Non-Statistical Rational Maps

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Some Notations

$f : M \to M$ continuous, M compact smooth manifold
$\mathcal{M}_1(M)$: the space of probability measures on M
Lebesgue: The normalized volume of M

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$$S_f^n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$$

$acc(\{S_f^n(x)\}_{n \in \mathbb{N}})$: The set of accumulation points in $\mathcal{M}_1(M)$

Definitions

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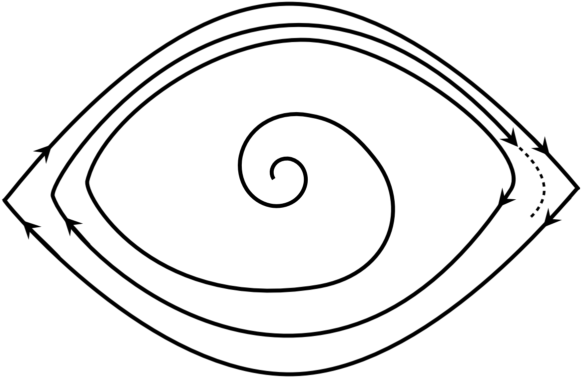
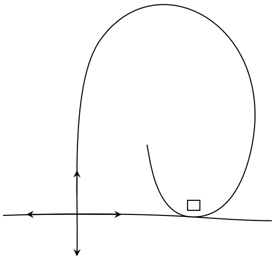
Definition: $x \in M$ is *non-statistical* for f if its Birkhoff averages does not converge.
Definition: A map f is *non-statistical* if there is a Leb-positive set non-statistical points.

The Bowen Eye



Non-Statistical Behavior Via Wandering Domain

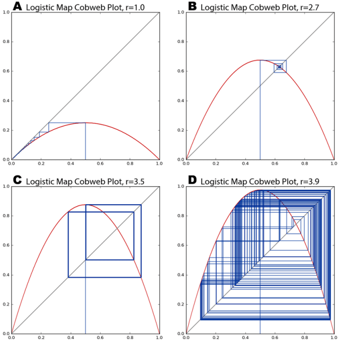
Colli and Vargas: Example of a C^∞ dissipative surface diffeomorphism haiving a wandering domain with non-statistical behavior
(Idea: Perturbation of a dissipative map with a thick Horseshoe and homoclinic tangency)



Kiriki and Soma: C^r-density ($r \neq \infty$) of maps with non-statistical wandering domain in Newhouse domains.
Question: What about examples in more rigid dynamics?

Non-Statistical Maps in Logistic Family

$f_r(x) = rx(1-x),\ r \in [0,4]$



Hofbauer and Keller: There exist uncountably many $r \in [0,4]$ for which f_r is non-statistical. Indeed for these parameters

$$\textit{for} \quad \textit{Leb.a.e.} \quad x \in [0,1], \quad acc(\{S_f^n(x)\}_{n\in\mathbb{N}}) = \mathcal{M}_1(f_r)$$

Remark: f_r has positive entropy so $\mathcal{M}_1(f_r)$ is a big set.

$f : \mathbb{C} \cup \infty =: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $f(x) = \frac{P(x)}{Q(x)}$,
f is degree d if the maximum of $\deg(P)$ and $\deg(Q)$ is d

$$Rat_d := \{f | f \text{ is a degree d rational map}\}$$

$f \in Rat_d$ has $2d - 2$ critical points (with multiplicity)

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$f \in Rat_d$ has $2d - 2$ critical points (with multiplicity)
Definition: A rational map is called *strictly post critically finite* if all of the critical points eventually are mapped to repelling periodic points.

$$\mathcal{S}_d := \{f \in Rat_d | f \text{ is strictly post critically finite}\}$$

Non-Statistical Rational Maps

Theorem

A (topological) generic rational map $f \in \overline{\mathcal{S}_d}$ is non-statistical.

Remarks:

- by [Rees,Astrong-Guathier-Mihalache-Vigny] $\overline{\mathcal{S}_d}$ has positive measure in Rat_d.
- These maps have no physical measure.
- The Julia set of these maps is $\hat{\mathbb{C}}$.

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Main Lemma

$f \in \overline{\mathcal{S}_d}$ and q repelling periodic point for f. Then for any $\epsilon > 0$, there is g arbitrary close to f with the following property:

$$\text{for} \quad Leb.a.e\ x \in \hat{\mathbb{C}} \quad \limsup_{n \to +\infty} dist(S_g^n(x), \delta_{O(q)}) < \epsilon.$$

Abstract Setting

$\Lambda \subset C^r(M, M)$ closed, $\mathcal{K} \subset \mathcal{M}_1(M)$

Definintion: The set Λ *statisitically bifurcates towards* $\mathcal{K}$ if for any $f \in \Lambda$ and $\epsilon > 0$ there is $g \in \Lambda$ arbitrary close to f such that

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Proposition

If Λ statistically bifurcates towards $\mathcal{K}$ then for a (topological) generic map $g \in \Lambda$ we have:

$$\text{for} \quad Leb.a.e\ x \in M \quad \liminf_{n\to+\infty} dist(S_g^n(x), \mathcal{K}) = 0.$$

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Main Theorem

For a generic map $f \in \overline{\mathcal{S}_d}$ we have

$$\text{for} \quad Leb.a.e.\ x \in M \quad \{\delta_{O(p)} | p \in Per(f)\} \subset acc(S_g^n(x)).$$

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- **Question:** Can this phenomenon happens generically in an open set of maps? Newhouse domains?
- **Question:**(Taken's Conjecture) Can this phenomenon be stable in some nontrivial family of dynamics?