1. On the regularity of maximal operators

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Abstract: In this talk we will discuss questions about the boundedness and continuity of classical and fractional maximal operators acting on Sobolev spaces and spaces of functions of bounded variation.

For $f \in L^1_{loc}(\mathbb{R}^d)$ and $0 \leq \beta < d$, we consider the centered fractional maximal function as

$$M_\beta f(x) = \sup_{r>0} \frac{1}{m(B_r(x))^{1-\frac{\beta}{d}}} \int_{B_r(x)} |f(y)| dy. \tag{1}$$

We could also consider the Non-centered fractional maximal function given by taking the supremum over all the balls that contain the point x instead of the balls centered at this point, we will denote this by $\widetilde{M}_\beta$. If $1 < p < \infty$, $0 < \beta < d/p$ and $q = dp/(d - \beta p)$, then $M_\beta : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded. When $p = 1$ one has a weak-type bound. Kinnunen and Saksman [Bull. London Math. Soc. 35 (2003), no. 4, 529–535] studied the regularity properties of such fractional maximal operators. One of the results they proved is that $M_\beta : W^{1,p}(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$ is bounded for p, q, β, d as described above (the same holds for the Non-centered fractional maximal function). I have been interested in the case $p = 1$.

In a joint work with my Ph.D. advisor Emanuel Carneiro [1], we established new bounds for the derivative of the Non-centered fractional maximal function, both in the continuous and in the discrete settings. More precisely we proved that if f is a function of bounded variation then $M_\beta f$ is absolutly continuous and $\|DM_\beta f\|_{L^q(\mathbb{R})} \leq C(\beta)\|Df\|_{L^1(\mathbb{R})}$. Very recently in [3] we extended this result to higher dimension for radial functions. The analogous results for the centered one are open questions.

Moreover, in dimension 1, for $0 \leq \beta < 1$ recently in [2] and [4] it was proved that the map $f \mapsto D\widetilde{M}_\beta f$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^q(\mathbb{R})$, here $q = 1/(1 - \beta)$. Something similar should be true for the centered one, and probably, also over the space of functions of bounded variation ($BV(\mathbb{R})$).

References:

References

[1] E. Carneiro and J. Madrid, *Derivative Bounds for Fractional Maximal Functions*, Transactions of the American Mathematical Society 369, No. 6 (2017), 4063–4092.

[2] E. Carneiro, J. Madrid and L. Pierce, *Endpoint Sobolev and BV Continuity for Maximal Operators,* Journal of Functional Analysis, 273, No. 10 (2017), 3262–3294

[3] H. Luiro and J. Madrid, *The Variation of the Fractional Maximal Function of a Radial Function,* Accepted to appear in International Mathematics Research Notices (2017).

[4] J. Madrid, *Endpoint Sobolev and BV Continuity for Maximal Operators II,* Preprint Submitted.

Questions:

Question 1: Do we have that if $f \in W^{1,1}(\mathbb{R})$ then $M_\beta f$ is absolutly continuous (or at least weakly differentiable) and $\|DM_\beta f\|_{L^q(\mathbb{R})} \leq C(\beta)\|Df\|_{L^1(\mathbb{R})}$?

Question 2: Is the map $f \mapsto DMf$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$?

Question 3: Is the map $f \mapsto \widetilde{M}f$ continuous from $BV(\mathbb{R})$ to $BV(\mathbb{R})$?

Question 4: Could we get similar results for maximal operators of convolution type?

Question 5: Could we get something more general on weighted spaces?