On the geometry of the space of standard subspaces of a Hilbert space

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DFG-AIMS Workshop “Global Differential Geometry”
May 21–25, 2018, Mbour, Senegal

Outline

Motivation

In Quantum Field Theory (QFT) (the theory of local observables), one studies families $\mathcal{M}(\mathcal{O})$ of von Neumann algebras on a Hilbert space $\mathcal{H}$ associated to space time regions $\mathcal{O} \subseteq M$. Among these regions, the test regions are enough to generate the whole theory. So one would like to understand their geometry.

By Tomita–Takesaki theory, every test region $\mathcal{O}$ specifies a standard subspace $V_\mathcal{O} \subseteq \mathcal{H}$. This leads to a set $\mathcal{V}$ of standard subspaces of $\mathcal{H}$ with

- an order structure (by inclusion)
- a duality operation (causal compl. for $\mathcal{O}$, symplectic compl. for $V_\mathcal{O}$)
- symmetries (generated by modular automorphism groups).

A key motivation to study the space of standard subspaces is to learn more about the geometry of test regions in QFT. This is of particular interest in curved (but homogeneous) settings and if the geometric structure of quantum space time is not a priori given.

Standard subspaces of a complex Hilbert space $\mathcal{H}$

A standard subspace $V \subseteq \mathcal{H}$ is a closed real subspace satisfying

$$V \cap iV = \{0\} \quad \text{and } \overline{V + iV} = \mathcal{H}.$$

Def: $\mathrm{Stand}(\mathcal{H})$= set of standard subspaces. We have obvious structures:

- partial order $\leq$ (by set inclusion $\subseteq$),
- duality $V' := (iV)^{\perp_{\mathbb{R}}}$ (symplectic complement)
  It is antitone: $V_1 \subseteq V_2 \Rightarrow V_2' \subseteq V_1'$ and involutive: $V'' = V$.
- symmetries: $\mathrm{AU}(\mathcal{H})$ (group of unitary and antiunitary ops on $\mathcal{H}$)

Parametrization by modular operators:

The antilinear involution $S \colon V + iV \to \mathcal{H}, S(x + iy) := x - iy$ is closed and has a polar decomposition $S = J\Delta^{1/2}$ with a conjugation J (antiunitary involution) and $\Delta > 0$ satisfying $J\Delta J = \Delta^{-1}$. We write

$$\mathrm{Mod}(\mathcal{H}) = \{(\Delta, J) \colon 0 < \Delta = \Delta^*, J\Delta J = \Delta^{-1}, J \text{ conjugation}\}$$

for the set of all such pairs.

We thus obtain a bijection

$$\mathrm{Stand}(\mathcal{H}) \to \mathrm{Mod}(\mathcal{H}), \quad V \mapsto (\Delta_V, J_V) \quad \text{with} \quad V = \mathrm{Fix}(J_V \Delta_V^{1/2}).$$

Exs: (a) $V = L^2([0,1],\mathbb{R}) \subseteq \mathcal{H} = L^2([0,1],\mathbb{C})$ with $\Delta_V = \mathbf{1}$ and $J_V f = \overline{f}$.

(b) $\mathcal{H} = L^2(\mathbb{R}^\times_+, \frac{dx}{x})$ with $\Delta f(x) = xf(x)$, $(Jf)(x) = \overline{f(x^{-1})}$ satisfies the modular relations and leads to

$$V = \Big\{ f \in \mathcal{H} \colon \int_0^\infty |f(x)|^2\, dx < \infty, \quad f(x^{-1}) = x^{1/2} \cdot \overline{f(x)} \Big\}.$$

Properties:

- $J_V V = V'$ and $\Delta_V^{it} V = V$ for $t \in \mathbb{R}$.
- Δ_V bd. $\Leftrightarrow V + iV = \mathcal{H} \Leftrightarrow [V_1 \subseteq V \subseteq V_2 \Rightarrow V_1 = V = V_2]$.
- $\Delta_V = \mathbf{1} \Leftrightarrow \mathcal{H} = V_{\mathbb{C}}$ with $J(x+iy) = x - iy$.
- Order on $\mathrm{Stand}(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_n(\mathbb{R})$ is trivial.

From operator algebras to standard subspaces

$B(\mathcal{H})$ $*$-algebra of bounded operators on $\mathcal{H}$
$\mathcal{S}' := \{A \in B(\mathcal{H}) \colon (\forall S \in \mathcal{S})\ AS = SA\}$ = commutant of $\mathcal{S} \subseteq B(\mathcal{H})$

- $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra if $\mathcal{M}$ is a $*$-alg. and $\mathcal{M} = \mathcal{M}''$
- $\Omega \in \mathcal{H}$ is cyclic for $\mathcal{M}$ if $\overline{\mathcal{M}\Omega} = \mathcal{H}$
- $\Omega \in \mathcal{H}$ is separating for $\mathcal{M}$ if $\mathcal{M} \to \mathcal{H}, M \mapsto M\Omega$ is injective.

Tomita-Takesaki-Theorem (1960s): If $\Omega \in \mathcal{H}$ is cyclic and generating for the von Neumann algebra $\mathcal{M}$, then

- $V_{\mathcal{M}} := \overline{\{M\Omega \colon M^* = M \in \mathcal{M}\}}$ is standard (hence defines (Δ, J))
- $J\mathcal{M}J = \mathcal{M}'$ (duality)
- $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ for $t \in \mathbb{R}$ (modular automorphisms).

$\mathrm{Stand}(\mathcal{H})$ **carries information on von Neumann algebras:**
Write $\mathrm{vN}(\Omega)$ for the set of von Neumann algs $\mathcal{M}$ with Ω cyclic and separating for $\mathcal{M}$. Then $\mathrm{vN}(\Omega) \to \mathrm{Stand}(\mathcal{H}), \mathcal{M} \mapsto V_{\mathcal{M}}$ is injective, $(V_{\mathcal{M}})' = V_{\mathcal{M}'}$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$ iff $V_{\mathcal{M}_1} \subseteq V_{\mathcal{M}_2}$.

Connections with Quantum Field Theory

In QFT one studies nets of von Neumann algebras $(\mathcal{M}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ in $B(\mathcal{H})$. Here $\mathcal{M}(\mathcal{O})$ corresponds to observables measurable in the "laboratory" $\mathcal{O} \subseteq M$. Requirements:

- $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$ (Isotony)
- $\mathcal{O}_1 \subseteq \mathcal{O}_2'$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$ (Locality) [$\mathcal{O}' =$ causal compl.]
- $(\forall g \in \mathrm{Aut}(M))(\exists U_g \in \mathrm{U}(\mathcal{H}))\ U_g\mathcal{M}(\mathcal{O})U_g^{-1} = \mathcal{M}(g\mathcal{O})$ (Covariance)

Given a unit vector $\Omega \in \mathcal{H}$ (vacuum state), we call $\mathcal{O} \subseteq M$ a test region if Ω is cyclic and separating for $\mathcal{M}(\mathcal{O})$. Then $V_{\mathcal{O}} := V_{\mathcal{M}(\mathcal{O})}$ is a standard subspace and we obtain $(\Delta_{\mathcal{O}}, J_{\mathcal{O}}) \in \mathrm{Mod}(\mathcal{H})$.
The technique of modular localization aims at the construction of the whole QFT from one $\mathcal{M}(\mathcal{O}_0)$, $\mathcal{O}_0$ a fixed test region, by using the group

$$G := \langle \Delta_{\mathcal{O}}^{it}, J_{\mathcal{O}} \mid t \in \mathbb{R}, \mathcal{O} \text{ test region}\rangle \subseteq \mathrm{AU}(\mathcal{H}).$$

Often the Bisognano–Wichmann Theorem implies that $g.V_{\mathcal{O}} = V_{g.\mathcal{O}}$ for an action of the group G on the space-time manifold M.
Problem: Understand space time geometry of orbits $G.V_{\mathcal{O}} \subseteq \mathrm{Stand}(\mathcal{H})$.

Σ-spaces (according to O. Loos, 1967)

Def.: Let Σ be a Lie group and M be a manifold. The structure of a Σ-space on M is given by a family of diffeomorphisms $(g_x)_{g\in\Sigma, x\in M}$ such that

$$g_x(x) = x, \quad g_x h_x = (gh)_x, \qquad g_x h_y = h_{g_x(y)} g_x.$$

For every $x \in M$ we have a Σ-action $g \mapsto g_x$ on M fixing x and every g_x is a symmetry of the whole structure.

Special cases: (a) $\Sigma = \mathbb{R}^\times$: dilation spaces

(b) $\Sigma = \{\pm 1\}$: reflection spaces

(c) $\Sigma = \{\pm 1\}$ and $x \in \mathrm{Fix}((-1)_x)$ isolated: symmetric spaces

Exs: (a) $M = G/H$, G Lie group, $\tau \in \mathrm{Aut}(G)$ involution and $H \subseteq G^\tau$. Then

$$(-1)_{xH}(yH) := x\tau(x^{-1}y)H$$

turns G/H into a reflection space; a symmetric space iff H is open in G^τ.

For $M = G$ we have $(-1)_g(h) = g\tau(g^{-1}h)$;

for $M = \mathbb{R}^n$ and $\tau(x) = -x$, we get $(-1)_x(y) = 2x - y$ (point reflections).

Homogeneous spaces as dilation spaces

Lemma: Let G be a Lie group, $\alpha\colon \Sigma \to \mathrm{Aut}(G)$ a homomorphism and $H \subseteq \mathrm{Fix}(\alpha(\Sigma))$. Then

$$r_{xH}(yH) := x\alpha_r(x^{-1}y)H, \qquad x, y \in G, r \in \Sigma$$

defines on G/H the structure of a Σ-space.

Ex: $M = \mathbb{R}^n$ is a dilation space with $r_x(y) = x + r(y - x)$.
Def.: If M is a reflection (dilation) space, then a morphism of reflection (dilation) spaces $\gamma\colon \mathbb{R} \to M$ is called a (dilation) geodesic.

Theorem ($\mathrm{Stand}(\mathcal{H})$ as a dilation space)

(a) $\mathrm{Stand}(\mathcal{H})$ *carries a natural structure of a dilation space, where*

$$(-1)_V W = J_V W' \quad \text{and} \quad (e^t)_V W = \Delta_V^{-it/2\pi} W, \qquad t \in \mathbb{R}.$$

(b) $\mathrm{Conj}(\mathcal{H})$ *is a symmetric space w.r.t.* $(-1)_J I = JIJ$.
(c) $q\colon \mathrm{Stand}(\mathcal{H}) \to \mathrm{Conj}(\mathcal{H}), V \mapsto J_V$ *is a morphism of reflection spaces.*
(d) *Continuous geodesics:* $\gamma(t) = U_{t/2}V$, *with* $U\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$ *a continuous homomorphism satisfying* $J_V U_t J_V = U_{-t}$.

Antiunitary representations

Def: A graded group is a pair (G, G_1), where $G_1 \leq G$ and $G/G_1 \cong \{\pm 1\}$.

Ex: The (antiunitary group) $(\mathrm{AU}(\mathcal{H}), \mathrm{U}(\mathcal{H}))$ of all unitary and antiunitary operators on $\mathcal{H}$. By Wigner's Theorem, $\mathrm{AU}(\mathcal{H})/\mathbb{T}$ is the symmetry group of quantum state space $\mathbb{P}(\mathcal{H})$ preserving transition probabilities.

Def: An antiunitary representation $(U, \mathcal{H})$ of a graded Lie group (G, G_1) is a morphism $U \colon G \to \mathrm{AU}(\mathcal{H})$ of graded groups, for which all orbit maps $U^\xi \colon G \to \mathcal{H}, g \mapsto U_g\xi$ are continuous.

Examples: (where $G_1 = G_0$ is the identity component):

- $\mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}_2$ (dilation group)
- the affine group of the line: $\mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$
- the projective group $\mathrm{PGL}_2(\mathbb{R})$, acting on $\mathbb{P}_1(\mathbb{R}) \cong \mathbb{S}^1$
- the proper Lorentz group $\mathrm{SO}_{1,d-1}(\mathbb{R})$.
- the proper Poincaré group $P(d)_+ \cong \mathbb{R}^d \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})$
- the proper conformal group of Minkowski space $\mathrm{SO}_{2,d}(\mathbb{R})$.

Brunetti–Guido–Longo (BGL) construction

If (U, G) is an antiunitary representation, we obtain a map

$$B\colon \operatorname{Hom}_{\rm gr}(\mathbb{R}^\times, G) \to \operatorname{Stand}(\mathcal{H}), \quad \gamma \mapsto V_\gamma,$$

determined uniquely by

$$V_\gamma = \operatorname{Fix}(J_\gamma \Delta_\gamma^{1/2}), \quad J_\gamma = U_{\gamma(-1)}, \quad \Delta_\gamma^{-it/2\pi} = U_{\gamma(e^t)}.$$

Note: For $v \in V_\gamma$, we have $U_{\gamma(e^{i\pi})} v = \Delta_\gamma^{1/2} v = J_\gamma v = U_{\gamma(-1)} v$.

Theorem (Dilation geodesics in $\operatorname{Stand}(\mathcal{H})$)

For an antiunitary rep. $(U, \mathcal{H})$ of $G = \operatorname{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ and $\gamma_x \colon \mathbb{R}^\times \to \operatorname{Aff}(\mathbb{R}), \gamma_x(r) = r_x$ (dilation structure of $\mathbb{R}$), we have:

- (a) $\eta\colon \mathbb{R} \to \operatorname{Stand}(\mathcal{H}), \eta(x) := V_{\gamma_x}$ *is a dilation geodesic (all obtained).*
- (b) *Borchers '92: η decreasing (incr.) iff $U_{(t,1)} = e^{itH}$ with $\pm H \geq 0$.*
- (c) *Wiesbrock '93: For a standard subspace $W \subseteq V$, there is a decreasing dilation geodesic from V to W iff $\Delta_V^{it} W \subseteq W$ for $t \geq 0$ ($W \hookrightarrow V$ is called a half-sided modular inclusion).*

Homogeneous spaces of standard subspaces

For every $\gamma \in \mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G)$, the subset

$$\mathcal{V}_\gamma := \{U_g V_\gamma \colon g \in G_1\} = U_{G_1} V_\gamma \subseteq \mathrm{Stand}(\mathcal{H})$$

is a dilation subspace and a G_1-homogeneous space G_1/H_γ, where $H_\gamma := \{g \in G_1 \colon U_g V_\gamma = V_\gamma\}$. The order on $\mathcal{V}_\gamma$ is non-trivial if and only if the semigroup

$$S_\gamma := \{g \in G_1 \colon U_g V_\gamma \subseteq V_\gamma\} \quad \text{with} \quad S_\gamma \cap S_\gamma^{-1} = H_\gamma$$

is not a group. Then the order on $\mathcal{V}_\gamma \cong G_1/H_\gamma$ is encoded in S_γ via

$$gV_\gamma \subseteq hV_\gamma \quad \text{iff} \quad h^{-1}g \in S_\gamma.$$

Ex: For antiunitary representations of $G = \mathrm{Aff}(\mathbb{R})$, the subset $\mathcal{V}_\gamma \cong \mathbb{R}$ is a dilation geodesic.
Order non-trivial $\Leftrightarrow$ positive/negative energy cond. $\pm H \geq 0$ (Borchers).
Then the order is determined by

$$\mathcal{V}_\gamma \cong \{[x, \infty) \colon x \in \mathbb{R}\} \subseteq 2^{\mathbb{R}} \quad \text{and} \quad S_\gamma = \{g \in \mathrm{Aff}(\mathbb{R}) \colon g(0) \geq 0\}.$$

Problem: Determine the semigroups S_γ? When do they have interior points? Describe the corresponding ordered homogeneous spaces M_γ.

Ex: (a) For antiunitary representations of $G = \mathrm{PGL}_2(\mathbb{R})$ (the Möbius group on $\mathbb{R}$) and $\gamma \colon \mathbb{R}^\times \to G$ hyperbolic (dilations of $\mathbb{R}$), the order is non-trivial iff U is a positive energy representation, i.e., $-i\mathrm{d}U(x) \geq 0$ for some $0 \neq x \in \mathfrak{g}$. Then

$$\mathcal{V}_\gamma \cong \{ \text{ proper open intervals } I \text{ in } \mathbb{S}^1\}, \qquad I' = \text{ interior of complement of } I$$

(conformal compactification of $\mathbb{R}$, as in CFT). Generated by pairs of monotone dilation geodesics.

(b) $G = P(2)_+ \cong \mathbb{R}^{1,1} \rtimes \mathrm{SO}(1,1)$, γ given by Lorentz boosts in $\mathrm{SO}(1,1)$. For positive energy representations $\mathcal{V}_\gamma \cong \mathbb{R}^2$ can be identified with the set of right wedges in $\mathbb{R}^{1,1}$, i.e., all translates of the standard right wedge $W_R = \{x_0 > |x_1|\}$.

$\mathcal{V}_\gamma$ is "flat". It is generated by two intersecting monotone dilation geodesics from V to W_1 and from W_2 to V satisfying $J_{W_1}J_{W_2} = J_V J_{W_2} J_{W_1} J_V$ (Wiesbrock condition).

Conformal compactifications of Jordan algebras

Let $\mathfrak{g}$ be a Lie algebra and $h \in \mathfrak{g}$ with $(\mathrm{ad}\, h)^3 = \mathrm{ad}\, h$. Then $\mathrm{ad}\, h$ defines a 3-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{with} \quad h \in \mathfrak{z}(\mathfrak{h}) \quad \text{for} \quad \mathfrak{h} := \mathfrak{g}_0.$$

Further $\tau := e^{\pi i\, \mathrm{ad}\, h}$ is an involution and we consider a graded Lie group $G = G_1 \rtimes \{\mathbf{1}, \tau\}$, $H := G^\tau$ and

$$\gamma \colon \mathbb{R}^\times \to Z(H) \subseteq H \subseteq G, \quad \gamma(-1) = \tau, \quad \gamma(e^t) = \exp(th).$$

Exs: We have this structures for the conformal group of

- Minkowski space $\mathbb{R}^{1,d-1}$: $\mathbb{R}^\times \subseteq H = \mathbb{R}^\times\, \mathrm{SO}_{1,d-1}(\mathbb{R}) \subseteq G = \mathrm{SO}_{2,d}(\mathbb{R})$.
- Jordan algebra $\mathrm{Sym}_n(\mathbb{R})$: $\mathbb{R}^\times \subseteq H = \mathrm{GL}_n(\mathbb{R}) \subseteq G = \mathrm{Sp}_{2n}(\mathbb{R})$.
- Jordan algebra $\mathrm{Herm}_n(\mathbb{C})$: $\mathbb{R}^\times \subseteq H = \mathrm{GL}_n(\mathbb{C}) \subseteq G = \mathrm{U}_{n,n}(\mathbb{C})$.
- simple euclidean Jordan algebras $\Rightarrow$ hermitian Lie groups of tube type.

Simple Jordan alg's: $\mathbb{R}^{1,d-1}, \quad \mathrm{Herm}_d(\mathbb{K}), \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}; \quad \mathrm{Herm}_3(\mathbb{O}).$

Let G be the conformal group of a simple eucl. Jordan alg. and γ as above.

Def: We call an antiunitary representation $(U, \mathcal{H})$ of G a positive energy representation if there exists $0 \neq x \in \mathfrak{g}$ with $-i\mathrm{d}U(x) \geq 0$.

Theorem (Identification Theorem)

$\mathcal{V}_\gamma \cong G/H$ is a symmetric space and for any non-trivial positive energy representation $(U, \mathcal{H})$ of G, the BGL map induces an order embedding

$$(G/H, \leq) \hookrightarrow \mathrm{Stand}(\mathcal{H}),$$

where the order on G/H is G-invariant and given by

$$gH \leq eH \quad \Longleftrightarrow \quad g \in S := \exp(C_+)\exp(C_-)H,$$

where $C_\pm \subseteq \mathfrak{g}_{\pm 1}$ are H-invariant closed convex cones (S is an Olshanski semigroup in Koufany decomposition).

Note: This is only for a very special choice of γ, which is far from generic.

Relations to QFT

The ordered space $(G/H, \leq)$ can be realized by subsets of the conformal completion $M = G/P$, $P := H\exp(\mathfrak{g}_{-1})$ of $\mathfrak{g}_1$ via $gH \mapsto g\exp(C_+)P$. In our context $M = G/P$ plays the role of the space-time manifold and the domains $\mathcal{O}_{gH} = g\exp(C_+)P$ the role of the test domains.

In QFT actually the following case appears:

$$G = \mathrm{SO}_{2,4}(\mathbb{R}), \quad M = G/P \cong \mathbb{S}^1 \times \mathbb{S}^3 \supseteq \mathbb{R}^{1,3}$$

(conformal completion of Minkowski space) and the set $\mathcal{W} := \{g\exp(C_+)P : g \in G\}$ is the set of conformal wedge domains. Affine test domains in Minkowski space $\mathbb{R}^{1,3}$ are the future cone

$$V_+ = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} \colon x_0 > |\mathbf{x}|\}$$

and the standard right wedge

$$W_R := \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}.$$

Open problems

- Characterize monotone geodesics $\gamma\colon \mathbb{R} \to \mathrm{Stand}(\mathcal{H})$: $\gamma(t) = U_{t/2}V$, $J_V U_t J_V = U_{-t}$. For dilation geodesics U this follows from the Borchers–Wiesbrock Theorem. Tools to be relevant here come from the theory of inner functions on complex half planes (see work of Longo–Witten (2011) on boundary QFTs).
- For an antiunitary representation $(U, \mathcal{H})$ of G, $\gamma \in \mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G)$, and the corresponding standard subspace V_γ, determine when the semigroup
$$S_\gamma := \{g \in G_1 \colon U_g V_\gamma \subseteq V_\gamma\}$$
has interior points. This should lead to an identification of the geometry of the ordered homogeneous spaces $\mathcal{V}_\gamma = U_{G_1} V_\gamma \subseteq \mathrm{Stand}(\mathcal{H})$.
- Borchers–Wiesbrock problem: Construct nets of local observables from finitely many modular groups and a single von Neumann algebra $\mathcal{M}$; this translates into finite configurations $\gamma_1, \ldots, \gamma_n$ of dilation geodesics and hence into antiunitary representation theory.

Concluding remarks

- Standard subspaces provide a context to study symmetries of systems of von Neumann algebras (nets of local observables) in terms of antiunitary reps of Lie groups
- This was used by Buchholz–Lechner–Summers to construct non-free QFTs on $\mathbb{R}^4$ by a deformation process (2011)
- One expects natural analytic extensions of antiunitary representations to play a role (combining complex semigroups and crown domains)
- All this applies to infinite dimensional groups (Virasoro, gauge groups ...)