Locality and renormalisation

Sylvie Paycha
joint work with Pierre Clavier, Li Guo and Bin Zhang

University of Potsdam
On leave from the Université Clermont-Auvergne

M'bour, May 21. 2018

GENERAL CONTEXT

SETUP AND OBJECTIVES

Data

- a (commutative) algebra $(\mathcal{A}, \star)$,
- an algebra of meromorphic germs at zero to be defined $\mathcal{M}$,
- an algebra morphism $\Phi : (\mathcal{A}, \star) \longrightarrow (\mathcal{M}, \cdot)$.

So Φ is multiplicative

$$\Phi\,(a_1 \star a_2) = \Phi(a_1) \cdot \Phi(a_2). \tag{1}$$

Our aim

Build a character $\Phi^{\mathrm{reg}} : (\mathcal{A}, \star) \longrightarrow (\mathbb{C}, \cdot)$

$$\Phi^{\mathrm{reg}} := \mathrm{ev}_0{}^{\mathrm{reg}} \circ \Phi : (\mathcal{A}, \star) \underset{\Phi}{\longrightarrow} (\mathcal{M}, \cdot) \underset{\mathrm{ev}_0{}^{\mathrm{reg}}}{\longrightarrow} (\mathbb{C}, \cdot)\,.$$

So Φ^{reg} should also be multiplicative and hence satisfy (1).

CIRCUMVENTING OBSTACLES

A first naive approach

- $\mathcal{M}$ meromorphic germs in one variable;
- Subtract the pole and evaluate the holomorphic part at the pole (here zero): ${\rm ev}_0{}^{\rm reg}\left(\sum_{j=1}^{k} a_j z^{-j} + h(z)\right) := h(0)$.
- Obstacle: ${\rm ev}_0{}^{\rm reg}(f_1 \cdot f_2) \neq {\rm ev}_0{}^{\rm reg}(f_1) \cdot {\rm ev}_0{}^{\rm reg}(f_2)$ so multiplicativity is destroyed: $1 = {\rm ev}_0{}^{\rm reg}\left(\frac{1}{z} \cdot z\right) \neq {\rm ev}_0{}^{\rm reg}\left(\frac{1}{z}\right) \cdot {\rm ev}_0{}^{\rm reg}(z) = 0$.

Alternative approach

- multivariate meromorphic germs: e.g. $f(z_1, z_2) = \frac{z_1}{z_2}$;
- independence/ locality/ orthogonality relation: $\frac{1}{z_1} \perp z_2$;
- a (partial) product on independent germs: $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}$.

Multivariate meromorphic germs

Multivariate meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms.
- Dependence set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Theorem (L. Guo, S.-P., B. Zhang/ N. Berline, M. Vergne 2015)

$\mathcal{M}(\mathbb{C}^k) = \mathcal{M}_-(\mathbb{C}^k) \oplus^\perp \mathcal{M}_+(\mathbb{C}^k)$, where $\mathcal{M}_-(\mathbb{C}^k) \ni \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$ with $\mathrm{Dep}(h) \perp \langle L_1, \cdots, L_n \rangle$ and $f_1 \perp f_2 \iff \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

Our protagonists

- Orthogonal projection $\pi_+ : \mathcal{M}(\mathbb{C}^k) \longrightarrow \mathcal{M}_+(\mathbb{C}^k)$.
- Regularised evaluator $ev_0{}^{\mathrm{reg}} := ev_0 \circ \pi_+ : \mathcal{M}(\mathbb{C}^k) \longrightarrow \mathbb{C}$.

APPLICATIONS

The algebra $\mathcal{A}$

1. pointed convex cones $\mathbf{C}$ in $\mathbb{R}^\infty$ equipped with the cartesian product (L. Guo, S.-P., B. Zhang 2017);
2. rooted forests $\mathbf{F}$ equipped with the concatenation product (P.Clavier, L. Guo, S.-P., B. Zhang 2018);
3. Feynman graphs Γ on manifolds equipped with the concatenation product (N.-V. Dang, B. Zhang 2017).

The map $\phi : \mathcal{A} \longrightarrow \mathcal{M}(\mathbb{C}^\infty)$

1. exponential integrals/sums on a cone $\mathbf{C}$: $\check{\mathbf{C}}^- \ni \vec{\epsilon} \longmapsto \int_{\vec{x}\in\mathbf{C}} e^{\langle\vec{\epsilon},\vec{x}\rangle}\, dx$ and $\check{\mathbf{C}}^- \ni \vec{\epsilon} \longmapsto \sum_{\vec{n}\in\mathbf{C}\cap\mathbb{Z}^\infty} e^{\langle\vec{\epsilon},\vec{n}\rangle}$;
2. branched zeta functions $s_{\mathbf{F}} \longmapsto \zeta_{\mathbf{F}}(s_{\mathbf{F}})$ indexed by forests $\mathbf{F}$;
3. Feynman amplitudes $(z_e, e \in \mathcal{E}(\Gamma)) \longmapsto \prod_e G^{z_e}$, with $G(z_e)$ the kernel of $(\Delta + m^2)^{-1+z_e}$, on each edge e of the graph Γ.

LOCALITY STRUCTURES

Locality in quantum field theory and beyond

Independence of events in QFT

An object is only directly influenced by its immediate surroundings. Two events situated in different locations do not influence each other.

Local functionals in QFT

Functionals F on fields ϕ of the form $F(\phi) = \int_M f\left(j_x^k(\phi)\right)\, dx$, where $j_x^k(\phi)$ is the equivalence class for the relation $\partial^\alpha \phi(x) = \partial^\alpha \psi(x)$ for any $|\alpha| \leq k$. Here, $\mathrm{Supp}\,(f\,(\psi)) \subset \mathrm{Supp}\,(\psi)$.

Locality also arises in

- analysis: local operators, local Dirichlet forms,
- analysis: locality in index theory.

Algebraic locality

Definition of locality

A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, called **locality relation** (or **independence relation**) of the locality set.

$$x_1 \top x_2 \iff (x_1, x_2) \in \top, \quad \forall x_1, x_2 \in X.$$

First examples of locality

- $X \top Y \iff X \cap Y = \emptyset$ on subsets X, Y of a set Z.
- $X \top Y \iff X \perp Y$ on subsets X, Y of an euclidean vector space V.

(almost-)Separation of supports

Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions $\phi, \psi \in \mathcal{D}(U)$ are independent i.e., $\phi \top \psi$ whenever $d\,(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)) > \epsilon$.

For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Further examples

Analysis: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty)$, $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \Longleftrightarrow P(A \cap B) = P(A)\, P(B).$$

Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$L_1 \top L_2 \Longleftrightarrow L_1 \pitchfork L_2 \Longleftrightarrow T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2.$.

Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \Longleftrightarrow m \wedge n = 1.$$

Partial binary operations

Operation on the graph of a locality relation

- Locality set: $(X, \top)$,
- Graph: $\top = \{(a, b) \in X^2, \quad a \top b\}$,
- Partial operation:

$$\begin{aligned} \star : X \times X \supset \top &\longrightarrow X \\ (a, b) &\longmapsto a \star b. \end{aligned}$$

Partial multiplicativity on meromorphic germs

The partial product on $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k \in \mathbb{N}} \mathcal{M}(\mathbb{C}^k)$:

$$\begin{aligned} \mathcal{M}(\mathbb{C}^\infty) \times \mathcal{M}(\mathbb{C}^\infty) \supset \top &\longrightarrow \mathcal{M}(\mathbb{C}^\infty) \\ \left(f = \frac{h(\vec{\ell})}{\vec{L}^{\vec{s}}}, \widetilde{f} = \frac{\widetilde{h}(\vec{\widetilde{\ell}})}{\vec{\widetilde{L}}^{\vec{\widetilde{s}}}} \right) &\longmapsto f \cdot \widetilde{f} = \frac{h(\vec{\ell}) \cdot \widetilde{h}(\vec{\widetilde{\ell}})}{\vec{L}^{\vec{s}} \cdot \vec{\widetilde{L}}^{\vec{\widetilde{s}}}}. \end{aligned}$$

Algebraic locality

Locality structures

- A locality semi-group is a locality set $(A, \top_A)$ with a product law $m_A : A \times A \supset \top_A \longrightarrow A$ compatible with the locality relation: $\forall U \subseteq A, \quad m_A((U^{\top_A} \times U^{\top_A}) \cap \top_A) \subset U^{\top_A}$ and locally associative.
- A locality vector space is a locality set $(V, \top_V)$ with a linear structure such that for any subset X of V, the set $X^{\top_V}$ is a linear subspace of V.
- A locality algebra is a locality set $(A, \top_A)$ equipped with a bilinear map $m_A : \top_A \longrightarrow A$ such that $(A, \top_A, m_A)$ is a locality algebra.

Counterexample

The abelian group $\mathbb{R}$ equipped with the relation $x \top y \Longleftrightarrow x + y \notin \mathbb{Z}$ can be equipped with the addition but it is not a locality semi-group: for $U = \{1/3\}$ we have $(1/3, 1/3) \in (U^\top \times U^\top) \cap \top$ but $1/3 + 1/3 = 2/3 \notin U^\top$.

Locality morphisms

Locality maps

$\Phi : (X, \top_X) \longmapsto (Y, \top_Y)$ is a locality map if $\Phi \otimes \Phi(\top_X) \subset \top_Y$

Locality of distribution kernels

Let $U \subset \mathbb{R}^n$. On $\mathcal{D}(U)$ define two locality relations:

- $\phi \top \psi \iff \mathrm{Supp}(\phi) \cap \mathrm{Supp}(\psi) = \emptyset$;
- For some symmetric kernel $K \in \mathcal{D}'(U \times U)$, define $\phi \top_K \psi \iff \int_U \phi(x)\, K(x, y)\, \psi(y)\, dx\, dy = 0$.

If K is local, then $Id : (\mathcal{D}(U), \top) \longrightarrow (\mathcal{D}(U), \top_K)$ is a locality map.

Locality morphisms

$\Phi : (A, \top_A, m_A) \longmapsto (B, \top_B, m_B)$ is moreover a locality morphism of locality algebras if $a_1 \top_A a_2 \Longrightarrow \Phi(m_A(a_1, a_2)) = m_B(\Phi(a_1), \Phi(a_2))$.

MULTIVARIATE REGULARISATION

A partially multiplicative renormalised map

Back to our two protagonists

- The orthogonal projection $\pi_+ : \left(\mathcal{M}(\mathbb{C}^k), \perp\right) \longrightarrow \left(\mathcal{M}_+(\mathbb{C}^k), \perp\right)$ is a locality morphism of locality algebras;
- The regularised evaluator ${ev_0}^{\mathrm{reg}} := ev_0 \circ \pi_+ : \left(\mathcal{M}_+(\mathbb{C}^k), \perp\right) \longrightarrow \mathbb{C}$ is a locality character.

Theorem

A locality morphism $\Phi : (\mathcal{A}, \top) \longrightarrow \left(\mathcal{M}(\mathbb{C}^k), \perp\right)$ gives rise to a locality character

$$\Phi^{\mathrm{reg}} := {ev_0}^{\mathrm{reg}} \circ \Phi : (\mathcal{A}, \top) \longrightarrow \mathbb{C}.$$

Conclusions

The renormalised map Φ^{reg} is partially multiplicative

$$a_1 \top_A 2 \Longrightarrow \Phi^{\mathrm{reg}}(a_1 \star a_2) = \Phi^{\mathrm{reg}}(a_1) \cdot \Phi^{\mathrm{reg}}(a_2). \qquad (2)$$

So provided $\Phi(\mathcal{A}) \subset \mathcal{M}(\mathbb{C}^\infty)$, we can renormalise while preserving partial multiplicativity.

Back to the examples: one can renormalise at poles

1. Exponential sums and integrals on rational convex cones equipped with an orthogonality independence relation (L. Guo, S.-P., B. Zhang 2017);
2. Branched zeta functions equipped with an orthogonality independence relation (P. Clavier, L. Guo, S.-P., B. Zhang 2018);
3. Feynman integrals associated with Feynman graphs on manifolds with a disjointness independence relation (N.-V. Dang, B. Zhang 2017).

P. Clavier, L. Guo, B. Zhang and S. P., An algebraic formulation of the locality principle in renormalisation, to appear in *Journal of the European Mathematical Society*.

L. Guo, B. Zhang and S.P., Renormalisation and the Euler-Maclaurin formula on cones, *Duke Math J.*, **166** (3) (2017) 537–571.

L. Guo, B. Zhang and S. P., A conical approach to Laurent expansions for multivariate meromorphic germs with linear poles, arXiv:1501.00426v2 (2017).

D. Manchon and S. P., Nested Sums of Symbols and Renormalized Multiple Zeta Values, Int. Math. Res. Notices (2010) 4628-4697. ArXiv: 0702135v3 [math.NT].

In preparation (with P. Clavier, L. Guo and B. Zhang):

- Locality morphisms from rooted trees to meromorphic functions with linear poles (tentative title).
- Renormalisation and locality: branched zeta values (tentative title).