}^2) + (d-a)f_{12}^1 + bf_{22}^1 - 3cf_{12}^2\Big]; \\
B &= \alpha_1^2\alpha_2\Big[\partial_1 b - 2bf_{11}^1 - (d-a)f_{11}^2\Big] + \alpha_2^3\Big[\partial_2 d - bf_{22}^1 + cf_{12}^2 - d(f_{12}^1 + f_{22}^2)\Big] \\
&\quad + \alpha_1\alpha_2^2\Big[\partial_2 b + \partial_1 d - 3bf_{12}^1 + cf_{11}^2 - (d-a)f_{12}^2 - d(f_{11}^1 + f_{12}^2)\Big]; \\
C &= \alpha_1^3\Big[-\partial_1 a + a(f_{11}^1 + f_{12}^2) - bf_{12}^1\Big] + \alpha_1\alpha_2^2\Big[-\partial_2 c + 2cf_{22}^2 - (d-a)f_{22}^1\Big] \\
&\quad + \alpha_1^2\alpha_2\Big[-\partial_2 a - \partial_1 c + a(f_{12}^1 + f_{22}^2) - bf_{22}^1 + 3cf_{12}^2 - (d-a)f_{12}^1\Big]; \\
D &= \alpha_1^3[-\partial_1 b + 2bf_{11}^1 + (d-a)f_{11}^2] + \alpha_1\alpha_2^2[-\partial_2 d + bf_{22}^1 - cf_{12}^2 + d(f_{12}^1 + f_{22}^2)] \\
&\quad + \alpha_1^2\alpha_2\Big[-\partial_2 b - \partial_1 d + 3bf_{12}^1 - cf_{11}^2 + d(f_{11}^1 + f_{12}^2) + (d-a)f_{12}^2\Big].
\end{aligned}
$$

Proof of Theorem 2

♣ Step 3. The matrix associated to $\mathcal{S}^\nabla(X)$ with respect to the basis $\{\partial_1, \partial_2\}$ is given by

$$(\mathcal{S}^\nabla(X)) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{22}$$

Its characteristic polynomial is given by

$$P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^2 - \lambda(A+D) + (AD - BC). \tag{23}$$

Proof of Theorem 2

♣ Step 4. Since (M, ∇) is affine Szabó, by Theorem 1, 0 is the only eigenvalue of the affine Szabó operator $\mathcal{S}^{\nabla}(X)$.

- Therefore,

$$\det(\mathcal{S}^{\nabla}(X)) = AD - BC = 0$$

and

$$trace(\mathcal{S}^{\nabla}(X)) = A + D = 0.$$

- The latter implies that

$$\begin{aligned}
&\partial_2 c - 2cf_{22}^2 + (d-a)f_{22}^1 = 0, \\
&-\partial_1 b + 2bf_{11}^1 + (d-a)f_{11}^2 = 0, \\
&\partial_1 a - \partial_2 b - \partial_1 d + 4bf_{12}^1 - 2cf_{11}^2 + (d-a)(f_{11}^1 + 2f_{12}^2) = 0, \\
&\partial_2 a + \partial_1 c - \partial_2 d + 2bf_{22}^1 - 4cf_{12}^2 + (d-a)(2f_{12}^1 + f_{22}^2) = 0.
\end{aligned}$$

Examples

Example

Consider on $\mathbb{R}^2$ the torsion free affine connection ∇ defined by

$$\nabla_{\partial_1}\partial_1 = (x_1 - x_2)\partial_1 \text{ and } \nabla_{\partial_2}\partial_2 = (x_1 + x_2 + 1)\partial_2. \tag{24}$$

After, a straightforward calculation, it is easy to check that $(\mathbb{R}^2, \nabla)$ is an affine Szabó manifold.

Example

Let consider on $\mathbb{R}^2$ the torsion free affine connection ∇ defined by

$$\nabla_{\partial_1}\partial_2 = x_2\partial_1 \text{ and } \nabla_{\partial_2}\partial_2 = x_1 x_2^2 \partial_1. \tag{25}$$

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After, a straightforward calculation, it is easy to check that $(\mathbb{R}^2, \nabla)$ is an affine Szabó manifold.

Affine 3-manifolds

Let M be a 3-dimensional smooth manifold and ∇ bea torsion free affine connection. We have

$$\begin{aligned}
\nabla_{\partial_1}\partial_1 &= f_{11}^1\partial_1 + f_{11}^2\partial_2 + f_{11}^3\partial_3,\\
\nabla_{\partial_1}\partial_2 &= f_{12}^1\partial_1 + f_{12}^2\partial_2 + f_{12}^3\partial_3,\\
\nabla_{\partial_1}\partial_3 &= f_{13}^1\partial_1 + f_{13}^2\partial_2 + f_{13}^3\partial_3,\\
\nabla_{\partial_2}\partial_2 &= f_{22}^1\partial_1 + f_{22}^2\partial_2 + f_{22}^3\partial_3,\\
\nabla_{\partial_2}\partial_3 &= f_{23}^1\partial_1 + f_{23}^2\partial_2 + f_{23}^3\partial_3,\\
\nabla_{\partial_3}\partial_3 &= f_{33}^1\partial_1 + f_{33}^2\partial_2 + f_{33}^3\partial_3,
\end{aligned}$$

where $f_{ij}^k = f_{ij}^k(x_1, x_2, x_3)$ are the Christoffel symbols.

Theorem 3 (Diallo and Massamba)

Theorem

Let M be a 3-dimensional smooth manifold and ∇ be a torsion free affine connection given by

$$\nabla_{\partial_i}\partial_k = \frac{1}{x_i}\partial_k; \quad \nabla_{\partial_j}\partial_k = \frac{1}{x_j}\partial_k; \quad \nabla_{\partial_i}\partial_j = \frac{x_k}{x_i x_j}\partial_k;$$

with $i \neq j \neq k; i, j, k = 1, 2, 3$ and $x_i \neq 0, x_j \neq 0, x_k \neq 0$. Then (M, ∇) is affine Szabó.

Example

Example

The following torsion free affine connections on $\mathbb{R}^3$ given by :

1. $\nabla_{\partial_1}\partial_2 = \frac{1}{x_2}\partial_1, \quad \nabla_{\partial_1}\partial_3 = \frac{1}{x_3}\partial_1, \quad \nabla_{\partial_2}\partial_3 = \frac{x_1}{x_2 x_3}\partial_1$;
2. $\nabla_{\partial_1}\partial_2 = \frac{1}{x_1}\partial_2, \quad \nabla_{\partial_1}\partial_3 = \frac{x_2}{x_1 x_3}\partial_2, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{x_3}\partial_2$;
3. $\nabla_{\partial_1}\partial_2 = \frac{x_3}{x_1 x_2}\partial_3, \quad \nabla_{\partial_1}\partial_3 = \frac{1}{x_1}\partial_3, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{x_2}\partial_3$,

are affine Szabó.

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2. $\nabla_{\partial_1}\partial_2 = \frac{1}{x_1}\partial_2, \quad \nabla_{\partial_1}\partial_3 = \frac{x_2}{x_1 x_3}\partial_2, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{x_3}\partial_2$;
3. $\nabla_{\partial_1}\partial_2 = \frac{x_3}{x_1 x_2}\partial_3, \quad \nabla_{\partial_1}\partial_3 = \frac{1}{x_1}\partial_3, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{x_2}\partial_3,$

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2. $\nabla_{\partial_1}\partial_2 = \frac{1}{x_1}\partial_2, \quad \nabla_{\partial_1}\partial_3 = \frac{x_2}{x_1 x_3}\partial_2, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{x_3}\partial_2\,;$
3. $\nabla_{\partial_1}\partial_2 = \frac{x_3}{x_1 x_2}\partial_3, \quad \nabla_{\partial_1}\partial_3 = \frac{1}{x_1}\partial_3, \quad \nabla_{\partial_2}\partial_3 = \frac{1}{x_2}\partial_3,$

are affine Szabó.

Theorem 4

Theorem

Let M and ∇ be the torsion free affine connection, whose nonzero coefficients of the connection are given by

$$\begin{aligned}
\nabla_{\partial_1}\partial_1 &= f_1(x_1, x_2, x_3)\partial_1 \\
\nabla_{\partial_1}\partial_2 &= f_2(x_1, x_2, x_3)\partial_1 \\
\nabla_{\partial_1}\partial_3 &= f_3(x_1, x_2, x_3)\partial_1
\end{aligned}$$

Then (M, ∇) is affine Szabó if and only if the Ricci tensor of (M, ∇) is cyclic parallel.

Example

Example

The following affine connections on $\mathbb{R}^3$ whose non-zero Christoffel symbols are given by :

1. $\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = -x_3\partial_1, \quad \nabla_{\partial_1}\partial_3 = x_2\partial_1$;
2. $\nabla_{\partial_1}\partial_1 = x_1\partial_1, \quad \nabla_{\partial_1}\partial_2 = 2x_3\partial_1, \quad \nabla_{\partial_1}\partial_3 = -2x_2\partial_1$;

are affine Szabó.

Example

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The following affine connections on $\mathbb{R}^3$ whose non-zero Christoffel symbols are given by :

1. $\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_1}\partial_2 = -x_3\partial_1, \quad \nabla_{\partial_1}\partial_3 = x_2\partial_1\,;$
2. $\nabla_{\partial_1}\partial_1 = x_1\partial_1, \quad \nabla_{\partial_1}\partial_2 = 2x_3\partial_1, \quad \nabla_{\partial_1}\partial_3 = -2x_2\partial_1\,;$

are affine Szabó.

Theorem 5

Theorem

Let M and ∇ be the torsion free affine connection given by

$$\nabla_{\partial_1}\partial_1 = f_1(x_1, x_2, x_3)\partial_2, \quad \nabla_{\partial_2}\partial_2 = f_2(x_1, x_2, x_3)\partial_3, \quad \nabla_{\partial_3}\partial_3 = f_3(x_1, x_2, x_3)\partial_1.$$

Then (M, ∇) is affine Szabó if at least one of the following conditions holds :

1. $f_1 = 0$, $f_2 = u(x_2)$ and $f_3 = v(x_2) + t(x_3)$.
2. $f_2 = 0$, $f_3 = t(x_3)$ and $f_1 = f(x_1) + g(x_3)$.
3. $f_3 = 0$, $f_1 = f(x_1)$ and $f_2 = h(x_1) + u(x_2)$.

Or at least one of the following conditons holds :

1. $f_1 = 0$, $f_2 = f(x_1) + g(x_2)$ and $f_3 = 0$.
2. $f_2 = 0$, $f_3 = v(x_2) + t(x_3)$ and $f_1 = 0$.
3. $f_3 = 0$, $f_1 = f(x_1) + g(x_3)$ and $f_2 = 0$.

Example

Example

The following torsion free affine connections on $\mathbb{R}^3$ whose non-zero Christoffel symbols are given by :

1. $\nabla_{\partial_1}\partial_1 = 0, \quad \nabla_{\partial_2}\partial_2 = x_2\partial_3, \quad \nabla_{\partial_3}\partial_3 = (x_2 + x_3^2)\partial_1$;
2. $\nabla_{\partial_1}\partial_1 = x_1^2\partial_2, \quad \nabla_{\partial_2}\partial_2 = (x_1 + x_2)\partial_3, \quad \nabla_{\partial_3}\partial_3 = 0$;

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are affine Szabó.

Theorem 6

Theorem

Define a torsion free connection on $\mathbb{R}^m$ by setting

$$\nabla_{\partial_i}\partial_j = \sum_{k>max(i,j)} \Gamma_{ij}^k(x_1, x_2, \cdots, x_{k-1})\partial_k \quad \text{for} \quad \Gamma_{ij}^k = \Gamma_{ji}^k.$$

Then $(\mathbb{R}^m, \nabla)$ is affine Szabó.

Theorem A. S. Diallo, S. Longwap and F. Massamba, *On three dimensional affine Szabó manifolds*, Balkan Journal of Geometry and its Applications, 22, (2017) (2), 21-36. 51

Theorem

Let (M, ∇) be a smooth torsion-free affine manifold. Then the following statements are equivalent :

1. (M, ∇) is affine Szabó.
2. The Riemann extension (T^*M, g_∇) of (M, ∇) is a pseudo-Riemannian Szabó manifold.

Example

Example

As an example, we have the following. The Riemann extension of the affine Szabó connection on $\mathbb{R}^3$ defined by

$$\nabla_{\partial_1}\partial_1 = x_1\partial_1, \quad \nabla_{\partial_1}\partial_2 = 2x_3\partial_1, \quad \nabla_{\partial_1}\partial_3 = -2x_2\partial_1, \tag{26}$$

is the pseudo-Riemannian metric of signature $(3,3)$ given by

$$\begin{aligned} g_\nabla &= 2dx_1 \otimes dx_4 + 2dx_2 \otimes dx_5 + 2dx_3 \otimes dx_6 \\ &\quad - 2x_1x_4dx_1 \otimes dx_1 - 4x_3x_4dx_1 \otimes dx_2 + 4x_2x_4dx_1 \otimes dx_3. \end{aligned} \tag{27}$$

After, a straightforward calculation, it easy to see that this metric is Szabó.

Theorem A. S. Diallo, S. Longwap and F. Massamba, *On twisted Riemannian extensions associated with Szabó metrics*, Hacettepe Journal of Mathematics and Statistics, 46, (2017) (4), 593-601. 53

Theorem

Let $(T^*M, g_{(,\phi)})$ be the cotangent bundle of an affine manifold (M, ∇) equipped with the twisted Riemannian extension. Then $(T^*M, g_{(,\phi)})$ is a pseudo-Riemannian Szabé manifold if and only $(M,)$ is affine Szabé for any symmetric $(0, 2)$-tensor field ϕ.

Example

Example

As an example we have the following. The twisted Riemannian extensions of the following affine Szabó

$$\nabla_{\partial_1}\partial_1 = u_1 u_3 \partial_2, \quad \nabla_{\partial_2}\partial_2 = 0, \quad \nabla_{\partial_3}\partial_3 = (u_1 + u_3)\partial_2 \tag{28}$$

is given by

$$\begin{aligned} g &= 2du_1 \otimes du_4 + 2du_2 \otimes du_5 + 2du_3 \otimes du_6 + 2\phi_{12} du_1 \otimes du_2 \\ &\quad + 2\phi_{13} du_1 \otimes du_3 + 2\phi_{23} du_2 \otimes du_3 + (\phi_{11} - 2u_1 u_3 u_5) du_1 \otimes du_1 \\ &\quad + \phi_{22} du_2 \otimes du_2 + [\phi_{33} - 2(u_1 + u_3)u_5] du_3 \otimes du_3, \end{aligned} \tag{29}$$

where $(u_1, u_2, \cdots, u_6)$ are coordinates in $\mathbb{R}^6$.

Conclusion

- From Theorem 1, the cyclic parallel of the Ricci tensor is a necessary conditon for an affine manifold to be affine Szabó manifold.
- In dimension 2, the cyclic parallel of the Ricci tensor is a sufficient conditon for an affine manifold to be Szabó.
- In dimension 3, there are affine manifolds which satisfy the cyclic parallel Ricci tensor but are not Szabó.
- In dimension 3, the classification seems to be very challenging. Partial results exists.

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Thanks

Thanks for your attention ! ! !