Deformation retract of Symp($\mathbb{C}P^1 \times \mathbb{C}P^1$) by the mean curvature flow

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Outline

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1. The mean curvature flow (MCF)

2. Hamiltonian property

3. Application : homotopy type of $\mathrm{Symp}(\mathbb{C}P^1 \times \mathbb{C}P^1)$
   - Towards Hofer geometry

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1. **The mean curvature flow (MCF)**
2. **Hamiltonian property**
3. **Application : homotopy type of $Symp(\mathbb{C}P^1 \times \mathbb{C}P^1)$**
   - Towards Hofer geometry

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1. The mean curvature flow (MCF)
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3. Application : homotopy type of $\mathcal{S}\mathrm{ymp}(\mathbb{C}P^1 \times \mathbb{C}P^1)$
   - Towards Hofer geometry

$$F_0 : \mathcal{M}^n \hookrightarrow (M^m, \bar{g})\ ;$$
$$\Big(\mathcal{M}\ ,\ g = F_0^* \bar{g}\Big) \equiv \Big(F_0\mathcal{M}\ ,\ \bar{g}_{\big|F_0\mathcal{M}}\Big)$$ and $TM = T\mathcal{M} \oplus N\mathcal{M}$.

MCF of $\mathcal{M}$ = family of immersions $F : \mathcal{M} \times [0, T) \longrightarrow M$ s. t.

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) & = H_{F(x,t)} \\ F(\cdot, 0) & = F_0(\cdot) \end{cases} \tag{1}$$

MC vector $H = trace_g(A)$, where A is the second fundamental form

$$\begin{array}{rccc} A: & \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) & \longrightarrow & \Gamma(N\mathcal{M}) \\ & (X, Y) & \longmapsto & A(X,Y) = (\overline{\nabla}_X Y)^{\perp} \end{array}$$

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$F : \mathcal{M} \times [0, T) \longrightarrow M$, the m.c.f. can be expressed in coordinates as

$$\frac{\partial F^\alpha}{\partial t} = g^{ij} \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} \left(\delta_{\alpha\beta} - \frac{\partial F^\beta}{\partial x^k} g^{kl} \frac{\partial F^\alpha}{\partial x^l} \right) \equiv \triangle_{g(t),\bar{g}} F. \tag{2}$$

Analytic view point : quasi linear second order system of parabolic equations

Short time existence is based on implicit function theorem

- closed surface
- Dirichlet boundary conditions
- sufficiently smooth initial data

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Assume $M = \mathbb{R}^{n+1}$ ($\mathcal{M}$ is an hypersurface)

Proposition

The evolution equation (1) has a smooth solution on a maximal time interval $0 \leqslant t < T < \infty$ and $\max\limits_{\mathcal{M}_t} |A|^2$ becomes unbounded as t goes to T.

$$\left(\frac{\partial}{\partial t} - \triangle\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4. \qquad (3)$$

Lemma

The function $U(t) = \max\limits_{\mathcal{M}_t} |A|^2$ is Lipschitz continuous and satisfies

$$\frac{1}{2(T-t)} \leqslant U(t) \leqslant \frac{C_0}{2(T-t)} \quad \text{if} \quad \exists\ C_0 \in \mathbb{R}_+ \qquad (4)$$

type I singularity

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Example

- $F_0 : S^n \longrightarrow \mathbb{R}^{n+1}$ and $F(x,t) = \phi(t) F_0(x)$.

$$\phi'(t) F_0(x) = \frac{\partial F(x,t)}{\partial t} = H = -\frac{n}{\phi(t)} F_0(x) \Longrightarrow \phi'(t) = -\frac{n}{\phi(t)} \tag{5}$$

  which solves in $\phi(t) = \sqrt{r_0^2 - 2nt}$ and the m.c.f can be written as $F(x,t) = \sqrt{r_0^2 - 2nt}\, F_0(x)$. $T = \frac{r_0^2}{2n}$ the singular time.

- $F_0 : S^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^{n+1}$ and $F(x,t) = \phi(t) F_0(x)$.

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$M \equiv \mathbb{R}^{n+1}$ and $\mathcal{M}$ =graph$f$ of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$

$$F_0 : \mathcal{M} \longrightarrow M \quad \text{equivalently } F_0 \equiv id \times f : \mathbb{R}^n \longrightarrow M$$

$F(x^1, \cdots, x^n, t) = \left(x^1, \cdots, x^n, f(x^1, \cdots, x^n, t)\right)$ is a parametrization of $(\mathcal{M}_t)$;

Denoting by $\nu = \frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}$ the unit upward normal,

$$\left\langle \frac{\partial F}{\partial t}(x,t), \nu \right\rangle \nu = H \tag{7}$$

m.c.f of the graph of f

$$\frac{\partial f}{\partial t} = \triangle f - \frac{Hess(\nabla f, \nabla f)}{\left(\sqrt{1+|\nabla f|^2}\right)^2} \tag{8}$$

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Higher codimension

For the graphical case, $f : (\mathcal{M}_1, g_1) \longrightarrow (\mathcal{M}_2, g_2)$,
$(M, \bar{g}) \equiv \big(\mathcal{M}_1 \times \mathcal{M}_2, g_1 \oplus g_2\big)$, $F \equiv id \times f : \mathcal{M}_1 \to M$, $\mathcal{M}$ = graph$f$

$$\frac{\partial f^\alpha}{\partial t} - g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^j}\frac{\partial f^\gamma}{\partial x^i} + \Gamma^l_{ij}\frac{\partial f^\alpha}{\partial x^l}\right) = 0, \tag{9}$$

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Singularity occurs when $|A|$ becomes unbounded. To continue the flow, one uses the backward heat-kernel and the monotonicity formula

The backward heat kernel

$$\rho_{x_0,t_0}(x,t) = \left(4\pi(t_0-t)\right)^{-\frac{n}{2}} \exp\left(-\frac{|x_0-x|^2}{4(t_0-t)}\right) \tag{10}$$

and the monotonicity formula says

$$\frac{d}{dt}\int \rho_{x_0,t_0}\, d\mu_t \leqslant 0, \forall\ t < t_0 \text{ and } \lim_{t\to t_0}\int \rho_{x_0,t_0}\, d\mu_t \text{ exists.} \tag{11}$$

(B. White) Each time $\exists\ \beta > 0$ s.t.

$$\lim_{t\to t_0}\int \rho_{x_0,t_0}\, d\mu_t < 1+\beta \tag{12}$$

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Fix $i : M \hookrightarrow \mathbb{R}^d$ and given $F : \mathcal{M} \longrightarrow M$, then $\overline{F} = i \circ F : \mathcal{M} \hookrightarrow \mathbb{R}^d$.
Pick a point $x_0 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$. Defined

$$\begin{array}{rccc} D_\lambda : & \mathbb{R}^d \times [0, t_0) & \longrightarrow & \mathbb{R}^d \times [-\lambda^2 t_0, 0) \\ & (x, t) & \longmapsto & \big(\lambda(x - x_0), \lambda^2(t - t_0)\big) \end{array}$$

$$\frac{d}{dt} F(x,t) = H = \overline{H} + V \quad \text{where} \quad V = -\sum_{i=1}^{n} \mathcal{A}_M(e_i, e_i) \tag{13}$$

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Proposition (K-W. Lee & Y-I Lee)

Assume that $k_{\mathcal{M}_1} \geqslant c_1$ and $k_{\mathcal{M}_2} \leqslant c_2$ for two constants c_1 and c_2 ($k_{\mathcal{M}_i}$ is a sectional curvature of $\mathcal{M}_i$). Suppose either $c_1 \geqslant 0$ and $c_2 \leqslant 0$ or $c_1 \geqslant c_2 \geqslant 0$ then the following hold :

1. *If $\dfrac{\det\left(g_1 + f^* g_2\right)_{ij}}{\det g_{ij}} < 4$, then the MCF of $\mathcal{M}$ =graphf remains the graph of a map and exists for all time.*
2. *Furthermore, if $c_1 > 0$ then the MCF converges smoothly to the graph of a constant map*

Outline

Symplectic manifold

- M is $2n$-dimensionnal manifold
- and

$$\begin{array}{rccc} \omega : & \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathcal{C}^\infty(M, \mathbb{R}) \\ & (X, Y) & \longmapsto & \omega(X, Y) \end{array}$$

is closed ($d\omega = 0$) and non-degenerated 2-form
($\omega(X, Y) = 0, \forall Y \in \mathfrak{X}(M) \Longleftrightarrow X = 0$)

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Hamiltonian function

Any smooth function $G : (M, \omega) \times [0, 1] \longrightarrow \mathbb{R}$ such that $\int_0^1 \int_M G(s, x)\omega^n \, dt = 0$ is called (normalized) Hamiltonian.

Associated time dependent vector field

$$G(s, x) \equiv G_s(x) \Longrightarrow X_s \qquad \text{s.t.} \qquad i_{X_s}\omega = \tilde{\omega}^{-1}(dG_s). \tag{15}$$
$$L_{X_s}\omega = 0 \qquad \text{i.e} \qquad X_s \qquad \text{is symplectic.}$$

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- A given diffeomorphism φ of (M,ω) is called hamiltonian if $\exists$ an hamiltonian isotopy $\{\varphi_s\}_{s\in[0,1]}$ such that $\varphi_0 = id$ and $\varphi_1 = \varphi$

- $\mathrm{Ham}(M,\omega) \equiv$ set of all hamiltonian diff. on M
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Obstruction of symplecto. to be hamilto.

Flux homomorphism

$$\widetilde{\mathrm{Flux}}: \widetilde{Symp_0}(M,\omega) \longrightarrow H^1(M,\mathbb{R})$$

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Let $f \in Ham(M, \omega)$ and $\{f_s\} \subset \mathrm{Symp}_0(M, \omega)$ such that $f_1 = f$.

The MCF of f gives rise to a 2-parameters family $\{f_{s,t}\}$ such that

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(S_1)\quad \begin{cases}
f_{1,0} = f \\
f_{0,t} = id_{\mathcal{M}}, \quad f_{1,t} = f_t \quad \text{for each } t < t_0 \\
Flux\{f_{s,t}\} = \left[\int_0^1 f^*_{s,t}(i_{X_{s,t}}\omega)ds \right] := [\mathcal{F}_t] \\
\frac{\partial}{\partial s} f_{s,t} = X_{s,t} \\
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Lemma

The flux form $\mathcal{F}_t$ of the isotopy $\{f_{s,t}\}_s$ satisfies : $\frac{\partial}{\partial t}\mathcal{F}_t = i_H\omega + dK_t$ *where* $K_t = \int_0^1 f_{s,t}^*\omega(X_{s,t}, H_{s,t})ds$ *and* f_t *is the time-one map of the isotopy* $\{f_{s,t}\}_{0\leqslant s\leqslant 1}$.

Since $X_{s,t}$ and $H_{s,t}$ are symplectic vector fields, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{F}_t &= \int_0^1 \frac{\partial}{\partial t}\left(f_{s,t}^* i_{X_{s,t}}\omega\right) ds \\
&= \int_0^1 \left(f_{s,t}^* L_{H_{s,t}} i_{X_{s,t}}\omega + f_{s,t}^* \frac{\partial}{\partial t} i_{X_{s,t}}\omega\right) ds \\
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The Maslov class

After J.M Morvan, one characterizes the Maslov class of a Lagrangian immersion in a (almost) Khaeler manifold as

$$[i_H\omega] \in H^1(M, \mathbb{R}). \tag{17}$$

Lemma

Let $\{f_s\}_{0\leqslant s\leqslant 1}$ be a symplectic isotopy to $f \in Symp(M, J, \omega)$ of a Kaehler manifold (M, J, ω). Assume that the m.c.f $(\mathcal{M}_t)_t$ with initial data $\mathcal{M}_0$=graphf, exists. Then the flux of $\{f_s\}_s$ along the m.c.f deforms to the Maslov class of $\mathcal{M}$.

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Theorem

Let (M, J, ω) be a compact connected Kaehler manifold and $f \in \mathrm{Symp}_0(M, \omega)$. Suppose $\mathcal{M}$ = graph$f$ has zero Maslov class. Assume that the m.c.f $(\mathcal{M}_t)_t$ of $\mathcal{M}$ exists. Then the flux of any symplectic isotopy to f is preserved along the flow.

In particular if $f \in Ham(M, J, \omega)$ then $f_t \in Ham(M, J, \omega)$ for each t, namely there is a (time dependent) fonction $\chi : M \times [0, 1] \to \mathbb{R}$ such that $H_t = J\nabla\chi_t$.

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Outline

- Dimension 2 :
  - $\mathrm{Symp}(M, \omega) \sim \mathit{Diff}^+(M)$, thus $\mathit{Symp}(S^2, \omega) \sim SO(3)$
  - $\mathrm{Symp}_0(T^2, \omega) \sim T^2$ extension of $SL(2, \mathbb{Z})$
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- Dimension 4 :
  - $\mathit{Symp}_c(\mathbb{R}^4, \omega_0)$ is contractible
  - $\mathrm{Symp}(\mathbb{C}P^1 \times \mathbb{C}P^1, \sigma \oplus \sigma) \sim \mathbb{Z}_2$ extension of $SO(3) \times SO(3)$
  - $\mathrm{Symp}(\mathbb{C}P^2, \omega_{FS}) \sim PU(3)$
  - Let $\mu > 1$, $\mathrm{Symp}_0(\mathbb{C}P^1 \times \mathbb{C}P^1, \mu\sigma \oplus \sigma)$ is path connected and its fundamental group is equal to $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

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Let (M, J, g, ω) be a Kaehler manifold and $\Lambda \in \mathbb{R}$

Any $f \in \mathcal{D}\mathrm{iff}(M)$ is called Λ-pinched if $\frac{1}{\Lambda^2} g \leqslant f^* g \leqslant \Lambda^2 g$

Theorem (M.-T. Wang and I. Mendos)

For each $n \in \mathbb{N}$, there exists a constant $\Lambda(n) > 1$ such that if $f : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ is a Λ-pinched symplecto. for some $0 < \Lambda < \Lambda(n)$, then

- *The m.c.f $(\mathcal{M}_t)_t$ of the graph of f in $\mathbb{C}P^n \times \mathbb{C}P^n$ exists for all time $t \geqslant 0$*
- *$\mathcal{M}_t$ is the graph of a symplecto. f_t for each $t \geqslant 0$*
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$f:(M^{2n},g,J,\omega) \longrightarrow (\widetilde{M}^{2n},\tilde{g},\tilde{J},\tilde{\omega})$, $\left(\mathcal{M}_t\right)_t$ the m.c.f of $\mathcal{M}$ = graph f

$$\begin{array}{ccc} \mathcal{M}_t & \hookrightarrow & \left(M \times \widetilde{M}, g \oplus \tilde{g}\right) \\ & & \pi_1 \swarrow \quad \searrow \pi_2 \\ & & M \qquad\quad \widetilde{M}. \end{array}$$

and $\Omega = \pi_1^* dvol_M$.

Choose the basis $(a_i)_i$ $\left(\text{thus } \left(E(a_i)\right)_i\right)$ on T_xM $\left(\text{on } T_{f(x)}\widetilde{M}\right)$

$$\left\{\begin{array}{rcl} e_i & = & \frac{1}{\sqrt{1+|d_xf(a_i)|^2}}\left(a_i, d_xf(a_i)\right) \\ & = & \frac{1}{\sqrt{1+\lambda_i^2}}\left(a_i, \lambda_i E(a_i)\right) \\ & & \\ e_{2n+i} = \mathcal{J}_{\left(x,f(x)\right)} e_i & = & \frac{1}{\sqrt{1+\lambda_i^2}}\left(J_x a_i, -\tilde{J}_{f(x)} E(a_i)\right) \\ & = & \frac{1}{\sqrt{1+\lambda_i^2}}\left(a_i, -\lambda_i E(a_i)\right). \end{array}\right.$$

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$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \triangle\right) * \Omega &= *\Omega \Big[Q(\lambda_i, h_{ijk}) + \\
&\quad \sum_{i,k} \frac{\lambda_i^2}{(1+\lambda_i^2)(1+\lambda_k^2)} \left(R_{ikik} - \lambda_k^2 \tilde{R}_{ikik}\right) \Big] \qquad (18)
\end{aligned}
$$

where

$$
Q(\lambda_i, h_{ijk}) = \sum_{i,j,k} h_{ijk}^2 - 2 \sum_k \sum_{i<j} (-1)^{i+j} \lambda_i \lambda_j (h_{i'ik} h_{j'jk} - h_{i'jk} h_{j'ik}), \qquad (19)
$$

$i' = i + (-1)^{i+1}$, and $R_{ikik} = R(a_i, a_k, a_i, a_k)$, $\tilde{R}_{ikik} = \tilde{R}(a_i, a_k, a_i, a_k)$

Assume $M = \widetilde{M}$, $E(a_i) = \sum_l b_i^l a_l$ therefore

$$\tilde{R}\Big(E(a_i), E(a_k), E(a_i), E(a_k)\Big) = b_i^l b_k^m b_i^n b_i^r R\Big(a_l, a_m, a_n, a_r\Big) \qquad (20)$$

Let us now take $M = \widetilde{M} = \mathbb{C}P^1 \times \mathbb{C}P^1$.
$E \equiv$ isometry, it can be represented by an element of $SU(2, \mathbb{C})$ as follows

$$E = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad (21)$$

where $\alpha = \alpha_1 + \iota\alpha_2$ and $\beta = \beta_1 + \iota\beta_2$ satisfy $|\alpha|^2 + |\beta|^2 = 1$.

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It follows the identification :

$$\begin{pmatrix} \alpha_1 & -\alpha_2 & -\beta_1 & -\beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & -\beta_1 \\ \beta_1 & -\beta_2 & \alpha_1 & \alpha_2 \\ \beta_2 & \beta_1 & -\alpha_2 & \alpha_1 \end{pmatrix} \equiv \begin{pmatrix} b_1^1 & b_1^2 & b_1^3 & b_1^4 \\ b_2^1 & b_2^2 & b_2^3 & b_2^4 \\ b_3^1 & b_3^2 & b_3^3 & b_3^4 \\ b_4^1 & b_4^2 & b_4^3 & b_4^4 \end{pmatrix}$$

which will help us to reduce the expression of $\tilde{R}_{ikik}$.

For brevity, let us introduce the following notations

$$\begin{aligned} \alpha \cdot \beta &= \alpha_1\beta_1 + \alpha_2\beta_2\,, & \bar{\alpha} \cdot \beta = \alpha_1\beta_1 - \alpha_2\beta_2, \\ \iota\alpha \cdot \beta &= \alpha_1\beta_2 - \alpha_2\beta_1\,, \qquad \text{and} & \iota\alpha \cdot \bar{\beta} = -\alpha_1\beta_2 - \alpha_2\beta_1 \end{aligned}$$

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where $\cdot$ stands for the dot product on $\mathbb{C}$ view as vector space.

$M = (\mathbb{C}P^1, J_1) \times (\mathbb{C}P^1, J_2)$ and take a canonical orthonormal basis (c_1, c_2) (resp. (c_3, c_4)) for the first factor (resp. the second factor) such that we construct $(a_i)_i$ as follows :

$$
(S') \begin{cases} a_1 &= \quad xc_1 + yc_3 \\ a_2 = (J_1 \oplus J_2)(a_1) &= \quad xc_2 + yc_4 \\ a_3 &= -yc_1 + xc_3 \\ a_4 = (J_1 \oplus J_2)(a_3) &= -yc_2 + xc_4 \end{cases}
$$

where x and y are real numbers satisfying $x^2 + y^2 = 1$.

With respect to this basis, we obtain

$$
\begin{aligned}
R_{1212} &= x^4 R^{(1)}_{1212} + y^4 R^{(2)}_{3434} = x^4 + y^4 = R_{3434} \\
R_{1414} &= x^2 y^2 \left(R^{(1)}_{1212} + R^{(2)}_{3434} \right) = 2x^2 y^2 \\
&= R_{2323} = R_{1234} = -R_{1423} \\
R_{1214} &= -x^3 y R^{(1)}_{1212} + x y^3 R^{(2)}_{3434} = -xy(x^2 - y^2) \\
&= R_{2123} = -R_{3234} = -R_{4143}
\end{aligned}
$$

otherwise $R_{ijkl} = 0$.

$$
\begin{aligned}
\tilde{R}_{1212} &= \tilde{R}\Big(E(a_1), E(a_2), E(a_1), E(a_2)\Big) \\
&= \left(|\alpha|^4 + |\beta|^4\right)\left(x^4 + y^4\right) + 4x^2y^2|\alpha|^2|\beta|^2 + 8x^2y^2(\bar{\alpha}\cdot\beta)^2 \\
&\quad +4(x^3y - xy^3)(\bar{\alpha}\cdot\beta)(|\alpha|^2 - |\beta|^2) \\
&= \tilde{R}\Big(E(a_3), E(a_4), E(a_3), E(a_4)\Big) = \tilde{R}_{3434}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\tilde{R}_{1313} &= \tilde{R}\Big(E(a_1), E(a_3), E(a_1), E(a_3)\Big) \\
&= 2(\iota\alpha\cdot\beta)^2\left(x^2 - y^2\right)^2 + 8x^2y^2(\alpha_1\alpha_2 + \beta_1\beta_2)^2 \\
&\quad +8(x^3y - xy^3)(\iota\alpha\cdot\beta)(\alpha_1\alpha_2 + \beta_1\beta_2) \\
&= \tilde{R}\Big(E(a_2), E(a_4), E(a_2), E(a_4)\Big) = \tilde{R}_{2424}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\tilde{R}_{1414} &= \tilde{R}\Big(E(a_1), E(a_4), E(a_1), E(a_4)\Big) \\
&= 2(\alpha \cdot \beta)^2(x^2 - y^2)^2 + 2x^2y^2\Big((\alpha_2^2 + \beta_1^2) - (\alpha_1^2 + \beta_2^2)\Big)^2 \\
&\quad +4(x^3y - xy^3)(\alpha \cdot \beta)\Big((\alpha_2^2 + \beta_1^2) - (\alpha_1^2 + \beta_2^2)\Big) \\
&= \tilde{R}\Big(E(a_2), E(a_3), E(a_2), E(a_3)\Big)\tilde{R}_{2323}.
\end{aligned} \tag{24}
$$

Lemma

$*\Omega$ satisfies :

$$\left(\frac{\partial}{\partial t} - \triangle\right) * \Omega = *\Omega\Bigg\{ Q(\lambda_i, h_{ijk}) + \frac{2\left(\lambda_3^2 - \lambda_1^2\right)^2}{\left(1+\lambda_1^2\right)^2\left(1+\lambda_3^2\right)^2}\Bigg(2x^2y^2\left(8(\alpha\cdot\beta)^2 + 4(\alpha_1\alpha_2 - \beta_1\beta_2)^2 - 1\right)$$
$$+4xy(x^2-y^2)(\alpha\cdot\beta)\left(\alpha_1^2+\beta_2^2-\alpha_2^2-\beta_1^2\right) + \left(1 - 2(\alpha\cdot\beta)^2\right)\Bigg)$$
$$+\frac{2\left(1-\lambda_1^2\lambda_3^2\right)^2}{\left(1+\lambda_1^2\right)^2\left(1+\lambda_3^2\right)^2}\Bigg(2x^2y^2\left(4(\iota\alpha\cdot\beta)^2 - 4\left(\alpha_1\alpha_2+\beta_1\beta_2\right)^2 - 1\right)$$
$$-8xy(x^2-y^2)(\iota\alpha\cdot\beta)\left(\alpha_1\alpha_2+\beta_1\beta_2\right) + \left(1-2(\iota\alpha\cdot\beta)^2\right)\Bigg)\Bigg\} \tag{25}$$

where $Q(\lambda_i, h_{ijk})$ is the quadratic form given in (19), α, β are complexes such that $|\alpha|^2 + |\beta|^2 = 1$ and x, y are reals satisfying $x^2 + y^2 = 1$.

Consider the function $\Theta(\lambda_1, \lambda_3)$ with the parameters x, y, α and β

$$\Theta(\lambda_1, \lambda_3) = 2\theta \frac{\left(\lambda_3^2 - \lambda_1^2\right)^2}{\left(1+\lambda_1^2\right)^2\left(1+\lambda_3^2\right)^2} + 2\kappa \frac{\left(1 - \lambda_1^2\lambda_3^2\right)^2}{\left(1+\lambda_1^2\right)^2\left(1+\lambda_3^2\right)^2} \quad (26)$$

where

$$\begin{aligned} \theta &= \left(1 - 2(\alpha\cdot\beta)^2\right) + 2x^2y^2\left(8(\alpha\cdot\beta)^2 + 4(\alpha_1\alpha_2 - \beta_1\beta_2)^2 - 1\right) \\ &+ 4xy(x^2 - y^2)(\alpha\cdot\beta)\left(\alpha_1^2 + \beta_2^2 - \alpha_2^2 - \beta_1^2\right) \end{aligned} \quad (27)$$

and

$$\begin{aligned} \kappa &= \left(1 - 2(\iota\alpha\cdot\beta)^2\right) + 2x^2y^2\left(4(\iota\alpha\cdot\beta)^2 - 4\left(\alpha_1\alpha_2 + \beta_1\beta_2\right)^2 - 1\right) \\ &- 8xy(x^2 - y^2)(\iota\alpha\cdot\beta)\left(\alpha_1\alpha_2 + \beta_1\beta_2\right). \end{aligned} \quad (28)$$

α, β are complexes such that $|\alpha|^2 + |\beta|^2 = 1$ and x, y are reals satisfying $x^2 + y^2 = 1$

$$\begin{cases} \Theta(1,1) & = & 0 \\ \nabla \cdot \Theta(1,1) & = & 0 \\ \frac{\partial^2}{\partial \lambda_1^2}\Theta(1,1) & = & \theta + \kappa = \frac{\partial^2}{\partial \lambda_3^2}\Theta(1,1) \\ \frac{\partial^2}{\partial \lambda_1 \partial \lambda_3}\Theta(1,1) & = & -\theta + \kappa \end{cases}$$

One easily check with Sylvester Criterion applied to the Hessain of Θ, that $\theta + \kappa$ and $\theta\kappa$ are positive. This implies that the Hessian of Θ is positive definite at $(\lambda_1, \lambda_3) = (1, 1)$. Thus Θ has a local minimal at $(1, 1)$ this infers $\Theta(\lambda_1, \lambda_3) \geqslant 0$

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Proposition

There exists $1 \lessapprox \tilde{\Lambda} < \frac{1}{5}\left(2\sqrt{10} + \sqrt{15}\right)$ *close to 1 such that if the inequality (29) holds for some* $1 < \tilde{\Lambda}_1 < \tilde{\Lambda}$

$$\frac{\tilde{\Lambda}_1}{2\left(1 + (\tilde{\Lambda}_1)^2\right)} \leqslant \min_{\mathcal{M}_0} *\Omega \tag{29}$$

then $\min_{\mathcal{M}_t} *\Omega$ *is non-decreasing in time. In particular,* $\mathcal{M}_t$ *is the graph of some* $f_t \in Symp(\mathbb{C}P^1 \times \mathbb{C}P^1)$.

Corollary

There exists $1 \lessapprox \tilde{\Lambda} < \frac{1}{5}\left(2\sqrt{10} + \sqrt{15}\right)$ close to 1 such that, if the initial symplectomorphism f is $\tilde{\Lambda}'$-pinched with

$$\tilde{\Lambda}' = \left[\frac{1}{2}\left(\tilde{\Lambda} + \frac{1}{\tilde{\Lambda}}\right)\right]^{\frac{1}{2}} + \sqrt{\frac{1}{2}\left(\tilde{\Lambda} + \frac{1}{\tilde{\Lambda}}\right) - 1}$$

then each f_t is $\tilde{\Lambda}$-pinched along the flow.

Theorem

Let f be a Λ-pinched symplecto. of $\mathbb{C}P^1 \times \mathbb{C}P^1$ for some $1 < \Lambda < \frac{1}{5}(2\sqrt{10} + \sqrt{15})$ sufficiently closed to 1. Then f deforms through symplecto. under the m.c.f ; the flow exists for all time and the sequence $(f_t)_t$ converges smoothly to a biholomorphic isometry of $\mathbb{C}P^1 \times \mathbb{C}P^1$ as t goes to infinity.

Remark

- Since λ_i tend to 1 as t goes to infinity, for all i, the limit map f^∞ is an isometry.
- Being symplectic is a closed property, so f^∞ is symplectic.
- Then at every $p \in \mathbb{C}P^1 \times \mathbb{C}P^1$,

$$f^\infty_* J = J f^\infty_* .$$

The same is true for the inverse of f^∞, and thus the map f^∞ is biholomorphic.

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Hofer defined the energy of an hamiltonian diffeo. φ as

$$E(\varphi) := \inf\left\{ \|G\| := \int_0^1 (\sup G - \inf G)\, dt \ : \ \varphi_1^G = \varphi \right\}$$

Lemma

The potential G_t of the time slice of the flow is given by :

$$G_t = G_0 + U_t + V_t \tag{30}$$

where $V_t = \int_0^t \int_0^1 f_{s,\tau}^* \omega(X_{s,\tau}, H_{s,\tau}) ds d\tau$ and U_t is such that $dU_t = \int_0^t f_\tau^* i_{H_\tau} \omega d\tau$.

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$$\frac{\partial}{\partial t}\mathcal{F}_t = f_t^* i_{H_t}\omega + d\int_0^1 f_{s,t}^*\omega\big(X_{s,t}, H_{s,t}\big). \tag{31}$$

Integrating in time we got :

$$\mathcal{F}_t = \mathcal{F}_0 + \int_0^t f_\tau^* i_{H_\tau}\omega d\tau + d\int_0^t\int_0^1 f_{s,\tau}^*\omega(X_{s,\tau}, H_{s,\tau})dsd\tau \tag{32}$$

Maslov class $= 0 \implies \exists U_t$ such that $dU_t = \int_0^t f_\tau^* i_{H_\tau}\omega d\tau$. Therefore

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