G_2-manifolds

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23. 5. 2018

Outline

- Holonomy
  Intro and general concepts
- G_2-geometry
  Properties, constructions, questions
- The ν-invariant
  Definition, properties, first examples
- Extra twisted connected sums
  Computation of the ν-invariant, consequences

Holonomy

Let (M, g) be a Riemannian manifold, $\dim M = m$, with Levi-Civita connection ∇.

Fix $p \in M$. For each loop $\gamma \colon [a, b] \to M$ at p, parallel translation with respect to ∇ defines an automorphism of the tangent space at p,

$$\Pi_\gamma \colon T_pM \to T_pM$$

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The holonomy group is the group of all these parallel translations

$$\mathrm{Hol}_p(M, g) = \left(\left\{ \Pi_\gamma \;\middle|\; \gamma \text{ loop at } p \right\}, \circ, \mathrm{id}\right)$$

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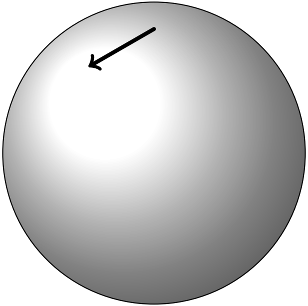
Holonomy—an example

Consider parallel translation along a spherical triangle



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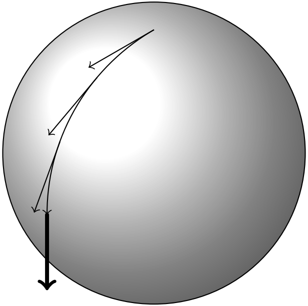

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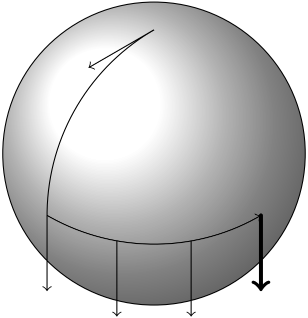

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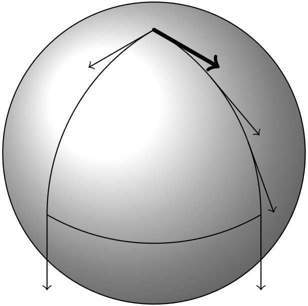

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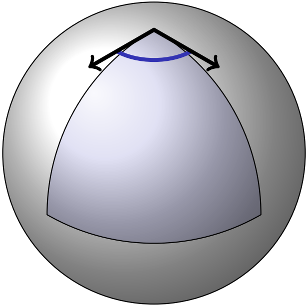

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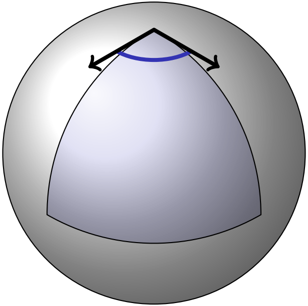
$$\mathrm{Hol}(S^2, g^{\mathrm{rnd}}) \cong SO(2)$$



Holonomy—Kähler manifolds

Let (M, g, J) be a Kähler manifold, $\dim M = 2n$, so J is a parallel complex structure on the tangent bundle TM
In particular, (M, J) is a complex manifold

Examples: complex projective space,
smooth complex projective varieties

Parallel translation commutes with J, hence

$$\mathrm{Hol}_p(M, g) \subset \mathrm{Aut}(T_pM, g_p, J_p) \cong U(n) \subset SO(2n)$$

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Hodge decomposition of de Rham cohomology on (M, g, J)

$$\Omega^\ell(M; \mathbb{C}) = \bigoplus_{p+q=\ell} \Omega^{p,q}(M)$$

$$H^\ell_{\mathrm{dR}}(M; \mathbb{C}) = \bigoplus_{p+q=\ell} H^{p,q}(M)$$

Holonomy—Calabi-Yau manifolds

Let (M, g, J) be a Kähler manifold, $\dim_{\mathbb{C}} M = n$
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Let P_{SO} denote the manifold of oriented orthonormal bases of TM. Then $SO(n)$ acts on P_{SO} by change of bases, and

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If $\Lambda^\bullet \mathbb{R}^n$ is a direct sum of G-invariant subspaces, then $\Lambda^\bullet T^*M$ is a direct sum of parallel subbundles

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Example. Hodge decomposition on Kähler manifolds

Holonomy—Parallel spinors

If the holonomy group G is connected and simply connected, then (M, g) is a spin manifold with spinor bundle

$$\mathcal{S}M = P_G \times_G \mathcal{S}\mathbb{R}^n \qquad \text{with} \qquad \mathcal{S}\mathbb{R}^n \cong \mathbb{R}^{2^{\left[\frac{n}{2}\right]}}$$

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Example. Let (M, g) be a Calabi-Yau manifold. Then

$$\mathcal{S}M \cong \Lambda^{0,\bullet} T^* M$$

The forms $1 \in \Omega^{0,0}(M)$ and $\bar{\Omega} \in \Omega^{0,n}(M)$ are parallel spinors and $\mathrm{ric} = 0$

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Theorem (Berger)

Let (M, g) *be a complete, simply connected Riemannian manifold and neither a product nor a symmetric space*

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$\mathrm{Hol}(M, g)$	dim	ric	*Structure*	N	*Name*
$SO(n)$	n			0	*general case*
$U(k)$	$2k$		J	0	*Kähler*
$SU(k)$	$2k$	0	J, Ω	2	*Calabi-Yau*
$Sp(\ell) \cdot Sp(1)$	4ℓ	const	$\langle I, J, K \rangle$	0	*Quat. Kähler*
$Sp(\ell)$	4ℓ	0	I, J, K, Ω	$\ell + 1$	*hyper Kähler*
G_2	7	0	$\varphi \in \Omega^3$	1	*exceptional*
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- Only special holonomy group for odd dimensions
- Only G_2 and Spin(7) holonomy have no direct relation to algebraic geometry

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Physical motivation

- In string theory, spacetime takes the form $\mathbb{R}^{3,1} \times V$, where V is a Calabi-Yau manifold
- In M-theory, spacetime takes the form $\mathbb{R}^{3,1} \times M$, where M is a G_2-manifold
- Possible relations to other physical theories

Hence, many fruitful interactions possible

G_2-Geometry—The group G_2

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The octonions split as $\mathbb{O} \cong \mathbb{R} \oplus \operatorname{Im} \mathbb{O}$ with $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$
The octonion multiplication induces a product and a norm

$$u \times v = \operatorname{Im}(u \cdot v) \quad \text{and} \quad \|u\| = \sqrt{-u^2} \quad \text{for } u, v \in \operatorname{Im} \mathbb{O}$$

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The group G_2 is defined as

$$G_2 = \operatorname{Aut}(\mathbb{O}) = \operatorname{Aut}(\operatorname{Im} \mathbb{O}, \times, \| \cdot \|)$$

G_2-geometry—The 3-form φ

Assume $\mathrm{Hol}(M, g) \subset G_2$. Then "$\times$" induces a parallel product on TM and a parallel 3-form $\varphi \in \Omega^3(M)$ with

$$\varphi(u, v, w) = \langle u \times v, w \rangle$$

Note that

$$\varphi \wedge \iota_u\varphi \wedge \iota_v\varphi = -6\langle u, v\rangle \, d\,\mathrm{vol}_g \qquad (*)$$

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Conversely, call an arbitrary $\varphi \in \Omega^3(M)$ positive if there exists a Riemannian metric g_φ on M such that (*) holds

Being positive is a point-wise, open condition on $\Lambda^3 T^* M$

Each positive 3-form φ defines a G_2-structure on M

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If φ above is closed and coclosed with respect to g_φ, then φ is parallel, also called torsion free, and $\mathrm{Hol}(M, g) \subset G_2$
Write (M, g) for (M, φ) if (M, g) has G_2-holonomy

G_2-geometry—Representation theory

If M has a G_2-structure, we have "Hodge decompositions"

$$\begin{aligned}
\Omega^0(M) &\cong \Omega^7(M) \cong \Omega_1(M) \\
\Omega^1(M) &\cong \Omega^6(M) \cong \qquad\qquad \Omega_7(M) \\
\Omega^2(M) &\cong \Omega^5(M) \cong \qquad\qquad \Omega_7(M) \oplus \Omega_{14}(M) \\
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G_2-manifolds are spin, and the spinor bundle decomposes

$$\mathcal{S}M \cong TM \oplus \underline{\mathbb{R}}$$

There exists a distinguished parallel spinor $\sigma \neq 0$, and $\mathrm{ric} = 0$

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Let $p_1(TM) \in H^4(M;\mathbb{R})$ denote the first Pontryagin class
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These are all known obstructions against holonomy G_2

G_2-geometry—The moduli space

Let M be a compact oriented 7-manifold and define

$$\mathcal{X} = \left\{ \varphi \in \Omega^3(M) \,\middle|\, \varphi \text{ is positive and torsion free } \right\}$$

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Constructions of (M, g) with $\mathrm{Hol}(M, g) = G_2$

- Bryant '87: first non-complete examples
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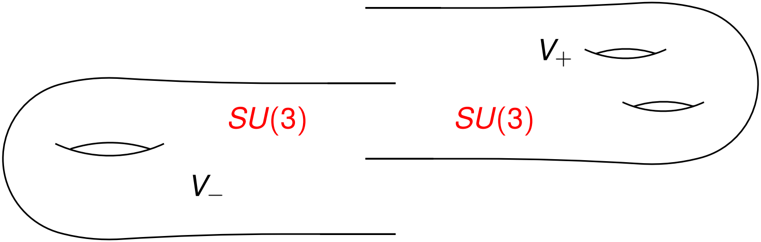

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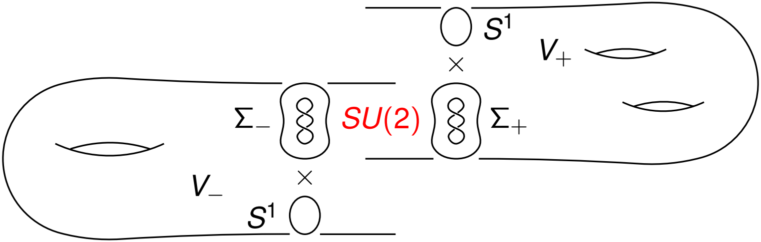

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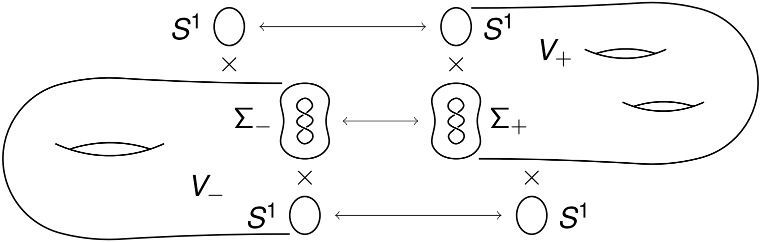

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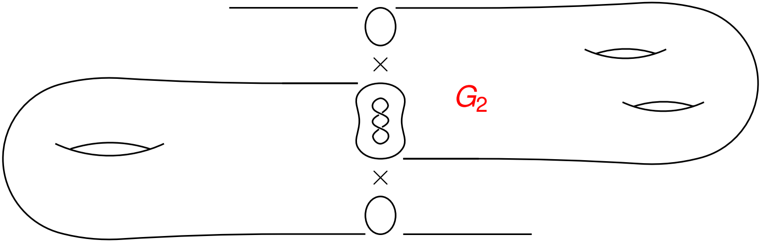

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  Singularities represent matter and forces in M-theory

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Recall: every G_2-structure leads
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Idea. Use nowhere vanishing spinors
to describe and distinguish G_2-structures

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Let σ_0, σ_1 be two nowhere vanishing spinors
Extend to a spinor $\bar{\sigma} \in \Gamma(\mathcal{S}^+(M \times [0,1]))$
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The ν-invariant—Cobordism definition

Idea. If M is spin, then M is the spin coboundary of some compact 8-manifold W (because $\Omega^7_{\mathrm{Spin}} = 0$)
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Definition (Crowley-Nordström)

Assume that $M = \partial W$ with W spin, compact. Define

$$\nu(M, \sigma) = \chi(W) - 3\,\text{sign}(W) - 2\#\bar{\sigma}^{-1}(0) \mod 48$$

- $\chi(W)$—Euler characteristic of W
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Theorem (Atiyah-Patodi-Singer)

Assume that $M = \partial W$ with W compact, then

$$\mathrm{sign}(W) = \int_W L(\nabla) - \eta(B_M)$$

- sign—signature
- $L(\nabla)$—Hirzebruch L-form in $\Omega^\bullet(W)$
- B_M—odd signature operator $*d \pm d*$ on $\Omega^{\mathrm{ev}}(M)$
- η—Atiyah-Patodi-Singer η-invariant

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Theorem (Atiyah-Patodi-Singer)

Assume that $M = \partial W$ with W compact, spin, then

$$\mathrm{ind}(D_W) = \int_W \hat{A}(\nabla) - \frac{\eta + h}{2}(D_M)$$

- D_W, D_M—spin Dirac operators on $\Gamma(\mathcal{S}^{\pm} W)$ and $\Gamma(\mathcal{S} M)$
- ind—Fredholm index
- $\hat{A}(\nabla)$—Atiyah $\hat{A}$-form in $\Omega^\bullet(W)$
- η—Atiyah-Patodi-Singer η-invariant
- $h = \dim\ker$

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Theorem (Gauß-Bonnet-Chern)

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$$\chi(W) = \int_W e(\nabla)$$

- χ—Euler characteristic
- $e(\nabla)$—Euler form in $\Omega^\bullet(W)$

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Theorem (Matthai-Quillen)

Assume that $M = \partial W$ with W compact, spin, then

$$\#\bar{\sigma}^{-1}(0) = \int_W e(\nabla^{\mathcal{S}^+W}) - \int_M \sigma^* \psi(\nabla^{\mathcal{S}M})$$

- $e(\nabla^{\mathcal{S}^+W})$—Euler form of the spinor bundle
- $\psi(\nabla^{\mathcal{S}M}, g^{SM})$—Mathai-Quillen form in $\Omega^\bullet(\mathcal{S}M)$

The ν-invariant—Analytic description

Magic formula

$$2e\left(\nabla^{\mathcal{S}^+W}\right) = e(\nabla) + 48\hat{A}(\nabla)^{[8]} - 3L(\nabla)^{[8]} \in \Omega^8(W)$$

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- $\psi(\nabla^{\mathcal{S}M}, g^{SM})$—Mathai-Quillen form in $\Omega^\bullet(\mathcal{S}M)$
- D_M—spin Dirac operator on $\Gamma(\mathcal{S}M)$
- B_M—odd signature operator $*d \pm d*$ on $\Omega^{\mathrm{ev}}(M)$
- h—dimension of the kernel
- η—Atiyah-Patodi-Singer η-invariant

$$\eta(A) = \left(\sum_{\lambda\in\mathrm{spec}(A)} \operatorname{sign}\lambda\ |\lambda|^t\right)\Bigg|_{t=0} = \int_0^\infty \operatorname{tr}\left(A\,e^{-tA^2}\right)\frac{dt}{\sqrt{\pi t}}$$

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- $\eta(D_M) \in \mathbb{R}$ is smooth on the G_2-moduli space $\mathcal{M}$

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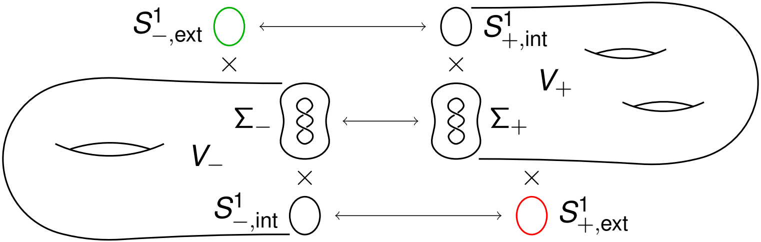
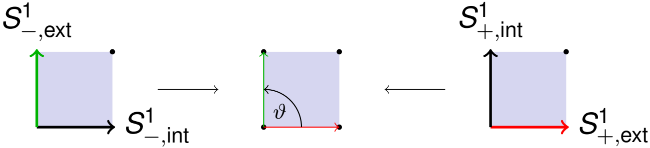
Answer. We will construct examples with $\bar{\nu}(M, g) \neq 0$
Using $\bar{\nu}(M, g)$, we will show that for some particular M, the G_2-moduli space $\mathcal{M}$ has several connected components

Extra twisted connected sums—Construction

Recall twisted connected sums

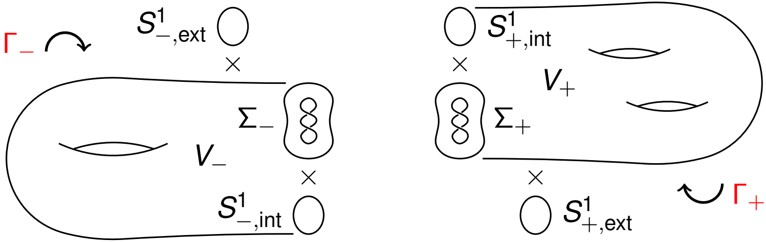


Gluing of tori at angle $\vartheta = \frac{\pi}{2}$



Extra twisted connected sums—Construction

Extra twisted connected sums



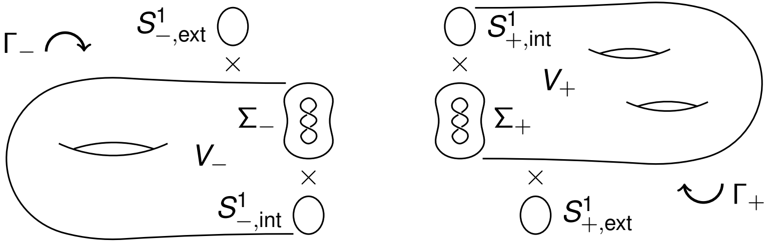
Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts both on $V_\pm$ and on $S^1_{\pm,\mathrm{ext}}$

The induced action on $\partial V_\pm$ has to fix $\Sigma_\pm$ pointwise

The actions on $S^1_{\pm,\mathrm{int}}$ and $S^1_{\pm,\mathrm{ext}}$ have to be free

Extra twisted connected sums—Construction

Extra twisted connected sums



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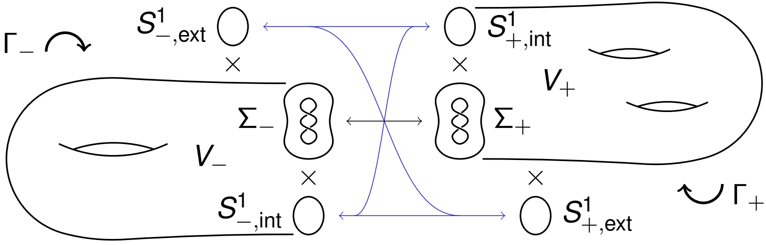

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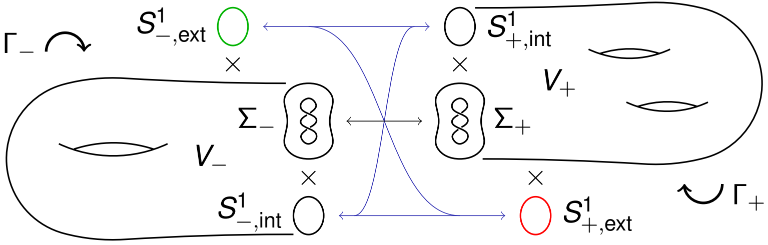
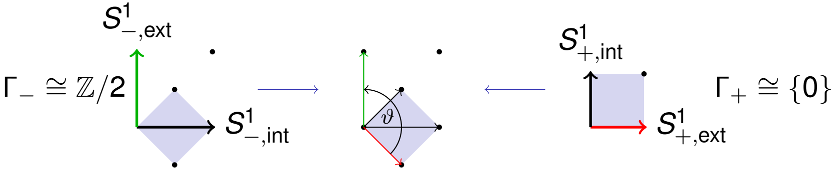
Then $(S^1_{\pm,\mathrm{int}} \times S^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ is again a flat 2-torus
If both the tori and the K3 surfaces are isometric,
we can glue $M_\pm = (V_\pm \times S^1_{\pm,\mathrm{ext}})/\Gamma_\pm$ at various angles ϑ

Extra twisted connected sums—Construction

Extra twisted connected sums



Modified gluing of tori at angle $\vartheta = \frac{3}{4}\pi$



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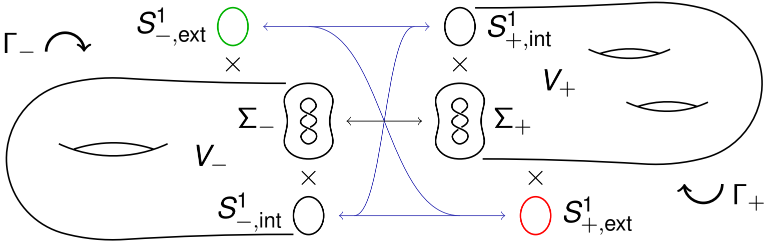
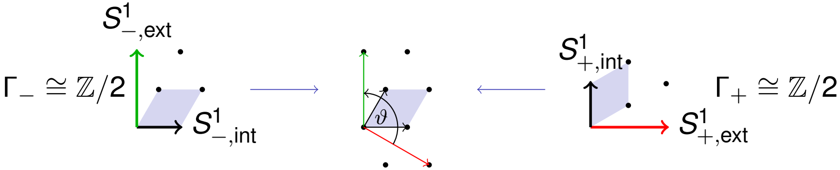


Modified gluing of tori at angle $\vartheta = \frac{2}{3}\pi$



Extra twisted connected sums—Construction

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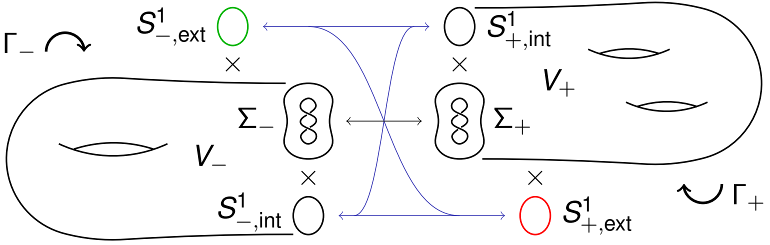
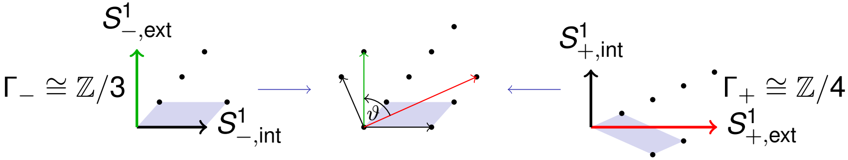


Modified gluing of tori at angle $\vartheta = \arccos\left(\frac{1}{\sqrt{6}}\right)$



Extra twisted connected sums—The gluing formula

Compute the extended ν-invariant on $M = M_+ \cup_X M_-$

Theorem (Bunke, Kirk-Lesch)

For suitable boundary conditions $L_\pm$,

$$\eta(D_M) = \eta_{\mathrm{APS}}(D_{M_+}; L_+) + \eta_{\mathrm{APS}}(D_{M_-}; L_-) + m_{\ker(D_X)}(L_+, L_-)$$
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Theorem (Crowley-G-Nordström)

Let (M, g) be an extra twisted connected sum with gluing angle ϑ and $\rho = \pi - 2\vartheta$, then there exists $N \in \mathbb{Z}$ such that

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One computes N from the maps $H^2(V_\pm) \to H^2(\Sigma_+) \cong H^2(\Sigma_-)$

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Note that $\frac{\rho}{\pi}$ can be irrational, e.g., for $\vartheta = \arccos\left(\frac{1}{\sqrt{6}}\right)$

Extra twisted connected sums—First results

Theorem (Crowley-G-Nordström)

If $\Gamma_\pm \cong \{0\}$ *or* $\mathbb{Z}/2$ *then*

$$\eta_{\mathrm{APS}}(D_{V_\pm \times S^1}; L_\pm) = \eta_{\mathrm{APS}}(B_{V_\pm \times S^1}; L_\pm) = 0$$

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From the gluing formula, we derive the value of $\bar{\nu}$

Corollary

If $\Gamma_\pm \cong \{0\}$ *or* $\mathbb{Z}/2$ *then*

$$\bar{\nu}(M, g) = -72\frac{\rho}{\pi} - 3N \operatorname{sign} \rho$$

Extra twisted connected sums—Examples I

Example (Crowley-G-Nordström)

There exists a spin 7-manifold M with

$$H^1(M) \cong H^2(M) \cong 0\,, \qquad H^4(M) \cong \mathbb{Z}^{97}\,, \quad \operatorname{div} p_1(TM) = 4$$

admitting three different G_2-holonomy metrics g_1, g_2, g_3 with

$$\bar\nu(M, g_1) = 0\,, \qquad \bar\nu(M, g_2) = 36\,, \qquad \bar\nu(M, g_3) = -36\,.$$

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The metric g_1 comes from a rectangular twisted connected sum
The metrics g_2, g_3 come from extra twisted connected sums
with $\Gamma_+ \cong \mathbb{Z}/2$, $\Gamma_- \cong \{0\}$ and with gluing angles $\frac{\pi}{4}$

Extra twisted connected sums—Examples II

Example (Crowley-G-Nordström)

There exists a spin 7-manifold M with

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In particular $\nu(M, \sigma_1) = \nu(M, \sigma_2) = \nu(M, \sigma_3) = 0$, and one can show that the underlying G_2-structures are homotopic.

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Let (M, g) be an extra twisted connected sum with $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ where k_+, $k_- \in \{1, 2\}$ and with $\vartheta \in \left\{\pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}, \frac{\pi}{2}\right\}$
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Note that $\Omega^7_{G_2} \cong \mathbb{Z}/3$
One can show that M is G_2-nullbordant if and only if $3 \mid \nu(M, \sigma)$
Hence there exists a compact W with a G_2-structure such that $\partial W = M$ if and only if $3 \mid \nu(M, \sigma)$

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Answer. No, there are examples with $3 \nmid \nu(M, \sigma)$.

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Note. Recall that if $k_+ \geq 3$ or $k_- \geq 3$, then there are gluing angles ϑ such that $\frac{\rho}{\pi} \notin \mathbb{Q}$
Because $\bar{\nu}(M, g) \in \mathbb{Z}$, expect more contributions to $\bar{\nu}(M, g)$

Extra twisted connected sums—Adiabatic limits

Let $M_\pm = V_\pm \times S^1_{\pm,\mathrm{ext}}$, rescale $S^1_{\pm,\mathrm{ext}}$ by $\varepsilon > 0$ to get $M_{\pm,\varepsilon}$
The limit $\varepsilon \to 0$ is called adiabatic limit

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Theorem (Bismut-Cheeger, Dai, G-Nordström)

Let $\gamma \in \Gamma_\pm$ be the generator that acts by $\frac{2\pi}{k_\pm}$ on $S^1_{\pm,\mathrm{ext}}$
Let $V_{\pm,j}$ be the set of isolated fixpoints of γ^j on $V_\pm$
Let $e^{i\alpha_{j,1}(p)}$, $e^{i\alpha_{j,2}(p)}$, $e^{i\alpha_{j,3}(p)}$ be the eigenvalues of γ^j on $T_pV_\pm$

$$\begin{aligned}\bar\nu(M_\pm) &= \lim_{\varepsilon\to 0}\Big(3\eta(B_{M_{\pm,\varepsilon}}, L_\pm) - 24\eta(D_{M_{\pm,\varepsilon}}, L_\pm)\Big)\\ &= \frac{3}{k_\pm}\sum_{j=1}^{k_\pm-1}\cot\frac{\pi j}{k_\pm}\sum_{p\in V_{\pm,j}}\frac{\cos\frac{\alpha_{j,1}(p)}{2}\cos\frac{\alpha_{j,2}(p)}{2}\cos\frac{\alpha_{j,3}(p)}{2} - 1}{\sin\frac{\alpha_{j,1}(p)}{2}\sin\frac{\alpha_{j,2}(p)}{2}\sin\frac{\alpha_{j,3}(p)}{2}}\\ &\in \mathbb{Q}\end{aligned}$$

Extra twisted connected sums—Adiabatic limits

Let $M_\pm = V_\pm \times S^1_{\pm,\mathrm{ext}}$, rescale $S^1_{\pm,\mathrm{ext}}$ by $\varepsilon > 0$ to get $M_{\pm,\varepsilon}$
The limit $\varepsilon \to 0$ is called adiabatic limit

Theorem (Bismut-Cheeger, Dai, G-Nordström)

Let $\gamma \in \Gamma_\pm$ be the generator that acts by $\frac{2\pi}{k_\pm}$ on $S^1_{\pm,\mathrm{ext}}$
Let $V_{\pm,j}$ be the set of isolated fixpoints of γ^j on $V_\pm$
Let $e^{i\alpha_{j,1}(p)}$, $e^{i\alpha_{j,2}(p)}$, $e^{i\alpha_{j,3}(p)}$ be the eigenvalues of γ^j on $T_pV_\pm$

$$\begin{aligned}
\bar\nu(M_\pm) &= \lim_{\varepsilon\to 0}\Bigl(3\eta(B_{M_{\pm,\varepsilon}}, L_\pm) - 24\eta(D_{M_{\pm,\varepsilon}}, L_\pm)\Bigr)\\
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&\in \mathbb{Q}
\end{aligned}$$

There are examples with $k_\pm \geq 3$ where $\bar\nu(M_\pm) \neq 0$

Extra twisted connected sums—Variational formula

The η-invariants of $M_{\pm,\varepsilon}$ depend on ε

Theorem (Bismut-Cheeger, Dai-Freed)

The variational formula for the η-invariant of a Dirac type operator a manifold with boundary consists of

- *the integral of a Chern-Simons form over the interior*
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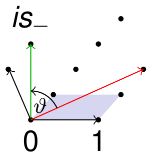
Integrate over ε to get the missing last contribution to $\bar{\nu}(M,g)$

Extra twisted connected sums—Matching the tori

Idea. Represent adiabatic limits as rays in the hyperbolic plane

Lattices $\Lambda_- \subset \Lambda \supset \Lambda_+$ in $\mathbb{C}$



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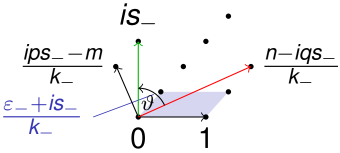
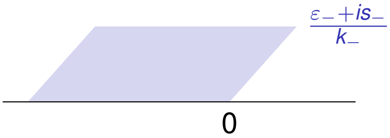


Hyperbolic plane $\mathcal{H} = \mathbb{R} + i\mathbb{R}_>$



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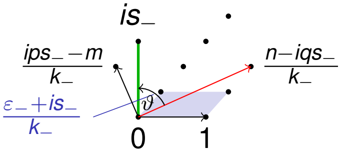

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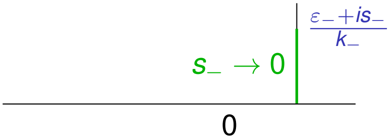

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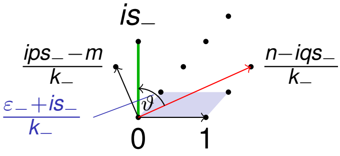

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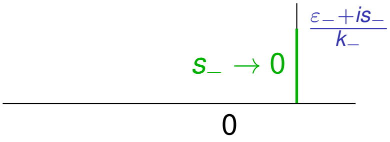
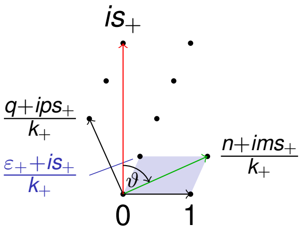
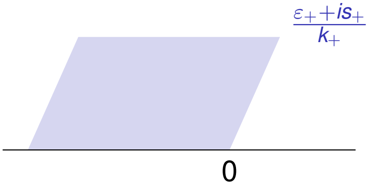


Conformal change of basis





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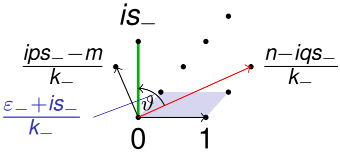

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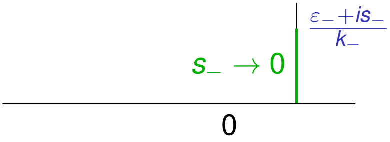

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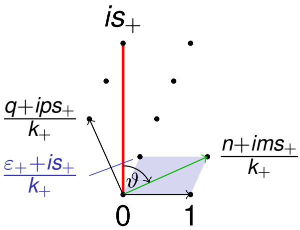

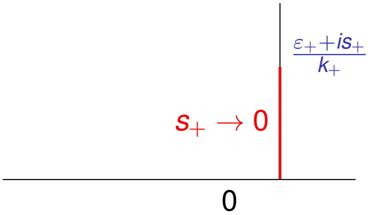

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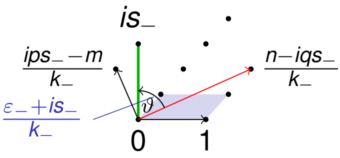

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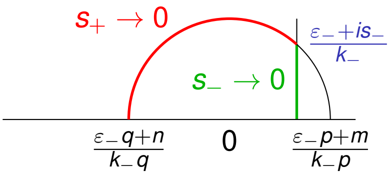

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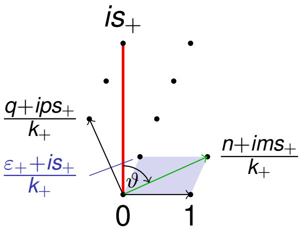
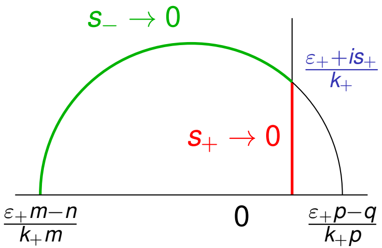


Möbius transformation



Extra twisted connected sums—Modular functions

The logarithm of the Dedekind η-function is given by

$$L(\tau) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty} \sum_{d|n} d^{-1}\, e^{2\pi i n \tau}$$

Theorem (G-Nordström-Zagier)

There exists a constant $c_{k_\pm,\varepsilon_\pm} \in \mathbb{Q}$ such that

$$\begin{aligned} &3\eta(B_{M_\pm}; L_\pm) - 24\eta(D_{M_\pm}; L_\pm) \\ &= \bar{\nu}(M_\pm) + \frac{144}{\pi}\left(iL\Big(\frac{s_\pm i + \varepsilon_\pm}{k_\pm}\Big) - iL\Big(\frac{s_\pm i - \varepsilon_\pm}{k_\pm}\Big) + c_{k_\pm,\varepsilon_\pm} \right) \end{aligned}$$

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Compute the variational term using the functional equations

$$L(\tau+1) = \frac{\pi i}{12} + L(\tau) \qquad \text{and} \qquad L\Big(-\frac{1}{\tau}\Big) = \frac{1}{2}\log\frac{\tau}{i} + L(\tau)$$

Extra twisted connected sums—Hyperbolic geometry

The exterior differential of the η-form $\tilde{\eta}(\mathbb{A})$ is

$$d\tilde{\eta}(\mathbb{A}) = -\frac{1}{4\pi} dA_{\mathsf{hyp}}$$

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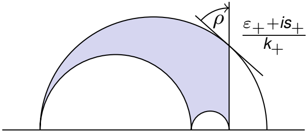

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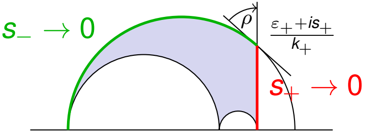

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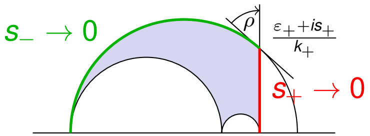

The bottom arcs do not contribute for symmetry reasons
Compute the variational term using Gauß-Bonnet
The corner angle cancels the irrational contribution from $-72\frac{\rho}{\pi}$
There are additional contributions from the cusps

Extra twisted connected sums—Results and questions

Example. The example with $\cos\vartheta = \frac{1}{\sqrt{6}}$ has $\bar{\nu}(M, g) = -65$
In particular, $3 \nmid \bar{\nu}(M, g)$, so it is not G_2-nullbordant

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- How many different G_2-metrics exist on one 7-manifold?
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Construct more examples

- Find more asymptotically cylindrical Calabi-Yau manifolds
- Understand their moduli space, make the K3 surfaces match
- Consider other constructions

Thanks for your attention!