Emergence of non-ergodic dynamic

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presentation including joint works in progress with

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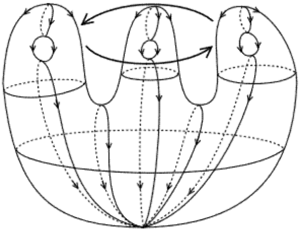
Given a smooth map f of a manifold M, we would like to describe the long term behavior of its orbits $(f^n(x))_n$ for most of its points $x \in M$.

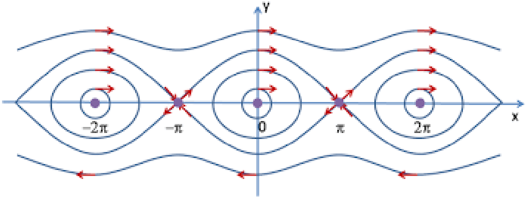
The simplest way would be to approximate the behavior of most of the points thanks to a finite number of points x_i for which we can compute the orbit $(f^n(x_i))_{n \geq 0}$.

This ways has been efficient to understand many systems, such as:

- the flow given by the gradient of a Morse functions, or more generally the Morse-Smale dynamics.

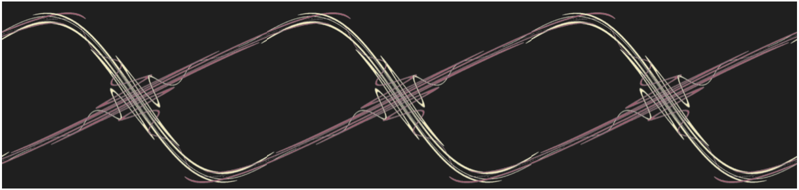


- The flow given by an integrable systems such as the pendulum.



Unfortunately this does not work every times for two reasons:

First, Poincaré discovered that systems can be sensitive to the initial condition at infinitely many points x. This means that for x' close to x, after a sufficiently large time n, the distance between $f^n(x)$ and $f^n(x')$ is large. The time n can be a logarithmic function of x and x'. As we never know with full precision the position of a point, we cannot predict its orbit.



Secondly, a system can need a large number $H(n)$ of points x_j so that for every $x \in M$, there exists $i \leq H(n)$ so that $f^m(x)$ and $f^m(x_i)$ are ϵ-close for every $m \leq n$. The number $H(n)$ can be exponential in n:

$$h = \lim_{n\to\infty} \frac{1}{n} \log H(n) > 0 \ .$$

The number h is the topological entropy of the system. It does not depend on ϵ-small.

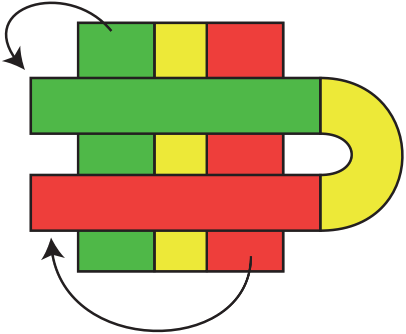


Figure: The Smale horshoe is the source of positive topological entropy by a theorem of Katok.

However, we are allowed to discard the study of a number of points of Lebesgue measures less than ϵ. Then we consider the number $H_{\mathrm{Leb}}(n)$ of points x_j so that for $1-\epsilon$-all the points $x \in M$, there exists $i \leq H_{\mathrm{Leb}}(n)$ so that $f^m(x)$ and $f^m(x_i)$ are ϵ-close for every $m \leq n$. It can happen:

$$h_{\mathrm{Leb}} = \limsup_{n \to \infty} \frac{1}{n} \log H_{\mathrm{Leb}}(n) > 0 \,.$$

The number h_{Leb} is the (Kolmogorov-Sinai) metric entropy of the system. It does not depend of ϵ-small.

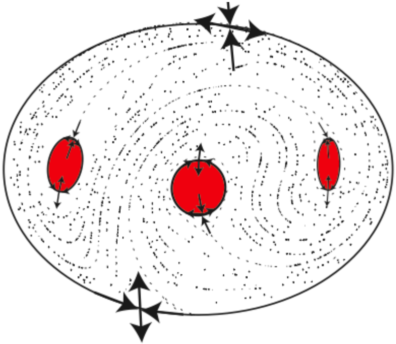


Figure: Chaotic Island obtained by surgery from an Anosov map

The systems of positive entropy are called chaotic. It is not possible to understand most of the orbits of such systems. However, one can try to understand the statistical behavior of most of the orbits.

The statistical behavior of the orbit of x for a dynamics f is given by the sequence of n^{th}-Birkhoff average:

$$S_f^n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \ .$$

We denote by $S_f(x)$ the set of cluster values of this sequence.

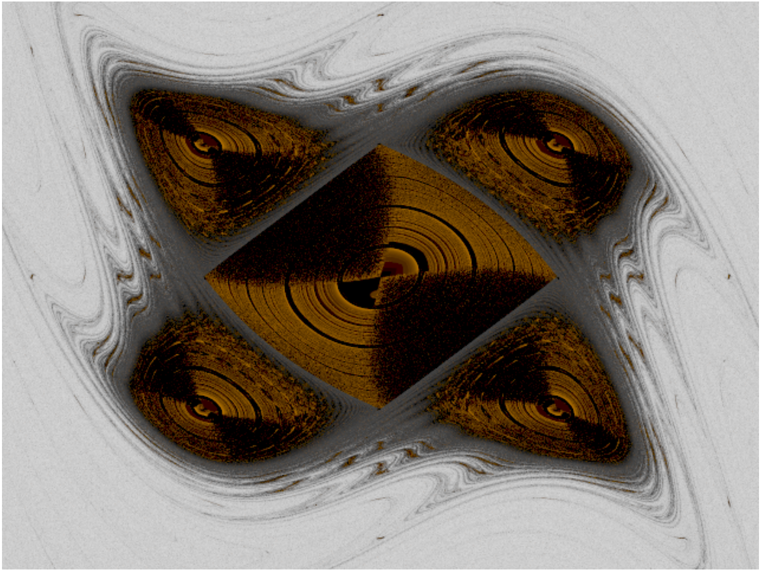
When the system is conservative (it leaves invariant the Lebesgue measure Leb), then for Leb a.e. point x, the sequence $(S_f^n(x))_n$ converges.

A conservative system is called ergodic if $\lim S_f^n(x)$ is equal to a same measure for Leb-a.e. $x \in M$. Then this measure is $\mathrm{Leb}(M) \cdot \mathrm{Leb}$.

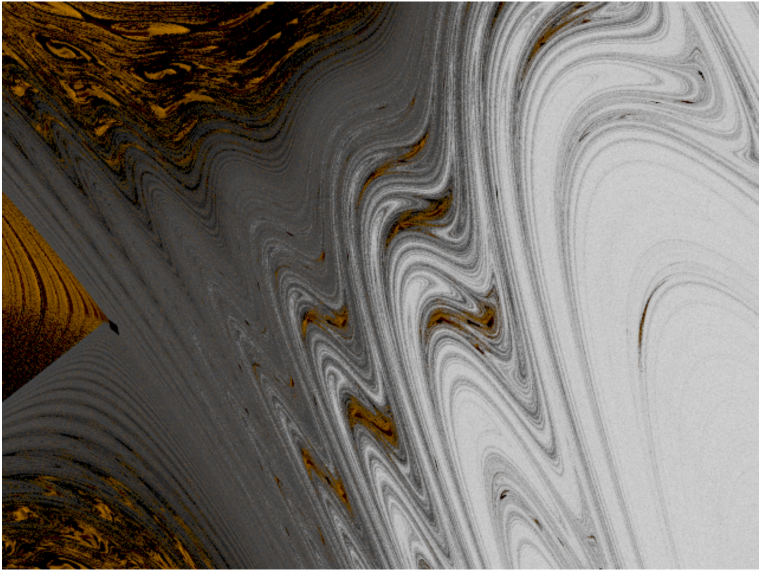
Bolzman ergodic hypothesis stated that a typical Hamiltonian dynamics, the system is ergodic: statistical behaviour of almost all the orbits is $\mathrm{Leb}(M) \cdot \mathrm{Leb}$.

This conjecture turned out to be wrong with the KAM theory. Nearby some integrable systems, there are (at least) infinitely many disjoint, invariant torii whose union forms a subset of positive measure.

Standard map (phase space)



Detail



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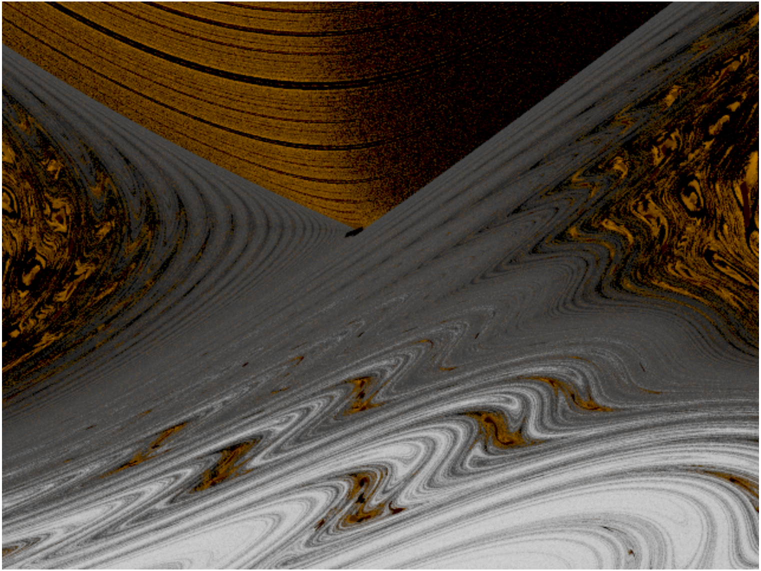


Still there are systems which are conservative and robustly ergodic. For instance the uniformly hyperbolic systems. More generally the partially hyperbolic, conservative systems are conjectured to be robustly ergodic by Pugh & Shub.

Also, the ergodic component which persists by KAM theorem nearby integrable systems are very well approximated by finitely many statistics.

In the 90's several conjectures (Tedeschini-Lalli & York, Pugh & Shub, Palis & Takens, Palis, Viana) suggested that for many kinds of (non-conservative) dynamical systems, there exists finitely many measures which approximate well the behavior of nearby all the points.

Those conjectures have been verified in many cases thanks to the works of Anosov, Sinai, Simányi, Shub, Pugh, Mañé, Marteens-Nikonov, Lyubich, Wilkinson, Alvez, Bonatti, Viana, Benedicks, Tsujii, Crovisier, Pujals, Obata ...

On the other hand, recently it has been shown that wild dynamics are typical in the sense of Kolmogorov in some cathegries of dynamical systems:

Theorem (Berger [1] [2])

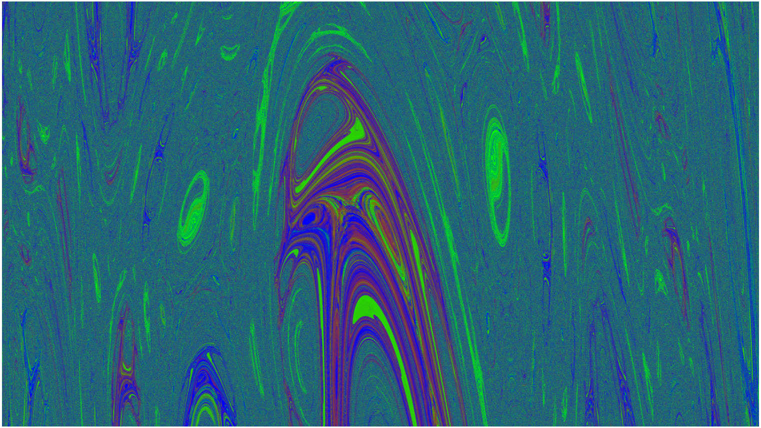
For every $1 \leq r < \infty$, there exists an open set of C^r-families $\mathcal{U}$ of C^r-self-mappings of M^n, so that a generic $(f_a)_a \in \mathcal{U}$ satisfies that for every a, the dynamics f_a displays infinitely many attracting cycles.

They display the Newhouse phenomenon: The sinks accumulate on the space of ergodic measures of a uniformly hyperbolic horseshoe.

The families (f_a) can be formed by diffeo if $n \geq 3$.

[1] Pierre Berger, Inventiones Mathematicae 2016

[2] Pierre Berger, Proceeding of the Steklov institute 2017



Wild dynamics are not negligible.

How to describe them?

How to describe their complexity?

Let M be a compact manifold. Let W be the 1-Wasserstein metric on the space of probability measures $\mathcal{M}(M)$ of M:

$$W(\mu,\nu) = \sup_{\phi\ 1-Lip} \int \phi d(\mu-\nu)\ .$$

Definition (B. [3])

The metric emergence $\mathcal{E}_{\mathrm{Leb}}(f)$ of $f \in C^0(M,M)$ is the function which at $\epsilon > 0$ gives the minimal number N of probability measures $(\mu_i)_{1\le i\le N}$ so that

$$\int_M W(S_f(x), \{\mu_i : 1 \le i \le N\}) d\mathrm{Leb} < \epsilon\ .$$

Conjecture (B. [3])

In many categories of differentiable mappings, a "typical map" f satisfies:

$$\limsup_{\epsilon\to 0} \frac{\log \mathcal{E}_{\mathrm{Leb}}(f)}{-\log \epsilon} = \infty\ . \qquad \text{(Super P)}$$

[3] Pierre Berger, Proceeding of the Steklov institute 2017

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- An ergodic map (such as a uniformly hyperbolic set, irrational rotation) has emergence 1 (and not super P)
- Newhouse phenomena has not finite emergence. Is it typically Super Pol?
- the identity of a d-manifold satisfies $\lim_{\epsilon \to 0} \frac{\log \mathcal{E}_{\mathrm{Leb}}(f)}{-\log \epsilon} = d$
- KAM implies that typical Hamiltonian system which displays a totally elliptic point is at least polynomial. How about (Super P)?

[4] Pierre Berger, Proceeding of the Steklov institute 2017

Theorem (Berger-Bochi)

For every $r \in [5, \infty]$, a generic map, surface, conservative mapping f is either weakly stable (none of its perturbations display an elliptic point) or its emergence is stretched exponential with exponent $d = 2$:

$$\limsup_{\epsilon \to 0} \epsilon^d \log \mathcal{E}_{Leb}(f)(\epsilon) \in (0, \infty) \tag{$\exp \cdot^d$}$$

Remark

In particular, a generic mapping which displays an elliptic point has super polynomial emergence.

Remark

We will see that the emergence of a mapping of a d-dimensional space cannot be more than $\exp \cdot^d$.

Remark

Weakly stable mappings are conjecturally uniformly hyperbolic (Mañé Like conjecture) and structurally stable (Lambda Lemma like conjecture).

A rational function R of the Riemannian sphere is in the bifurcation locus if it can be perturbed to displays $d = deg\ R$-different elliptic points. When $d \geq 2$, the bifurcation locus $Biff \subset Rat(\S)$ is a closed set of positive Lebesgue measure: $\mathrm{Leb}\,(Biff) > 0$ by a Theorem of Rees and Astorg - Gauthier - Mihalache - Vigny.

Theorem (Talebi, in progress)

For $R \in Biff$ generic, for Lebesgue a.e. z in the sphere, the covering number of $S_R(z)$ is super exponential. In particular, the metric Emergence of R is super exponential.

Comparing Emergence and Entropy.

If (X, d) is a totally bounded metric space, the covering number $H_d(\epsilon)$ is the minimal number of ϵ-balls necessary to cover X.

If μ is a probability measure on X, the quantization number $H_{d\,\mu}(\epsilon)$ is the minimal number N of points $(x_i)_{1\leq i\leq N}$ such that:

$$\int_X d(x, \{x_i : i\}) d\mu(x) < \epsilon \,.$$

Proposition

The quantization number $H_{d\,\mu}(\epsilon)$ is equal to the minimal cardinality of the support of an atomic measure which is ϵ-close to μ for the Wassertstein metric W.

It holds $H_{d\,\mu} \leq H_d$

Let $\mathcal{M}_f(X)$ be the subset of f-invariant, probability measures, let $\mathcal{M}_e(X)$ be the subset of ergodic ones.

Let $\mu \in \mathcal{M}_f(X)$. Then the Birkhoff average S takes it value in $\mathcal{M}_e(X)$ for μ-a.e. x. The ergodic decomposion of μ is the pull back $S_*\mu$ of μ. It is a probability measure on $\mathcal{M}_e(X)$.

The topological emergence $\mathcal{E}_{top}(f)$ is the covering number of $\mathcal{M}_e(X)$.

The metric emergence $\mathcal{E}_\mu(f)$ is the quantization number of $S_*\mu$.

Theorem (Variational Principle, Berger-Bochi)

If X has box dimension d, then

$$\sup_{\mu \in \mathcal{M}_f(X)} \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_\mu(f)(\epsilon)}{-\log \epsilon} = \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_{top}(f)(\epsilon)}{-\log \epsilon}$$

For every n, let $d_n(x, y) := \sup_{0 \leq k \leq n} d(f^k(x), f^k(y))$.

Let $H_n(\epsilon)$ be the covering number of (X, d_n). Let $H_{n\,\mu}(\epsilon)$ be the quantization number of (X, d_n) for an (ergodic) measure μ.

The topological entropy is $h_{top}(f) := \limsup_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log H_n(\epsilon)$.

The metric entropy is $h_{\mu}(f) := \limsup_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log H_{n\,\mu}(\epsilon)$.

Theorem (Variational Principle)

If X has box dimension d, then

$$\sup_{\mu \in \mathcal{M}_f(X)} h_{\mu}(f) = h_{top}(f) \ .$$

Theorem (Kloeckner, Bolley-Guillin-Villani)

If X is a compact space with box dimension d, then the covering number $H_d(\epsilon)$ of $\mathcal{M}(X)$ satisfies:

$$\lim_{\epsilon \to 0} \frac{\log \log H_d(\epsilon)}{-\log \epsilon} = d \ .$$

Corollary

The emergence of a map f of a manifold M satisfies:

$$\limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_\mu(\epsilon)}{-\log \epsilon} \leq \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_{top}(\epsilon)}{-\log \epsilon} \leq \dim M \ .$$

Theorem (Margulis-Ruelle inequality)

If f is of class $C^{1+\alpha}$, then $h_\mu \leq \int \log \|Df\| d\mu \leq \log \|Df\|_\infty$.

Conjecture (Entropy)

Positive metric entropy is typical.

Theorem (Herman-Berger-Turaev)

Every C^∞-surface, conservatif diffeo which displays an elliptic periodic point can be approximated to a surface, conservatif diffeomorphism with positive metric entropy.

Conjecture (Emergence)

Super polynomial emergence is typical.

Theorem (Berger-Bochi)

For every $\infty \geq r \geq 5$, a generic C^r-surface, conservative diffeomorphism which displays an elliptic point has metric Emergence super polynomial.

There are C^∞-conservative mappings with maximal emergence and entropy zero. There are C^∞-conservative mappings with positive entropy and which are ergodic (and so trivial emergence).

Theorem (Berger–Turaev, in progress)

Let $\mathcal{U}$ be the open set of C^∞-sympletic mappings of a manifold M^{2n} which displays a totally elliptic periodic point. Then a generic family $(f_a)_a \in C^\infty(\mathbb{R}^k, \mathcal{U})$ satisfies that for every $a \in \mathbb{R}^k$:

$$\limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_{\mathrm{Leb}}(f_a)(\epsilon)}{-\log \epsilon} = 2n$$

In particular, super-polynomial emergence is Kolmogorov-typical in $\mathcal{U}$.