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Lectures on Holographic Renormalization

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Abstract

We provide a pedagogical introduction to the method of holographic renormalization, in its Hamiltonian incarnation. We begin by reviewing the description of local observables, global symmetries, and ultraviolet divergences in local quantum field theories, in a language that does not require a weak coupling Lagrangian description. In particular, we review the formulation of the Renormalization Group as a Hamiltonian flow, which allows us to present the holographic dictionary in a precise and suggestive language. The method of holographic renormalization is then introduced by first computing the renormalized two-point function of a scalar operator in conformal field theory and comparing with the holographic computation. We then proceed with the general method, formulating the bulk theory in a radial Hamiltonian language and deriving the Hamilton-Jacobi equation. Two methods for solving recursively the Hamilton-Jacobi equation are then presented, based on covariant expansions in eigenfunctions of certain functional operators on the space of field theory couplings. These algorithms constitute the core of the method of holographic renormalization and allow us to obtain the holographic Ward identities and the asymptotic expansions of the bulk fields.

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1 Introduction

The gauge/gravity duality [1] stipulates a mathematical equivalence between a theory of (quantum) gravity and a local quantum field theory (QFT), without gravity, on a lower dimensional space. The best studied examples of such holographic dualities typically involve gravity in an asymptotically anti de Sitter (AdS) space and a dual QFT 'living' on the boundary of AdS. This mathematical equivalence is reflected in a precise map between physical observables on the two sides of the duality. For local observables, this map is summarized in the prescription for computing QFT correlation functions from the gravity dual, originally proposed in [2, 3]. Namely, for every local, single-trace and gauge-invariant operator $\mathcal{O}(x)$ there is a field, Φ, in the dual 'bulk' gravity theory. The generating functional of connected correlation functions of $\mathcal{O}(x)$, $W[J]$, is then identified with the bulk on-shell action

$$W[J] \sim S_{\text{on-shell}}[\Phi]|_{\Phi\sim J}\,, \tag{1.1}$$

evaluated on solutions of the bulk equations of motion subject to Dirichlet boundary conditions on the AdS boundary. The arbitrary function that is kept fixed at the boundary is identified with the source $J(x)$. This statement is an operational definition of the holographic dictionary, allowing

one to compute, in principle, any local QFT observable from the bulk theory. However, there are a number of practical and conceptual obstacles.

The most obvious technical difficulty is that both sides of (1.1) actually involve infinite quantities. On the QFT side, we know that the generating functional of composite operators generically possesses ultraviolet (UV) divergences, even in a conformal field theory (CFT). We will see an explicit example of this phenomenon later on. On the gravity side, the on-shell action is also generically divergent, due to the infinite volume of AdS space. In order to make sense of (1.1), therefore, one must somehow remove the divergences from both sides and identify the remaining finite expressions. On the QFT side the procedure for systematically and consistently removing the UV divergences is known as *renormalization*. *Holographic renormalization* [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] is the analogous procedure for the gravity side of the duality.

A more conceptual drawback of the identification (1.1) is that it only maps certain objects on the two sides of the duality, such as the on-shell action and the generating function. However, the bulk fields, or indeed the equations of motion in the bulk are not given any concrete meaning on the QFT side, except from the indirect role in evaluating the on-shell action. As we shall see, both the Renormalization Group (RG) of local QFTs and the dual gravitational theories admit a Hamiltonian description that allows us to formulate the holographic dictionary more precisely.

These lecture notes are organized as follows. In section 2 we discuss local QFT observables and global symmetries in a language that does not assume a weak coupling or Lagrangian description. Moreover, we put forward a Hamiltonian formulation of the Renormalization Group of local QFTs that directly parallels the description of the holographic dual bulk theory later on. We end section 2 with a concrete example of UV divergences in the two-point function of a scalar operator in a CFT. In section 3 we carry out explicitly the holographic computation for the two-point function on a fixed AdS background and reproduce the renormalized result obtained from the CFT calculation. The Hamiltonian formulation of the holographic dictionary is presented in section 3.2. Section 4 discusses at length the radial Hamiltonian formulation of the bulk dynamics for Einstein-Hilbert gravity coupled to a self interacting scalar. In Section 5 we present two algorithms for recursively solving the radial Hamilton-Jacobi equation, which constitutes the core of holographic renormalization. Given the solution of the Hamilton-Jacobi equation derived in section 5, in section 6 we provide general expressions for the renormalized one-point functions in the presence of sources and derive the holographic Ward identities. Finally, in section 7 we show how the asymptotic expansions of the bulk fields can be obtained systematically from the solution of the Hamilton-Jacobi equation. Some background material is presented in the appendices. In particular, appendix B is a self contained review of Hamilton-Jacobi theory in classical mechanics.

2 Local QFT observables and the local Renormalization Group

Before we delve into the details of the holographic dictionary and the computation of QFT observables from the bulk gravitational theory, it is instructive to review some basic aspects of QFTs and to put them in a language that will later help us make contact with the holographic dual bulk theory. In particular, since the gauge/gravity duality relates the strongly coupled regime of local QFTs to the bulk gravity theory, it is crucial to describe the local QFT observables and their properties in a way that is valid at strong coupling. Ideally we would like to discuss local QFT observables without assuming the existence of a microscopic Lagrangian description of the QFT.

2.1 QFT correlation functions and the generating functional

The basic objects of a local QFT are correlation functions of local operators, $\mathcal{O}(x)$, namely

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle. \tag{2.1}$$

In particular, if we know all correlation functions of all local operators of a local QFT, then in most cases we know all there is to know about this theory[2]. In a generic theory, even if there is only a finite number of local operators present in a given QFT, the number of correlation functions that we need to know can be infinite. So, instead of having to deal with an infinite number of correlation functions, it is useful to introduce the generating function of correlation functions, $Z[J]$, as a book keeping device. For a single local operator $\mathcal{O}(x)$, the generating function takes the form

$$Z[J] = \sum_{k=0}^{\infty}\frac{1}{k!}\int d^dx_1\int d^dx_2\dots\int d^dx_k J(x_1)J(x_2)\dots J(x_k)\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_k)\rangle, \tag{2.2}$$

where d is the spacetime dimension. Given $Z[J]$, any correlation function of the operator $\mathcal{O}(x)$ can be extracted by multiple functional differentiation:

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_k)\rangle = \left.\frac{\delta^k Z[J]}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_k)}\right|_{J=0}. \tag{2.3}$$

These definitions straightforwardly generalize to a set of local operators $\{\mathcal{O}_1(x),\mathcal{O}_2(x),\cdots\}$, with the corresponding generating functional $Z[J_1,J_2,\cdots]$ depending on the sources $J_1(x),J_2(x),\cdots$. Moreover, the definition of the generating functional through (2.2) is completely general and it does not assume a Lagrangian description of the theory. Of course, if the theory admits a Lagrangian description, then the generating functional $Z[J]$ has the standard path integral representation

$$Z[J] = \int \mathcal{D}\phi\; e^{i\int d^dx\mathcal{L}(\phi)+\int d^dxJ(x)\mathcal{O}(x)}, \tag{2.4}$$

where ϕ here stand for the elementary Lagrangian fields.

An alternative but equivalent way to encode all local observables is in terms of the generating function of *connected* correlation functions

$$W[J] = \log Z[J], \tag{2.5}$$

or

$$W[J] = \sum_{k=0}^{\infty}\frac{1}{k!}\int d^dx_1\int d^dx_2\dots\int d^dx_k J(x_1)J(x_2)\dots J(x_k)\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_k)\rangle_c, \tag{2.6}$$

where $\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_k)\rangle_c$ are now connected correlation functions. The first derivative of the generating function (2.6) corresponds to the one-point function of the dual operator *in the presence of an arbitrary source*, namely

$$\langle\mathcal{O}(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)}. \tag{2.7}$$

Taking further derivatives with respect to the source we can obtain any desired correlation function of the operator $\mathcal{O}(x)$. In particular, the one-point function in the presence of sources (2.7) encodes

[2] Sometimes, additional global observables must be specified to uniquely identify a theory [14]

the same local information as the generating function (2.6). This fact will be crucial for the discussion of the holographic dictionary later on.

Another important aspect of (2.7) is that it amounts to a prescription for the insertion of the local operator $\mathcal{O}(x)$ in any correlation function and so, in effect, it provides a *definition* of the local operator $\mathcal{O}(x)$. This is indeed the point of view adopted in the so called *local Renormalization Group* formulation of QFT [15], where local operators are defined as derivatives of the generating function with respect to the corresponding local coupling. For example, the stress tensor, a $U(1)$ current and a scalar operator are *defined* through the relations

$$\mathcal{T}_{ij}(x) = -\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}(x)}, \tag{2.8a}$$

$$\mathcal{J}^i(x) = -\frac{1}{\sqrt{g}}\frac{\delta W}{\delta A_i(x)}, \tag{2.8b}$$

$$\mathcal{O}(x) = -\frac{1}{\sqrt{g}}\frac{\delta W}{\delta \varphi(x)}, \tag{2.8c}$$

where g_{ij} is a general background metric on the space where the QFT is defined, and A_i is an Abelian background gauge field. The indices $i, j = 1, 2, \cdots, d$ run over all coordinates parameterizing the space where the QFT is defined.

2.2 The local Renormalization Group as a Hamiltonian flow

The expressions (2.8) for the one-point functions in the presence of sources bare striking resemblance to the expression for the canonical momenta in classical Hamilton-Jacobi (HJ) theory. In particular, the one-point functions (2.8) look mathematically identical to the expressions (B.8) or (B.14) for the canonical momenta in appendix B, where we review some basic aspects of HJ theory that we will use repeatedly throughout these lectures.

This analogy turns out to be particularly useful for developing the holographic dictionary and can be formalized as follows [16]. Let $\mathcal{Q}$ be the space of functions (more generally tensors) on the spacetime, Σ, where the QFT resides (e.g. $\mathbb{R}^d$). The sources $J^\alpha(x)$ are coordinates on $\mathcal{Q}$, which is the analogue of the configuration space in classical mechanics. Let us extend this configuration space to $\mathcal{Q}_{\text{ext}} = \mathcal{Q} \times \mathbb{R}$, by appending an abstract "time" τ to the generalized coordinates $J^\alpha(x)$ as in appendix B in the case of a time-dependent Hamiltonian. Accordingly, an abstract Hamiltonian operator, $\mathbb{H}$, must be introduced as conjugate momentum to τ. Note that $\mathbb{H}$ is a global operator, i.e. it does not depend on x.[3] The extended phase space is then parameterized by the variables

$$\{\mathcal{O}_\alpha(x), \mathbb{H}; J^\alpha(x), \tau\}, \tag{2.9}$$

and it is isomorphic to the cotangent bundle $T^*\mathcal{Q}_{\text{ext}}$, which is endowed with the pre-symplectic form

$$\Theta = \int d^dx\, \mathcal{O}_\alpha(x)\delta J^\alpha(x) - \mathbb{H}d\tau, \tag{2.10}$$

and the canonical symplectic closed 2-form

$$\Omega = \int d^dx\, \delta\mathcal{O}_\alpha(x) \wedge \delta J^\alpha(x) - d\mathbb{H} \wedge d\tau, \tag{2.11}$$

that can be written locally as $\Omega = \delta\Theta$.

[3] To make contact with [16] one can introduce a Hamiltonian density, $\mathbb{h}(x)$, through $\mathbb{H} = \int d^dx\, \mathbb{h}(x)$.

Any functional, $F[J;\tau]$, provides a closed section of the cotangent bundle, $s : \mathcal{Q}_{\text{ext}} \longrightarrow T^*\mathcal{Q}_{\text{ext}}$, given locally by

$$s = \delta F[J;\tau]. \tag{2.12}$$

It follows that

$$\Theta \circ s = \delta F[J;\tau], \tag{2.13}$$

or equivalently

$$\mathcal{O}_\alpha = \frac{\delta F[J;\tau]}{\delta J^\alpha}, \quad \mathbb{H} = -\frac{\partial F[J;\tau]}{\partial \tau}, \tag{2.14}$$

while

$$\Omega \circ s = \int d^dx \int d^dx' \frac{\delta^2 F[J;\tau]}{\delta J^\beta(x')\delta J^\alpha(x)} \delta J^\beta(x') \wedge \delta J^\alpha(x) - \frac{\partial^2 F[J;\tau]}{\partial \tau^2} d\tau \wedge d\tau = 0. \tag{2.15}$$

As follows from the Hamilton-Jacobi theorem (see appendix B), the τ-evolution of all the variables is then governed by Hamilton's equations

$$\dot{J}^\alpha = \frac{\delta \mathbb{H}}{\delta \mathcal{O}_\alpha}, \quad \dot{\mathcal{O}}_\alpha = -\frac{\delta \mathbb{H}}{\delta J^\alpha}, \quad \dot{\mathbb{H}} = \frac{\partial \mathbb{H}}{\partial \tau}. \tag{2.16}$$

Note that the functional derivatives in (2.14) and (2.16)are partial derivatives.

There are two different closed sections of the cotangent bundle $T^*\mathcal{Q}_{\text{ext}}$ one can naturally define for any local QFT. Taking τ to be related to some generic energy scale μ via $\tau = \log(\mu/\mu_o)$, where μ_o is some constant reference scale, the bare and renormalized generating functions, respectively $W[J]$ and $W_{\text{ren}}[J;\tau]$, provide two distinct closed sections of the cotangent bundle $T^*\mathcal{Q}_{\text{ext}}$. The difference between these two functionals is that $W_{\text{ren}}[J;\tau]$ is RG invariant, i.e. given $\sigma : \mathbb{R} \longrightarrow \mathcal{Q}$, its total derivative with respect to τ vanishes, $\dot{W}_{\text{ren}}[\sigma(\tau);\tau] = 0$, while $W[J]$ is *not* an RG invariant. The total derivative of $W[J]$ with respect to τ gives, by construction, the Legendre transform of the Hamiltonian $\mathbb{H}$, i.e. the associated Lagrangian[4]

$$\dot{W}[J] = \mathbb{L} = \int d^dx \dot{J}^\alpha \mathcal{O}_\alpha - \mathbb{H} = \int d^dx \beta^\alpha \mathcal{O}_\alpha - \mathbb{H}, \tag{2.17}$$

where $\beta^\alpha = \dot{J}^\alpha$ are the beta functions of the couplings J^α. Moreover, $W[J]$ depends on τ only through the couplings J^α, while $W_{\text{ren}}[J;\tau]$ can also depend *explicitly* on τ through the conformal anomaly. Through (2.14), these two sections define different local operators and Hamiltonians, which are related through a canonical transformation [17].

Renormalized RG Hamiltonian

Taking $F[J;\tau] = W_{\text{ren}}[J;\tau]$, the first equation in (2.14) is just the renormalized version of the local RG definition of local operators that we saw above in (2.8), namely[5]

$$\mathcal{O}^{\text{ren}}_\alpha = \frac{\delta W_{\text{ren}}[J;\tau]}{\delta J^\alpha}. \tag{2.18}$$

[4]Note that in [16] only the RG invariant $W_{\text{ren}}[J;\tau]$ is considered, written in terms of the bare and renormalized couplings. $W[J]$ is not discussed at all in that reference.

[5]The way we have defined the operators $\mathcal{O}_\alpha$ and $\mathbb{H}$ in this subsection, they are in fact densities with respect to the background metric g_{ij}, i.e. we have not divided by $\sqrt{g}$ as in (2.8). Moreover, $\mathcal{O}_\alpha$ include the stress tensor.

The second equation in (2.14), with $F[J;\tau] = W_{\rm ren}[J;\tau]$, can be viewed as a *definition* of the Hamiltonian $\mathbb{H}_{\rm ren}$ in QFT. In particular, we conclude that $\mathbb{H}_{\rm ren}$ is numerically equal to the conformal anomaly,

$$\boxed{\mathbb{H}_{\rm ren} = -\frac{\partial W_{\rm ren}[J;\tau]}{\partial \tau} = -\int d^d x \sqrt{g}\mathcal{A},} \tag{2.19}$$

where $\mathcal{A}$ is the conformal anomaly.

Bare RG Hamiltonian

Taking $F[J;\tau] = W[J]$, on the other hand, provides a section of $T^*\mathcal{Q}$. The first equation in (2.14) is then identical to the local RG expressions (2.8), while the second equation in (2.14) implies that the bare RG Hamiltonian vanishes identically

$$\boxed{\mathbb{H} = -\frac{\partial W[J]}{\partial \tau} = 0.} \tag{2.20}$$

As we mentioned above, the bare and renormalized Hamiltonians, as well as the corresponding local operators, are related by a canonical transformation whose generating function (in the sense of canonical transformations) is given by the local counterterms, $W_{\rm ct}[J;\tau]$, [17]. Note that the explicit τ-dependence of $W_{\rm ren}[J;\tau]$ is entirely due to the local counterterms and, in particular, the conformal anomaly. Under this canonical transformation

$$W[J] \longrightarrow W_{\rm ren}[J;\tau] = W[J] + W_{\rm ct}[J;\tau]. \tag{2.21}$$

RG equations

The RG equations for the generating functions $W[J]$ and $W_{\rm ren}[J;\tau]$ are respectively

$$\mathbb{L} = \dot{W} = \int d^d x \beta^\alpha \mathcal{O}_\alpha \Leftrightarrow \mathbb{H} = 0, \tag{2.22a}$$

$$0 = \dot{W}_{\rm ren} = \int d^d x \beta^\alpha \mathcal{O}^{\rm ren}_\alpha + \frac{\partial W_{\rm ren}}{\partial \tau} = \int d^d x \beta^\alpha \mathcal{O}^{\rm ren}_\alpha + \int d^d x \sqrt{g}\mathcal{A}. \tag{2.22b}$$

The first of these equations is just the HJ equation (2.20). Comparing the second equation with the HJ equation (2.19) we conclude that the renormalized Hamiltonian takes the form

$$\boxed{\mathbb{H}_{\rm ren} = \int d^d x \beta^\alpha \mathcal{O}^{\rm ren}_\alpha,} \tag{2.23}$$

where the sum in this expression is over all operators in the theory, including the stress tensor. Given the beta functions as functions of the local running couplings J^α, this Hamiltonian is linear in the canonical momenta, i.e. in $\mathcal{O}^{\rm ren}_\alpha$ [16]. The standard renormalization procedure in QFT is equivalent to determining the beta functions as functions of the local running couplings and $W_{\rm ren}[J;\tau]$ through the HJ equation (2.19), i.e.

$$\boxed{\left(\int d^d x \beta^\alpha[J] \frac{\delta}{\delta J^\alpha} + \frac{\partial}{\partial \tau}\right) W_{\rm ren}[J;\tau] = 0.} \tag{2.24}$$

This is the standard RG equation.

Given $\beta^\alpha[J]$ one can integrate the first Hamilton equation in (2.16) to obtain

$$\mathbb{H} = \int d^d x \beta^\alpha[J]\mathcal{O}_\alpha + \mathcal{F}[J;\tau], \tag{2.25}$$

for some unspecified $\mathcal{F}[J;\tau]$. Combining this relation with the fact that $\mathbb{H}$ and $\mathbb{H}_{\rm ren}$ are related by a canonical transformation generated by $W_{\rm ct}[J;\tau]$, namely

$$\mathbb{H} - \mathbb{H}_{\rm ren} + \frac{\partial W_{\rm ct}}{\partial \tau} = 0, \tag{2.26}$$

we deduce that

$$\mathcal{F}[J;\tau] = \left(\int d^d x \beta^\alpha[J]\frac{\delta}{\delta J^\alpha} - \frac{\partial}{\partial \tau}\right) W_{\rm ct}[J;\tau], \tag{2.27}$$

and hence

$$\boxed{\mathbb{H}[O_\alpha, J^\beta] = \int d^d x \beta^\alpha[J]\mathcal{O}_\alpha + \left(\int d^d x \beta^\alpha[J]\frac{\delta}{\delta J^\alpha} - \frac{\partial}{\partial \tau}\right) W_{\rm ct}[J;\tau].} \tag{2.28}$$

However, if the beta functions are not just functions of the running couplings, but depend linearly on the local operators $\mathcal{O}_\alpha$, i.e.

$$\beta^\alpha[O, J] = \mathcal{G}^{\alpha\beta}[J]\mathcal{O}_\beta, \tag{2.29}$$

then the first of Hamilton's equations in (2.16) gives

$$\mathbb{H} = \frac{1}{2}\int d^d x\, \mathcal{G}^{\alpha\beta}[J]\mathcal{O}_\alpha\mathcal{O}_\beta + \widetilde{\mathcal{F}}[J;\tau], \tag{2.30}$$

for some unspecified $\widetilde{\mathcal{F}}[J;\tau]$. Notice that if the beta functions take the form (2.29), then the RG flow is a *gradient flow*, since $\beta^\alpha = \mathcal{G}^{\alpha\beta}\delta W/\delta J^\beta$. As we shall see, this form of the beta functions and of the Hamiltonian $\mathbb{H}$ are directly related to the bulk holographic description of the theory.

2.3 Global symmetries and Ward identities

A general property of QFTs is that they typically possess a number of global symmetries. For example, a relativistic QFT on flat Minkowski space possesses Poincaré symmetry. If the theory is additionally scale invariant, then it will generically possess conformal symmetry. Such theories are known as conformal field theories (CFTs) and the fact that they are conformally invariant allows us to make sense of them on curved backgrounds that are conformally related to flat Minkowski space. Other examples of global symmetries include internal symmetries such as $SU(2)$ isospin (for massless up and down quarks) or supersymmetry.

In QFTs that admit a classical Lagrangian description, global symmetries manifest themselves as invariances of the classical action and lead via Noether's theorem to conserved currents. For example, Poincaré invariance of the classical action implies that the stress-energy tensor, $\mathcal{T}_{ij}$, is conserved, i.e.

$$\partial^i \mathcal{T}_{ij} = 0. \tag{2.31}$$

Similarly, global internal symmetries lead to conserved currents $\mathcal{J}^i$,

$$\partial_i \mathcal{J}^i = 0. \tag{2.32}$$

At the quantum level these currents become quantum operators and their classical conservation laws imply relations among certain correlation functions that involve these currents. These identities,

relating various correlation functions as a result of the classical Noether theorem, are known as *Ward identities*.

It is often the case, however, that some of the classical symmetries are broken at the quantum level. This happens because in a QFT various quantities contain ultraviolet divergences which must be regulated and renormalized to yield a well defined quantity. However, there may not exist a regulator that preserves all of the classical symmetries of the theory, which leads to the breaking of some symmetries at the quantum level. This breaking of the classical symmetries at the quantum level leads to the so-called *quantum anomalies* in the Ward identities.

A particularly elegant way to derive the Ward identities of a quantum field theory, without relying on a classical Lagrangian description of the theory, is to work with the generating functional of correlation functions and gauge the global symmetries by promoting the sources of the corresponding conserved currents to gauge fields. Among all operators in any QFT there is always the stress tensor, $\mathcal{T}_{ij}$, and let us assume that there is in addition an internal $U(1)$ symmetry giving rise to a current, $\mathcal{J}^i$, in the spectrum of operators. Moreover, to be generic, let us suppose that there is also a scalar operator, $\mathcal{O}$, transforming trivially both under the Poincaré group and the $U(1)$ symmetry, but has definite scaling dimension Δ. The generating functional of connected correlation functions will then be a function of the sources, g_{ij}, A_i, φ, respectively for the stress tensor, the current of the internal symmetry, and for the scalar operator, as well as for all other operators in the theory which we will not need to consider:

$$W[g, A, \varphi, \ldots]. \tag{2.33}$$

As we would now do in a classical Lagrangian description of the theory to derive Noether's theorem, we gauge the global symmetries by promoting the Poincaré transformations to diffeomorphisms and the internal global symmetry to a local gauge symmetry, while promoting the sources[6] ${g_{(0)}}^{ij}$ and $A_{(0)i}$ to gauge fields of the corresponding local symmetries. In a classical Lagrangian description this would amount to introducing 'minimal couplings' in the Lagrangian. Under infinitesimal diffeomorphisms, parameterized by the vector $\xi^i(x)$, the sources then transform as

$$\delta_\xi g^{ij}_{(0)} = -(D^i_{(0)}\xi^j + D^j_{(0)}\xi^i), \quad \delta_\xi A_{(0)i} = A_{(0)j}D_{(0)i}\xi^j + \xi^j D_{(0)j}A_{(0)i}, \quad \delta_\xi \varphi_{(0)} = \xi^j D_{(0)j}\varphi_{(0)}, \tag{2.34}$$

while under infinitesimal $U(1)$ gauge transformations, parameterized by the gauge function $\alpha(x)$, they transform as

$$\delta_\alpha g_{(0)ij} = 0, \quad \delta_\alpha A_{(0)i} = D_{(0)i}\alpha(x), \quad \delta_\alpha \varphi_{(0)} = 0, \tag{2.35}$$

where $D_{(0)i}$ denotes the covariant derivative with respect to the metric $g_{(0)ij}$. The Ward identities now can be stated very simply and generally as

$$\delta_\xi W = 0, \quad \delta_\alpha W = 0, \quad \forall \quad \xi^i, \alpha, \tag{2.36}$$

respectively following from the Poincaré and $U(1)$ symmetries. We can manipulate these expressions a bit further to bring the Ward identities in a more familiar form. Starting with the $U(1)$ Ward identity we have

$$\begin{aligned} \delta_\alpha W = 0 &\Leftrightarrow \int d^dx \left(\delta_\alpha {g_{(0)}}^{ij} \frac{\delta W}{\delta {g_{(0)}}^{ij}} + \delta_\alpha A_{(0)i} \frac{\delta W}{\delta A_{(0)i}} + \delta_\alpha \varphi_{(0)} \frac{\delta W}{\delta \varphi_{(0)}} \right) = 0 \\ &\Leftrightarrow \int d^dx D_{(0)i}\alpha(x) \frac{\delta W}{\delta A_{(0)i}} = 0 \Leftrightarrow \int d^dx \alpha(x) D_{(0)i} \left(\frac{\delta W}{\delta A_{(0)i}} \right) = 0, \end{aligned} \tag{2.37}$$

[6] The subscript (0) here is intended to help make contact with the holographic computation later.

where we have integrated by parts in the last step and have dropped the boundary term. Since $\alpha(x)$ is arbitrary, it follows that the $U(1)$ Ward identity is equivalent to the identity

$$D_{(0)i}\left(\frac{\delta W}{\delta A_{(0)i}}\right) = 0. \tag{2.38}$$

We can now repeat this exercise for diffeomorphisms to obtain

$$\delta_\xi W = 0 \Leftrightarrow D^i_{(0)}\left(2\frac{\delta W}{\delta {g_{(0)}}^{ij}}\right) - F_{(0)ij}\frac{\delta W}{\delta A_{(0)i}} + \frac{\delta W}{\delta \varphi_{(0)}} D_{(0)j}\varphi_{(0)}(x) = 0, \tag{2.39}$$

where $F_{(0)ij} = \partial_i A_{(0)j} - \partial_j A_{(0)i}$ is the field strength of the gauge field $A_{(0)i}$.

In terms of the one-point functions in the presence of sources the above Ward identities take the simple form

$$\boxed{D_{(0)i}\langle \mathcal{J}^i(x)\rangle = 0,} \tag{2.40}$$

$$\boxed{D^i_{(0)}\langle \mathcal{T}_{ij}(x)\rangle - \langle \mathcal{J}^i(x)\rangle_s F_{(0)ij} + \langle \mathcal{O}(x)\rangle D_{(0)j}\varphi_{(0)}(x) = 0,} \tag{2.41}$$

following respectively from $U(1)$ and Poincaré invariance.

Finally, let us consider Weyl transformations, i.e. *local* scale transformations, parameterized by the Weyl factor $\sigma(x)$. Under infinitesimal Weyl transformations the sources transform as

$$\delta_\sigma {g_{(0)}}^{ij} = -2\delta\sigma(x){g_{(0)}}^{ij}, \quad \delta_\sigma A_{(0)i} = 0, \quad \delta_\sigma \varphi_{(0)} = -(d-\Delta)\delta\sigma(x)\varphi_{(0)}, \tag{2.42}$$

where Δ is the conformal dimension of the operator $\mathcal{O}(x)$ and we focus here on a CFT since scale invariance is not a symmetry of a generic QFT. As we have seen, even if our theory is a conformal field theory, the generating functional of renormalized correlation functions will not be in general invariant under such a Weyl transformation. The variation of the generating functional with respect to Weyl transformations defines the *conformal anomaly*

$$\delta_\sigma W = \int d^d x \sqrt{g_{(0)}}\delta\sigma(x)\mathcal{A}, \tag{2.43}$$

where the anomaly density, $\mathcal{A}$ is a local function of the sources. Using the above transformation of the sources, this then leads to the trace Ward identity

$$\boxed{\langle \mathcal{T}^i_i(x)\rangle = -(d-\Delta)\varphi_{(0)}\langle \mathcal{O}(x)\rangle + \mathcal{A}.} \tag{2.44}$$

We recognize this Ward identity as the local version of the RG equation (2.24), at a fixed point of the renormalization group.

2.4 UV divergences and renormalization of composite operators

Let us now address in more detail the question of renormalization in QFT with a simple example. This will allow us to directly compare with a holographic calculation in the next subsection in order to get a first idea of the holographic dictionary.

Consider a CFT with a scalar operator $\mathcal{O}_\Delta(x)$ of conformal dimension Δ. Conformal symmetry determines the two-point function up to an overall constant, namely

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{c(g,\Delta)}{|x-y|^{2\Delta}}, \tag{2.45}$$

where c is an arbitrary constant, depending on the dimension Δ and possibly any coupling constants, g, of the CFT, that we could absorb into the normalization of the operator $\mathcal{O}_\Delta$, but we will not. Depending on the conformal dimension, Δ, this correlator may suffer from short distance singularities. Consider the case $\Delta = d/2 + k + \epsilon$, where ϵ is an infinitesimal parameter and k is a non-negative integer. Iterating the identity

$$\frac{1}{|x-y|^{2\Delta}} = \frac{1}{2(\Delta-1)(2\Delta-d)}\Box\frac{1}{|x-y|^{2\Delta-2}}, \qquad |x-y| \neq 0, \tag{2.46}$$

where $\Box = \delta^{ij}\partial_i\partial_j$, $k+1$ times, we find

$$\begin{aligned}\frac{1}{|x-y|^{2\Delta}} &= \frac{1}{2\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(d/2+\epsilon)}{2^{2k}\Gamma(k+1+\epsilon)\Gamma(d/2+k+\epsilon)}\frac{1}{d-2+2\epsilon}\Box^{k+1}\frac{1}{|x-y|^{d-2+2\epsilon}} \\ &\sim \frac{-1}{2\epsilon}\frac{\omega_{d-1}\Gamma(d/2)}{2^{2k}\Gamma(k+1)\Gamma(d/2+k)}\Box^k\delta^{(d)}(x-y),\end{aligned} \tag{2.47}$$

where $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit $(d-1)$-sphere and we have used the identity $\Box(x^2)^{-d/2+1} = -(d-2)\omega_{d-1}\delta^{(d)}(x)$. We thus find that there is a pole at $\Delta = d/2+k$, or $\epsilon = 0$. To produce a well defined distribution we subtract the pole and define [18]

$$\begin{aligned}\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_{\text{ren}} &= c(g,\Delta)\lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(d/2+\epsilon)}{2^{2k}\Gamma(k+1+\epsilon)\Gamma(d/2+k+\epsilon)}\right. \\ &\qquad \left.\frac{1}{d-2+2\epsilon}\Box^{k+1}\frac{1}{|x|^{d-2}}\left(\frac{1}{|x|^{2\epsilon}}-\mu^{2\epsilon}\right)\right\} \\ &= \frac{-c_k}{2(d-2)}\Box^{k+1}\frac{1}{|x|^{d-2}}\left\{\log\left(\mu^2x^2\right)+a(k)\right\},\end{aligned} \tag{2.48}$$

where

$$c_k \equiv c(g,\Delta)\frac{\Gamma(d/2)}{2^{2k}\Gamma(k+1)\Gamma(d/2+k)}. \tag{2.49}$$

The constant $a(k)$ reflects the scheme dependence in the subtraction of the pole. Here we have defined the subtraction in such a way so that $a = 0$, but other subtraction schemes, such as minimal subtraction, lead to a non-zero a. The renormalized correlator agrees with the bare one away from coincident points but is also well-defined at $x^2 = 0$. To allow a direct comparison of the renormalized two-point function with the result we will obtain below from the bulk calculation, it is useful to write down its Fourier transform. Using the identity

$$\int d^dx e^{ip\cdot x}\frac{1}{|x|^{d-2}}\log\left(\mu^2x^2\right) = -\frac{4\pi^{d/2}}{\Gamma(d/2-1)}\frac{1}{p^2}\log(p^2/\bar{\mu}^2), \tag{2.50}$$

where $\bar{\mu} = 2\mu/\gamma$ and $\gamma = 1.781072\ldots$ is the Euler constant, we obtain

$$\langle\mathcal{O}_\Delta(p)\mathcal{O}_\Delta(-p)\rangle_{\text{ren}} = c_k\frac{(-1)^{k+1}}{2(d-2)}\frac{4\pi^{d/2}}{\Gamma(d/2-1)}p^{2k}\log(p^2/\bar{\mu}^2). \tag{2.51}$$

3 The holographic dictionary

3.1 A first look at the holographic dictionary and holographic renormalization

In order to compute the above scalar two-point function holographically, we consider a self interacting scalar field in a fixed Euclidean background with the action

$$S = \int d^{d+1}x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right). \tag{3.1}$$

We will take the metric to be of the form

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j, \tag{3.2}$$

where $i, j = 1, 2, \ldots, d$ run over the field theory directions, and the induced metric on the constant r slices is given by

$$\gamma_{ij}(r,x) = e^{2A(r)}\hat{g}_{ij}(x), \tag{3.3}$$

with

$$A(r) = r, \quad \hat{g}_{ij}(x) = \delta_{ij}, \tag{3.4}$$

for AdS. This metric is diffeomorphic to the upper-half plane or Poincaré coordinates metric

$$ds^2 = \frac{dz_0^2 + d\vec{z}^2}{z_0^2}. \tag{3.5}$$

Our first task is to obtain the radial Hamiltonian for this model, interpreting the radial coordinate r as Hamiltonian 'time'. The action can be written in the form

$$S = \int^r dr' L = \int^r dr' d^d x \sqrt{\gamma}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi + V(\phi)\right). \tag{3.6}$$

The canonical momentum conjugate to ϕ then is

$$\pi = \frac{\delta L}{\delta\dot{\phi}} = \sqrt{\gamma}\dot{\phi}. \tag{3.7}$$

The HJ equation can be derived from the relation

$$\dot{\mathcal{S}} = L = \int d^d x\left(\dot{\phi}\frac{\delta\mathcal{S}}{\delta\phi} + \dot{\gamma}_{ij}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\right), \tag{3.8}$$

where Hamilton's principal function (see appendix B), $\mathcal{S}$, has no explicit r dependence since the Lagrangian is diffeomorphism covariant. Writing

$$\pi = \sqrt{\gamma}\dot{\phi} = \frac{\delta\mathcal{S}}{\delta\phi}, \tag{3.9}$$

this equation becomes

$$\int d^d x\left[\sqrt{\gamma}\left(\frac{1}{2}\left(\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\phi}\right)^2 - \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - V(\phi)\right) + 2\dot{A}\gamma_{ij}\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\right] = 0. \tag{3.10}$$

This is the HJ equation for the scalar field in a fixed gravitational background, which can be rewritten in the more useful form

$$\sqrt{\gamma}\left(\frac{1}{2}\left(\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\phi}\right)^2 - \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - V(\phi)\right) + 2\dot{A}\delta_\gamma\mathcal{L} = \partial_i v^i, \tag{3.11}$$

where

$$\mathcal{S} = \int d^d x\mathcal{L}, \tag{3.12}$$

and

$$\delta_\gamma = \int d^d x\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}. \tag{3.13}$$

The term $\partial_i v^i$ on the RHS is a total derivative that can be arbitrary, but which generically needs to be taken into account when trying to solve (3.11). It is not difficult to solve this equation iteratively, for example in a derivative expansion, for a general potential $V(\phi)$. However, for the present discussion it suffices to consider the simple –yet far from trivial– case of a free scalar field with the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{3.14}$$

The great simplification that results from this potential is that we can solve the corresponding HJ equation exactly, to all orders in transverse derivatives.

The HJ equation (3.11) in this case becomes

$$\sqrt{\gamma}\left(\frac{1}{2}\left(\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\phi}\right)^2 - \frac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2\right) + 2\delta_\gamma\mathcal{L} = \partial_i v^i. \tag{3.15}$$

Inserting an ansatz of the form

$$\mathcal{S} = \frac{1}{2}\int d^dx\sqrt{\gamma}\phi f(-\Box_\gamma)\phi, \tag{3.16}$$

we find that it solves the HJ equation, provided the function $f(x)$ satisfies [19]

$$f^2(x) + df(x) - m^2 - x - 2xf'(x) = 0. \tag{3.17}$$

The general solution of this equation is

$$f(x) = -\frac{d}{2} - \frac{\sqrt{x}\left(K_k'(\sqrt{x}) + cI_k'(\sqrt{x})\right)}{K_k(\sqrt{x}) + cI_k(\sqrt{x})}, \tag{3.18}$$

where $k = \Delta - d/2 > 0$, c is an arbitrary constant, and $I_k(x)$ and $K_k(x)$ denote the modified Bessel function of the first and second kind respectively. Using the asymptotic behaviors as $x \to 0$

$$K_0(x) \sim -\log x, \quad K_k(x) \sim \frac{\Gamma(k)}{2}\left(\frac{x}{2}\right)^{-k}, \quad k > 0, \qquad I_k(x) \sim \frac{1}{\Gamma(k+1)}\left(\frac{x}{2}\right)^k, \tag{3.19}$$

we see that $K_k(x)$ dominates in $f(x)$ as $x \to 0$, unless $|c| \to \infty$. In particular, we find

$$f(x) \overset{x\to 0}{\sim} \begin{cases} -\frac{d}{2} + k = -(d-\Delta), & |c| < \infty, \\ -\frac{d}{2} - k = -\Delta, & |c| \to \infty. \end{cases} \tag{3.20}$$

Since,

$$\dot{\phi} = \frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\phi}, \tag{3.21}$$

we see that the two asymptotic solutions for $f(x)$ correspond to $\phi \sim e^{-(d-\Delta)r}$ and $\phi \sim e^{-\Delta r}$ respectively, which are precisely the asymptotic behaviors of the two linearly independent solutions of the equation of motion. The solution for $f(x)$ with $|c| < \infty$ corresponds to the asymptotically dominant mode. Hence, in order to make the variational problem well defined for generic solutions of the equation of motion we have no choice but demand that $|c| < \infty$.

Expanding the solution for $f(x)$ with $|c| < \infty$ for small x and taking k to be an integer we obtain,

$$\begin{aligned} f(x) &= -(d-\Delta) + \frac{x}{(2\Delta-d-2)} - \frac{x^2}{(2\Delta-d-2)(2\Delta-d-4)} + \cdots + \frac{(-1)^k}{2^{2k-1}\Gamma(k)^2}x^k\log x \\ &\quad + \left(a(k) - \frac{c}{2^{2k-2}\Gamma(k)^2}\right)x^k + \cdots, \end{aligned} \tag{3.22}$$

where $a(k)$ is a known function of k, whose explicit form we will not need, and the dots denote asymptotically subleading terms. A number of comments are in order here. Firstly, this solution depends explicitly on the undetermined constant $|c| < \infty$. Secondly, this solution seems to lead to a non-local boundary term due to the logarithmic term. And finally, one may worry that higher terms in this asymptotic expansion need to be considered. Fortunately, all these issues can be addressed by noticing that the contribution of the last term to the boundary term is proportional to

$$\int d^d x \sqrt{\gamma} \phi (-\Box_\gamma)^k \phi, \tag{3.23}$$

which, taking into account the asymptotic behavior of the scalar and of the induced metric, can be easily seen to have a finite limit as $r \to \infty$. Such terms, therefore, correspond to adding finite local contributions to the boundary term S_b. We conclude that higher order terms in the asymptotic expansion of $f(x)$ need not be considered since they would give rise to a vanishing contribution to S_b in the limit $r \to \infty$. Moreover, the arbitrariness in the value of c is not a problem because different values of c lead to boundary terms S_b which differ by a finite local term. Any value of $|c| < \infty$, therefore, is equally acceptable since the corresponding boundary term makes the variational problem well defined. Finally, coming to the apparent non-locality of the boundary term we have deduced above, we notice that the logarithmic term can be written as

$$(-\Box_\gamma)^k \log(-\Box_\gamma) = (-\Box_\gamma)^k \left(\log(\mu^2 e^{-2r}) + \log(-\Box_\delta/\mu^2) \right), \tag{3.24}$$

where μ^2 is an arbitrary scale and $\Box_\delta = \partial_i \partial_i$ denotes the Laplacian in the flat transverse space. Crucially, the non-local part gives rise to a finite contribution in Hamilton's principal function and so it can be omitted from counterterms. The most general local boundary term that makes the variational problem well defined is therefore [12, 19]

$$\begin{aligned} S_{\rm ct}[\gamma, \phi, r] \;=\; & -\frac{1}{2} \int d^d x \sqrt{\gamma} \phi \left(-(d-\Delta) + \frac{-\Box_\gamma}{(2\Delta - d - 2)} - \frac{(-\Box_\gamma)^2}{(2\Delta - d - 2)(2\Delta - d - 4)} + \cdots \right. \\ & \left. + \frac{(-1)^k}{2^{2k-1}\Gamma(k)^2} (-\Box_\gamma)^k \log(\mu^2 e^{-2r}) + \xi (-\Box_\gamma)^k \right) \phi, \end{aligned} \tag{3.25}$$

where we have allowed for a local finite boundary term with arbitrary coefficient ξ. Notice that although it is possible to find counterterms that remove the UV divergences and are also local in transverse derivatives, this is only at the cost of introducing explicit dependence in the radial coordinate, r. This is precisely the origin of the holographic conformal anomaly [4].

The renormalized action on the UV cut-off r_o is defined as

$$S_{\rm ren} := S_{\rm reg} + S_{\rm ct}, \tag{3.26}$$

and it admits a finite limit, $\widehat{S}_{\rm ren}$, as the cut-off is removed:

$$\widehat{S}_{\rm ren} = \lim_{r_o \to \infty} S_{\rm ren}. \tag{3.27}$$

In this case, ignoring the scheme dependent contact terms, we obtain

$$S_{\rm ren} = \frac{(-1)^k}{2^{2k}\Gamma(k)^2} \int d^d x \phi_{(0)} (-\Box)^k \log(-\Box/\bar{\mu}^2) \phi_{(0)}. \tag{3.28}$$

The holographic dictionary identifies $S_{\rm ren}$ with the renormalized generating function of connected correlators, $W_{\rm ren}[J]$, and $\phi_{(0)}$ with the source J. We therefore deduce that the renormalized two-point function of the dual scalar operator takes the form

$$\langle \mathcal{O}_\Delta(p) \mathcal{O}_\Delta(-p) \rangle_{\rm ren} = \frac{(-1)^{k+1}}{2^{2k-1}\Gamma(k)^2} p^{2k} \log(p^2/\bar{\mu}^2), \tag{3.29}$$

which agrees with the CFT calculation in (2.51). Comparing the coefficients, we determine

$$c(g,\Delta) = \frac{2k\Gamma(d/2+k)}{\pi^{d/2}\Gamma(k)}, \tag{3.30}$$

which turns out to be precisely the correct coefficient consistent with the Ward identities.

3.2 The holographic dictionary in Hamiltonian language

The local RG description of QFTs that we discussed above allows us to formulate the holographic dictionary in a more precise language, identifying all quantities in the bulk theory with QFT quantities. In particular, we identify the following objects on the two sides of the gauge/gravity duality:

Radial coordinate	r	$\leftrightarrow$	$\tau = \log\mu$	RG "time"
Induced fields	ϕ	$\leftrightarrow$	J	Running local couplings (sources)
Regularized action	$S_{\rm reg}[\phi]$	$\leftrightarrow$	$W[J]$	Generating function
Renormalized action	$S_{\rm ren}[\phi]$	$\leftrightarrow$	$W_{\rm ren}[J]$	Renormalized generating function
Radial Hamiltonian	H	$\leftrightarrow$	$\mathbb{H}$	RG Hamiltonian
Radial momenta	π_ϕ	$\leftrightarrow$	$\langle\mathcal{O}\rangle$	Running local operators
Non-normalizable modes	$\phi_{(0)}$	$\leftrightarrow$	$J_R\big\vert_\infty$	Renormalized couplings at ∞
Renormalized momenta	$\widehat{\pi}_{(\Delta)}$	$\leftrightarrow$	$\langle\mathcal{O}\rangle\big\vert_\infty$	Bare operators

This table should serve as a guide in order to interpret all calculations in the bulk theory that we are going to describe in the next sections.

4 Radial Hamiltonian formulation of gravity theories

The holographic dictionary consists in a precise map between observables on the two sides of the duality. From the point of view of the bulk gravitational theory, the physical observables correspond to the symplectic space of asymptotic data, which is the key to formulating a well posed variational problem [17]. As we will now review, a general systematic construction of the symplectic space of asymptotic data proceeds by formulating the bulk dynamics in Hamiltonian language, with the radial coordinate identified with the Hamiltonian "time". As we saw in the previous section, this formulation of the bulk dynamics parallels the real space renormalization group of the dual QFT.

For concreteness, let us consider Einstein-Hilbert gravity in a $d+1$-dimensional non-compact manifold $\mathcal{M}$ coupled to a scalar field described by the action

$$S = -\frac{1}{2\kappa^2}\left(\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(R[g] - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right) + \int_{\partial\mathcal{M}} d^d x\sqrt{\gamma}2K\right). \tag{4.1}$$

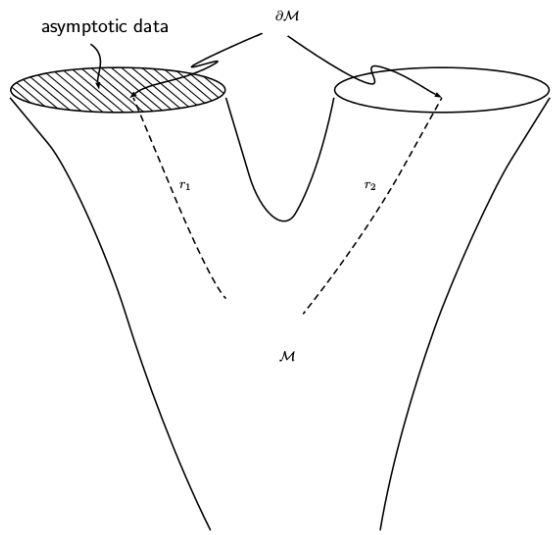

Figure 1: A non-compact manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$ consisting of two disconnected components. The Hamiltonian formulation of the bulk dynamics in the vicinity of the two disconnected components must be done separately, using two different radial coordinates, r_1 and r_2, emanating respectively from each disconnected component of the boundary. The Hamiltonian analysis need only be applicable in an open neighborhood of each boundary component, which is sufficient in order to construct the symplectic space of asymptotic data on each component, as well as the appropriate boundary terms required to render the variational problem well posed.

Here, $\kappa^2 = 8\pi G_{d+1}$ is the gravitational constant in $d+1$ dimensions and the boundary term is the standard Gibbons-Hawking term for Einstein-Hilbert gravity [20], which, as we shall see, is required in order to formulate the dynamics in a Hamiltonian language.[7] Moreover, throughout these lectures we will work in Euclidean signature, but the entire analysis can be straightforwardly adapted to Lorentzian signature.

The radial Hamiltonian formulation of the bulk dynamics starts with picking a radial coordinate r such that $r \to \infty$ corresponds to the location of the boundary $\partial\mathcal{M}$ of $\mathcal{M}$. This radial coordinate need not be a Gaussian normal coordinate, nor should it be a good coordinate throughout $\mathcal{M}$. Instead, r need only cover an open chart $\mathcal{M}_\epsilon$ in the vicinity of $\partial\mathcal{M}$ in $\mathcal{M}$. Moreover, if $\partial\mathcal{M}$ consists of multiple disconnected components then a different radial coordinate must be used in the vicinity of each boundary component and different Hamiltonian descriptions must be applied to describe the various asymptotic regimes, as is illustrated in Figure 1.

Having picked a radial coordinate r emanating from (a component of) the boundary $\mathcal{M}$, the radial Hamiltonian formulation of the dynamics proceeds as in the standard ADM formalism [22], except that the Hamiltonian "time" r is a spacelike coordinate instead of a timelike one. All tensor fields are decomposed in components along and transverse to the radial coordinate r. In particular, the metric is parameterized in terms of the lapse function N, the shift vector N_i, and the induced metric γ_{ij} on the hypersurfaces Σ_r of constant radial coordinate r as

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j, \tag{4.2}$$

[7] We emphasize that, contrary to what is often claimed, the Gibbons-Hawking term *does not* render the variational problem well posed in a non-compact manifold. It does so in a *compact* space, but in a non-compact manifold additional boundary terms are required [21, 17].

where $i, j = 1, \ldots, d$. The metric $g_{\mu\nu}$ is therefore replaced in the Hamiltonian description by the three fields $\{N, N_i, \gamma_{ij}\}$ on Σ_r. Moreover, the curvature tensors of the metric $g_{\mu\nu}$ can be expressed in terms of the (intrinsic) curvature tensors of the hypersurfaces Σ_r and the extrinsic curvature, K_{ij}, describing the embedding of $\Sigma_r \hookrightarrow \mathcal{M}$. The latter is defined as

$$K_{ij} = \frac{1}{2}\left(\mathcal{L}_n g\right)_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right), \tag{4.3}$$

where the dot $\dot{}$ denotes a derivative w.r.t. the radial coordinate r, D_i denotes the covariant derivative w.r.t. the induced metric γ_{ij}, and the unit normal to Σ_r, n^μ, is given by $n^\mu = \left(1/N, -N^i/N\right)$. Using the expressions for the inverse metric and the Christoffel symbols given in appendix A one finds that the Ricci scalar takes the form

$$R[g] = R[\gamma] + K^2 - K_{ij}K^{ij} + \nabla_\mu \zeta^\mu, \tag{4.4}$$

where $R[\gamma]$ is the Ricci scalar of the induced metric γ_{ij}, $K = \gamma^{ij}K_{ij}$ denotes the trace of the extrinsic curvature, and $\zeta^\mu = -2Kn^\mu + 2n^\rho \nabla_\rho n^\mu$. From the identities in appendix A follows that $\zeta^r = -2K/N$ and, hence, the Gibbons-Hawking term in (4.1) precisely cancels the total derivative term in Ricci curvature (4.4). This allows us to write the action as an integral over a radial Lagrangian as

$$S = \int dr L, \tag{4.5}$$

where

$$L = -\frac{1}{2\kappa^2}\int_{\Sigma_r} d^d x \sqrt{\gamma} N \left(R[\gamma] + K^2 - K^i_j K^j_i - \frac{1}{2N^2}\left(\dot{\varphi} - N^i \partial_i \varphi\right)^2 - \frac{1}{2}\gamma^{ij}\partial_i\varphi\partial_j\varphi - V(\varphi)\right). \tag{4.6}$$

Note that, as we anticipated earlier, the Gibbons-Hawking term is required for the radial Hamiltonian formulation of the bulk dynamics. This observation can be utilized in order to derive the correct Gibbons-Hawking term for general bulk Lagrangians, such as, for example, that describing a scalar field *conformally* coupled to Einstein-Hilbert gravity [23].

From the radial Lagrangian (4.6) we read off the canonical momenta conjugate to the induced metric γ_{ij} and the scalar φ

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}), \tag{4.7a}$$

$$\pi_\varphi = \frac{\delta L}{\delta \dot{\varphi}} = \frac{1}{2\kappa^2}\sqrt{\gamma}\, N^{-1}\left(\dot{\varphi} - N^i\partial_i\varphi\right). \tag{4.7b}$$

However, the Lagrangian (4.6) does not depend on the radial derivatives (generalized velocities), $\dot{N}$ and $\dot{N}_i$, of the shift function and lapse vector and so their conjugate momenta vanish identically. This means that the lapse function and the shift vector are not dynamical fields, but rather Lagrange multipliers, whose equations of motion lead to *constraints*. The separation of variables into dynamical fields and Lagrange multipliers through the ADM decomposition (4.2) is one of the main advantages of the Hamiltonian formulation of the bulk dynamics.

The Legendre transform of the Lagrangian (4.6) gives the Hamiltonian

$$H = \int_{\Sigma_r} d^d x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_\varphi \dot{\varphi}\right) - L = \int_{\Sigma_r} d^d x \left(N\mathcal{H} + N_i \mathcal{H}^i\right), \tag{4.8}$$

where

$$\mathcal{H} = 2\kappa^2\gamma^{-\frac{1}{2}}\left(\pi^i_j\pi^j_i - \frac{1}{d-1}\pi^2 + \frac{1}{2}\pi_\varphi^2\right) + \frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \frac{1}{2}\partial_i\varphi\partial^i\varphi - V(\varphi)\right), \tag{4.9a}$$

$$\mathcal{H}^i = -2D_j\pi^{ij} + \pi_\varphi\partial^i\varphi. \tag{4.9b}$$

It follows that Hamilton's equations for the Lagrange multipliers N and N_i impose the constraints

$$\mathcal{H} = \mathcal{H}^i = 0, \tag{4.10}$$

and, hence, the Hamiltonian vanishes identically on the constraint surface. This is a direct consequence of the diffeomorphism invariance of the bulk theory [24]. In particular, the constraints $\mathcal{H} = 0$ and $\mathcal{H}^i = 0$ are first class constraints that, through the Poisson bracket, generate diffeomorphisms along the radial direction and along Σ_r, respectively.

4.1 Hamilton-Jacobi formalism

From the expressions (4.8) and (4.9) we observe that the Hamiltonian does not depend explicitly on the radial coordinate r, but only through the induced fields on Σ_r. This is a consequence of the diffeomorphism invariance of the action (4.1) and it implies that the HJ equation takes the form

$$H = 0, \tag{4.11}$$

which is equivalent to the two constraints (4.10), where the canonical momenta are expressed as gradients of Hamilton's principal function $\mathcal{S}$ (see appendix B)

$$\pi^{ij} = \frac{\delta\mathcal{S}}{\delta\gamma_{ij}}, \quad \pi_\varphi = \frac{\delta\mathcal{S}}{\delta\varphi}. \tag{4.12}$$

This form of the canonical momenta turns the constraints (4.10) into functional partial differential equations for $\mathcal{S}$. The momentum constraint, $\mathcal{H}^i = 0$, implies that $\mathcal{S}[\gamma,\varphi]$ is invariant with respect to diffeomorphims on the radial slice Σ_r. The Hamiltonian constraint, $\mathcal{H} = 0$, takes the form

$$\frac{2\kappa^2}{\sqrt{\gamma}}\left(\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\frac{\delta\mathcal{S}}{\delta\gamma_{ij}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}} + \frac{1}{2}\left(\frac{\delta\mathcal{S}}{\delta\varphi}\right)^2\right) + \frac{\sqrt{\gamma}}{2\kappa^2}\left(R[\gamma] - \frac{1}{2}\partial_i\varphi\partial^i\varphi - V\right) = 0, \tag{4.13}$$

and dictates the radial evolution of the induced fields on Σ_r.

As is reviewed in appendix B, a solution $\mathcal{S}[\gamma,\varphi]$ of the HJ equation leads to a solution of Hamilton's equations, and hence of the second order equations of motion. In particular, given a solution $\mathcal{S}[\gamma,\varphi]$ of the HJ equation, equating the expressions (4.7) and (4.12) for the canonical momenta (this corresponds to the first of Hamilton's equations) leads to the first order flow equations

$$\dot{\gamma}_{ij} = 4\kappa^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{kl}\gamma_{ij}\right)\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\gamma_{kl}}, \tag{4.14a}$$

$$\dot{\varphi} = \frac{2\kappa^2}{\sqrt{\gamma}}\frac{\delta\mathcal{S}}{\delta\varphi}. \tag{4.14b}$$

Integrating these first order equations one obtains the corresponding solution of the second order equations of motion. Crucially, to determine the most general solution of the equations of motion one need *not* find the most general solution of the HJ equation. The HJ equation is a (functional) partial differential equation and so its general solution contains arbitrary integration functions of

the induced fields. However, the general solution of the equations of motion is parameterized by $2n$ integration constants,[8] where n is the number of generalized coordinates, i.e. of induced fields on Σ_r. The general solution of the equations of motion, therefore, can be obtained from a *complete integral* of the HJ equation, which is a principal function $\mathcal{S}[\gamma, \varphi]$ containing n integration *constants* (functions of the transverse coordinates only) [24]. Another n integration constants are obtained by integrating the first order equations (4.14), which leads to a solution of the equations of motion with $2n$ integration constants, i.e. the general solution.

Another important aspect of HJ theory reviewed in appendix B is that the regularized action, defined as the on-shell action evaluated with the radial cut-off Σ_r, i.e.

$$S_{\text{reg}}[\gamma(r,x), \varphi(r,x)] = \int^r dr' \, L|_{\text{on-shell}} \,, \tag{4.15}$$

is naturally a functional of the induced fields γ_{ij} and φ on Σ_r and satisfies the HJ equation (4.13). If the regularized action is evaluated on the general solution of the equations of motion, then S_{reg} contains n integration constants and so it corresponds to a complete integral of the HJ equation. If, however, S_{reg} is evaluated on solutions of the equations of motion that satisfy certain conditions in the deep interior of $\mathcal{M}$, such as regularity conditions, then it will generically contain less than n integration constants and so it will not correspond to a complete integral of the HJ equation.

Recapitulating the last two paragraphs, we have seen that the $2n$ integration constants parameterizing the general solution of the equations of motion are divided into two distinct sets of integration constants in the HJ formalism: n integration constants parameterize a complete integral of the HJ equation, while the remaining n arise as integration constants of the first order equations (4.14). As we shall see later, the integration constants parameterizing a complete integral of the HJ equation correspond generically to the *normalizable* modes of the asymptotic solutions of the equations of motion, while the integration constants coming from the flow equations correspond to the *non-normalizable modes*.[9] Moreover, we have argued that the regularized action (4.15), evaluated on the general solution of the equations of motion gives rise to a complete integral of the HJ equation. Combining these two facts leads to an observation that is fundamental to holographic renormalization and its relation to HJ theory. In order for a theory to be (holographically) renormalizable, the near-boundary divergences of the regularized action (4.15) must be the same for all solutions of the equations of motion and should not depend on the details of the solutions in the deep interior of $\mathcal{M}$. This means that the near-boundary divergences of any complete integral of the HJ equation must be the same, and hence independent of the n integration constants parameterizing a complete integral of the HJ equation. We therefore arrive at the following definition:

Definition 4.1 *(Holographic renormalizability)*
A gravity theory in a non-compact manifold that admits a radial Hamiltonian description is holographically renormalizable if:

(i) The near boundary divergences of any complete integral of the radial HJ equation are the same, so the difference between any two complete integrals is free of divergences.

[8] In order to distinguish them from arbitrary integration functions of the HJ partial differential equation, we refer to arbitrary functions of the transverse coordinates arising from the integration of the radial equations of motion as "integration constants".

[9] Since under certain conditions both modes can be normalizable, more generally the distinction is between asymptotically subleading and dominant modes, respectively.

(ii) The common divergent terms of all complete integrals are local functionals of the induced fields on the radial cut-off Σ_r, i.e. analytic functions of the induced fields and polynomial in transverse derivatives.

The first of these conditions is equivalent with the existence of a well defined symplectic space of asymptotic solutions of the equations of motion and it is required in order to render the variational problem in $\mathcal{M}$ well posed [21, 17]. The second condition, however, is necessary only due to the holographic interpretation of the near boundary divergences of the regularized action as the UV divergences of the generating functional of a *local* quantum field theory. As is discussed in [17], a free scalar field in $\mathbb{R}^{d+1}$ is an example of a system that satisfies condition (i), but not (ii). In cases when condition (i) is not met, there are two possibilities for making progress. One option is to treat the mode(s) that causes condition (i) to be violated perturbatively, and proceed as one would in conformal perturbation theory in the presence of an irrelevant operator. This approach was discussed in general in [25, 26] and explicit examples can be found in [27, 28, 29, 30, 31, 32]. Such an analysis is often sufficient, but it is also possible to treat the modes that violate condition (i) non-perturbatively. This requires constructing a well-defined symplectic space of asymptotic solutions of the equations of motion and generically involves some rearrangement of the bulk degrees of freedom, such as a Kaluza-Klein reduction. This approach, which is discussed in [17], is the holographic dual of following the RG flow in the presence of the irrelevant operator in reverse until a new UV "fixed point" is found. The new "fixed point" in this case is defined in terms of the symplectic space of asymptotic solutions of the bulk equation of motion, and almost in all cases it involves asymptotically non-AdS backgrounds.

Assuming that both conditions of Definition 4.1 hold, as we will assume from now on, the UV divergences of any complete integral of the HJ equation, and hence of the regularized action, can be removed by adding the negative of the divergent part of any solution of the HJ equation as a boundary term in the original action (4.1). Namely, we define the counterterms as

$$S_{\text{ct}} = -\mathcal{S}_{\text{local}}, \tag{4.16}$$

where $\mathcal{S}_{\text{local}}$ is the divergent part of any complete integral of the HJ equation, which, by condition (ii) of the above definition, is a local functional of the induced fields on the radial slice Σ_r. In the next section we will give a precise definition of $\mathcal{S}_{\text{local}}$, and discuss procedures for systematically determining these terms by solving the HJ equation. Before we turn to the systematic construction of $\mathcal{S}_{\text{local}}$, however, we should emphasize one last important point. Although the local and divergent part of the HJ solution is unique, the above discussion suggests that it is possible to add further *finite* and local boundary terms to the bulk action (4.1), corresponding to (a very special choice of) the integration constants of a complete integral of the HJ equation. More generally, therefore, the counterterms will be defined as

$$S_{\text{ct}} = -\left(\mathcal{S}_{\text{local}} + \mathcal{S}_{\text{scheme}}\right), \tag{4.17}$$

where $\mathcal{S}_{\text{scheme}}$ denotes these extra finite terms, which we will discuss in more detail in the next sections. These terms, an example of which is the term proportional to ξ in (3.25), do not cancel divergences, but they correspond to choosing a renormalization scheme [8]. Once the local counterterms, S_{ct}, have been determined, the renormalized action on the radial cut-off is given by

$$S_{\text{ren}} := S_{\text{reg}} + S_{\text{ct}} = \int d^d x \left(\gamma_{ij}\Pi^{ij} + \varphi\Pi_\varphi\right), \tag{4.18}$$

where the renormalized canonical momenta Π^{ij} and Π_φ are arbitrary functions that correspond to the integration constants parameterizing an asymptotic complete integral of the HJ equation. As

we shall see explicitly later, the holographic dictionary relates Π^{ij} and Π_φ with the renormalized one-point functions of the dual operators.

5 Recursive solution of the Hamilton-Jacobi equation

The main task in carrying out the procedure of holographic renormalization is determining the local functional $\mathcal{S}_{\rm local}$, as well as the asymptotic expansions for the induced fields on Σ_r. There is a number of methods to obtain these, differing in generality and efficiency. The approach of [4, 8, 9, 10] does not rely on the HJ equation and its first objective is to obtain the asymptotic expansions for the induced fields by solving asymptotically the second order equations of motion. Evaluating the regularized action on these asymptotic solutions and then inverting the asymptotic expansions in order to express the result in terms of induced fields on the cut-off Σ_r leads to an explicit expression for $\mathcal{S}_{\rm local}$. This method is general but it is unnecessarily complicated. In particular, as we shall see, it is much more efficient to first obtain $\mathcal{S}_{\rm local}$ by solving the HJ equation, and only then derive the asymptotic expansions of the induced fields by integrating the first order equations (4.14), instead of the second order equations. Moreover, deriving the holographic Ward identities is much simpler in the radial Hamiltonian language since they follow directly from the first class constraints (4.10).

The method of [6, 11] does use the HJ equation to obtain $\mathcal{S}_{\rm local}$, but it does so by postulating an ansatz consisting of all possible local and covariant terms that can potentially contribute to the UV divergences with arbitrary coefficients. Inserting this ansatz in the HJ equation leads to equations for the coefficients that can be solved to determine $\mathcal{S}_{\rm local}$. For simple cases this approach is practical since the possible terms in $\mathcal{S}_{\rm local}$ can be easily guessed. However, this method becomes impractical for more complicated systems where $\mathcal{S}_{\rm local}$ contains more than a couple of terms, or when it is not easy to guess all terms (e.g. for asymptotically Lifshitz backgrounds). In particular, if n is the number of independent terms in the ansatz for $\mathcal{S}_{\rm local}$, the number of equations for the arbitrary coefficients in the ansatz one obtains from the HJ equation is generically of order $n(n+1)/2$, which grows much larger than n very fast. The system of equations determining the coefficients in the ansatz is therefore overdetermined, but all equations need to be checked to ensure that the solution is consistent.

A systematic algorithm for solving the HJ equation recursively, without relying on an ansatz, was developed in [13]. This method is based on a formal expansion of the principal function $\mathcal{S}$ in eigenfunctions of the dilatation operator of the dual theory at the UV, and can be applied to any background that possesses some kind of asymptotic scaling symmetry. Besides asymptotically locally AdS backgrounds, this includes backgrounds with non-relativistic Lifshitz symmetry [33, 34]. This method was generalized to relativistic backgrounds that do not necessarily possess an asymptotic scaling symmetry in [35], while a further generalization to include non-relativistic backgrounds was carried out in [36, 31]. This latter generalization involves an expansion of $\mathcal{S}$ in simultaneous eigenfunctions of *two* commuting operators. However, here we will focus on the simpler cases discussed in [13] and [35], which involve an expansion in eigenfunctions of a single operator.

The initial steps in the recursive algorithms of [13] and [35] are common, and they just rely on the fact that we seek a solution $\mathcal{S}$ of the HJ equation in the form of a covariant expansion in eigenfunctions of a –yet unspecified– functional operator $\mathfrak{d}$. Namely, we formally write

$$\mathcal{S} = \mathcal{S}_{(\alpha_0)} + \mathcal{S}_{(\alpha_1)} + \mathcal{S}_{(\alpha_2)} + \cdots , \tag{5.1}$$

where each term is an eigenfunction of $\mathfrak{d}$, i.e.

$$\mathfrak{d}\mathcal{S}_{(\alpha_k)} = \lambda_k \mathcal{S}_{(\alpha_k)}, \tag{5.2}$$

with an eigenvalue λ_k. α_k denotes a convenient label that counts the order of the expansion. In order to obtain a recursive algorithm for determining $\mathcal{S}_{(\alpha_k)}$ it is necessary to also introduce a density $\mathcal{L}$ such that

$$\mathcal{S} = \int_{\Sigma_r} d^d x \mathcal{L}[\gamma, \varphi]. \tag{5.3}$$

We then have

$$\mathcal{L} = \mathcal{L}_{(\alpha_0)} + \mathcal{L}_{(\alpha_1)} + \mathcal{L}_{(\alpha_2)} + \cdots , \tag{5.4}$$

where $\mathcal{S}_{(\alpha_k)} = \int_{\Sigma_r} d^d x \mathcal{L}_{(\alpha_k)}$. Note that the densities $\mathcal{L}_{(\alpha_k)}$ are only defined up to total derivative terms and they are not necessarily eigenfunctions of the operator $\mathfrak{d}$. They are egenfunctions up to total derivatives.

An important identity that is crucial in the construction of the recursion algorithm follows from the expressions (4.12) for the canonical momenta. Namely, for arbitrary variations we have

$$\pi^{ij}\delta\gamma_{ij} + \pi_\varphi \delta\varphi = \delta\mathcal{L} + \partial_i v^i(\delta\gamma, \delta\varphi), \tag{5.5}$$

for some vector field $v^i(\delta\gamma, \delta\varphi)$. Specializing this to the operator $\mathfrak{d}$ gives

$$\pi^{ij}_{(\alpha_k)}\mathfrak{d}\gamma_{ij} + \pi_{\varphi(\alpha_k)}\mathfrak{d}\varphi = \mathfrak{d}\mathcal{L}_{(\alpha_k)} + \partial_i v^i_{(\alpha_k)}(\mathfrak{d}\gamma, \mathfrak{d}\varphi) = \lambda_k \mathcal{L}_{(\alpha_k)} + \partial_i \widetilde{v}^i_{(\alpha_k)}(\mathfrak{d}\gamma, \mathfrak{d}\varphi), \tag{5.6}$$

where

$$\pi^{ij}_{(\alpha_k)} = \frac{\delta \mathcal{S}_{(\alpha_k)}}{\delta\gamma_{ij}}, \quad \pi_{\varphi(\alpha_k)} = \frac{\delta \mathcal{S}_{(\alpha_k)}}{\delta\varphi}, \tag{5.7}$$

and $\widetilde{v}^i_{(\alpha_k)}$ is a vector field, generically different from $v^i_{(\alpha_k)}$ due to the fact that the action of $\mathfrak{d}$ on $\mathcal{L}_{(\alpha_k)}$ may involve a total derivative. Since $\mathcal{L}$ is defined only up to a total derivative, however, without loss of generality we can choose the total derivatives in such a way so that

$$\boxed{\pi^{ij}_{(\alpha_k)}\mathfrak{d}\gamma_{ij} + \pi_{\varphi(\alpha_k)}\mathfrak{d}\varphi = \lambda_k \mathcal{L}_{(\alpha_k)}.} \tag{5.8}$$

This identity will be crucial in the construction of the recursion algorithm.

5.1 The induced metric expansion

To proceed with the recursion algorithm we need to pick a suitable operator $\mathfrak{d}$. The choice of such an operator is not unique, but it has to satisfy certain consistency criteria. Here we will discuss two specific choices. The first one is the operator

$$\boxed{\delta_\gamma = \int 2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}},} \tag{5.9}$$

which was introduced in [35]. The covariant expansion in eigenfunctions of this operator treats the scalar field non-perturbatively. In particular the resulting asymptotic solution of the HJ equation is expressed in terms of a generic scalar potential $V(\varphi)$, without the need to explicitly specify $V(\varphi)$. As a result, this expansion of the solution of principal function $\mathcal{S}$ is valid even for (relativistic) asymptotically non-AdS backgrounds, such as non-conformal branes [37].

It is easy to see that the covariant expansion in eigenfunctions of the operator (5.9) is a derivative expansion.[10] Choosing the label $\alpha_k = 2k$ to count derivatives, the corresponding eigenvalue is

[10] For the gravity-scalar system the expansion in eigenfunctions of (5.9) is indeed a derivative expansion. However, in general this is not the case. A counterexample is a Maxwell field.

$\lambda_k = d - 2k$, where d is the contribution of the volume element. The zero order solution, therefore takes the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} U(\varphi), \tag{5.10}$$

for some "superpotential" $U(\varphi)$. Inserting this ansatz into the Hamiltonian constraint we find that $U(\varphi)$ satisfies the equation

$$\boxed{2(U')^2 - \frac{d}{d-1} U^2 - V(\varphi) = 0.} \tag{5.11}$$

As for the full HJ equation, we only need to obtain an asymptotic solution of this equation, around the value of φ near the boundary. As we have emphasized already, the recursive algorithm for solving the HJ equation we are describing here applies equally to asymptotically AdS and non-AdS backgrounds. The form of the scalar potential, therefore, is largely unrestricted, and we will keep both $V(\varphi)$ and $U(\varphi)$ general in the subsequent discussion. However, before we proceed it is instructive to have a closer look at the explicit form of $V(\varphi)$ and $U(\varphi)$ in the case of asymptotically AdS backgrounds.

In order for the theory (4.1) to admit an AdS solution, corresponding to $\varphi = 0$, the scalar potential must admit a Taylor expansion of the form

$$V(\varphi) = -\frac{d(d-1)}{\ell^2} + \frac{1}{2} m^2 \varphi^2 + \cdots , \tag{5.12}$$

where ℓ is the AdS radius of curvature and the scalar mass must satisfy the Breitenlohner-Freedman (BF) bound [38]

$$m^2 \ell^2 \geq -(d/2)^2, \tag{5.13}$$

in order for the AdS vacuum to be stable with respect to scalar perturbations. Moreover, the mass is related to the dimension Δ of the dual operator through the quadratic equation

$$m^2 \ell^2 = -\Delta(d - \Delta). \tag{5.14}$$

Seeking a solution of (5.11) in the form of a Taylor expansion in φ, one finds two distinct solutions of the form[11]

$$U(\varphi) = -\frac{d-1}{\ell} - \frac{1}{4\ell} \mu \varphi^2 + \cdots , \tag{5.15}$$

where μ takes the two possible values Δ or $d - \Delta$. However, only a solution of the form

$$U(\varphi) = -\frac{d-1}{\ell} - \frac{1}{4\ell} (d - \Delta) \varphi^2 + \cdots , \tag{5.16}$$

can be used as a counterterm since only this solution removes the divergences from all possible solutions involving a non-trivial scalar [39].

Given the superpotential $U(\varphi)$ that determines the zero order solution in the covariant expansion of the HJ equation, we insert the formal expansion in eigenfunctions of the operator (5.9) in the HJ equation and match terms of equal eigenvalue using the identity (5.8), which leads to the linear recursion equations

$$\boxed{2U'(\varphi) \frac{\delta}{\delta \varphi} \int d^d x \mathcal{L}_{(2n)} - \left(\frac{d - 2n}{d - 1} \right) U(\varphi) \mathcal{L}_{(2n)} = \mathcal{R}_{(2n)}, \quad n > 0,} \tag{5.17}$$

[11] The overall sign of U is determined by requiring that the first order equations (4.14) imply the correct leading asymptotic behavior for the scalar, namely $\varphi \sim e^{-(d-\Delta)r}$. Moreover, when the scalar mass saturates the BF bound, one of the two asymptotic solutions for $U(\varphi)$ contains logarithms. We refer to [39] for the explicit form of the function $U(\varphi)$ in that case.

where

$$\mathcal{R}_{(2)} = -\frac{\sqrt{\gamma}}{2\kappa^2}\left(R[\gamma] - \frac{1}{2}\partial_i\varphi\partial^i\varphi\right), \tag{5.18}$$

$$\mathcal{R}_{(2n)} = -\frac{2\kappa^2}{\sqrt{\gamma}}\sum_{m=1}^{n-1}\left(\pi_{(2m)}{}^i_j\pi_{(2(n-m))}{}^j_i - \frac{1}{d-1}\pi_{(2m)}\pi_{(2(n-m))} + \frac{1}{2}\pi_{\varphi(2m)}\pi_{\varphi(2(n-m))}\right), \quad n > 1.$$

Note that if $U'(\varphi) = 0$, i.e. $U(\varphi)$ is a constant, then these recursion equations become algebraic. When $U'(\varphi) \neq 0$, these equations are first order linear inhomogeneous functional differential equations. The general solution, therefore, is the sum of the homogeneous solution and a unique inhomogeneous solution. The homogeneous solution takes the form

$$\mathcal{L}^{\text{hom}}_{(2n)} = \mathcal{F}_{(2n)}[\gamma]\exp\left(\frac{1}{2}\left(\frac{d-2n}{d-1}\right)\int^{\varphi}\frac{d\bar{\varphi}}{U'(\bar{\varphi})}U(\bar{\varphi})\right), \tag{5.19}$$

where $\mathcal{F}_{(2n)}[\gamma]$ is a local covariant functional of the induced metric of weight $d-2n$. It can be easily shown that these homogeneous solutions contribute only to the finite part of the on-shell action, and so we are not interested in them [35]. We are, therefore, only interested in the *inhomogeneous* solution of (5.17), which formally takes the form

$$\boxed{\mathcal{L}_{(2n)} = \frac{1}{2}e^{-(d-2n)A(\varphi)}\int^{\varphi}\frac{d\bar{\varphi}}{U'(\bar{\varphi})}e^{(d-2n)A(\bar{\varphi})}\mathcal{R}_{(2n)}(\bar{\varphi}),} \tag{5.20}$$

where

$$A = -\frac{1}{2(d-1)}\int^{\varphi}\frac{d\bar{\varphi}}{U'(\bar{\varphi})}U(\bar{\varphi}). \tag{5.21}$$

If $\mathcal{R}_{(2n)}$ does not involve derivatives of the scalar field with respect to the transverse coordinates, then evaluating the integral (5.20) is straightforward since it reduces to an ordinary integral. When $\mathcal{R}_{(2n)}$ does contain derivatives of the scalar field, however, some care is required in evaluating this integral. Table 1 in [35] provides general integration identities for up to and including four transverse derivatives, in all possible tensor combinations. This allows one to determine $\mathcal{L}_{(2n)}$ for $n \leq 2$, which suffices for $d \leq 4$.

The recursive procedure to successively determine $\mathcal{L}_{(2n)}$ proceeds as follows. For $n = 1$, $\mathcal{R}_{(2)}$ is given explicitly in (5.18) and so $\mathcal{L}_{(2)}$ can be immediately obtained from (5.20). The result is given in Table 2 of [35]. Having obtained the solution for $\mathcal{L}_{(2)}$, the relations (5.7) give the corresponding canonical momenta, which allow one to evaluate the next $\mathcal{R}_{(2n)}$ using (5.18). Inserting this back in (5.20) and performing the integral gives the next order solution for $\mathcal{L}_{(2n)}$. For $n = 2$ the general result is given in Table 3 of [35].

The order at which the recursive procedure stops depends on the leading asymptotic behavior of the fields. For asymptotically locally AdS backgrounds the recursion stops at order $n = [d/2]$, i.e. the integer part of $d/2$, since higher order terms are UV finite and arbitrary integration constants, parameterizing a complete integral of the HJ equation, enter in the solution. In that case, therefore, the counterterms are defined as

$$S_{\text{ct}} := -\sum_{n=0}^{[d/2]}\mathcal{S}_{(2n)}. \tag{5.22}$$

For even d, the last term in this sum gives rise to explicit cut-off dependence through a logarithmic divergence. The way this arises in this approach is as follows. The recursive procedure described above must be done keeping d as an arbitrary parameter. Denoting by $2k$ the final value of d, the

recursion is carried out up to order $n = k$, where one finds that the solution $\mathcal{L}_{(2k)}$ contains a factor of $1/(d-2k)$, which is singular when we set d to its integer value $2k$. This singularity is then removed by the replacement

$$\frac{1}{d-2k} \to r_o, \tag{5.23}$$

where r_o is the radial cut-off [13, 35]. After this replacement one sets $d = 2k$ in the counterterms, which now contain a term which explicitly depends on r_o. This term is identified with the holographic conformal anomaly [4].

5.2 Dilatation operator expansion

We next turn to the covariant expansion developed in [39], which is an expansion in eigenfunctions of the dilatation operator

$$\boxed{\delta_D = \int d^d x \left(2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + (\Delta - d)\varphi \frac{\delta}{\delta \varphi} \right),} \tag{5.24}$$

where Δ is the conformal dimension of the scalar operator dual to φ. As we pointed out earlier, this expansion is less general than the expansion in eigenfunctions of δ_γ that we just discussed, since it is applicable only to backgrounds with an asymptotic scaling symmetry, but for such backgrounds it is technically simpler than the induced metric expansion. For an application of this expansion to backgrounds with asymptotic Lifshitz symmetry we refer the interested reader to [33, 34].

The dilatation operator (5.24) can be motivated as follows. Since the bulk theory is diffeomorphism invariant, the Hamiltonian does not explicitly depend on the radial coordinate r. It follows that the solution $\mathcal{S}$ of the HJ equation also only depends on the radial coordinate through the induced fields, i.e. $\mathcal{S} = \mathcal{S}[\gamma, \varphi]$. Hence, the radial derivative can be represented by the functional operator

$$\partial_r = \int d^d x \left(\dot{\gamma}_{ij}[\gamma, \varphi] \frac{\delta}{\delta \gamma_{ij}} + \dot{\varphi}[\gamma, \varphi] \frac{\delta}{\delta \varphi} \right). \tag{5.25}$$

Using the leading asymptotic form of the induced fields appropriate for asymptotically locally AdS backgrounds, namely (setting the AdS radius of curvature, ℓ, to 1)

$$\gamma_{ij} \sim e^{2r} g_{(0)ij}(x), \qquad \varphi \sim e^{-(d-\Delta)r} \varphi_{(0)}(x), \tag{5.26}$$

where $g_{(0)ij}(x)$ and $\varphi_{(0)}(x)$ are arbitrary sources, implies that

$$\dot{\gamma}_{ij} \sim 2\gamma_{ij}, \qquad \dot{\varphi} \sim -(d-\Delta)\varphi. \tag{5.27}$$

Inserting these expressions in the covariant representation (5.25) of the radial derivative we obtain

$$\partial_r \sim \int d^d x \left(2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + (\Delta - d)\varphi \frac{\delta}{\delta \varphi} \right) \equiv \delta_D, \tag{5.28}$$

where δ_D is the dilatation operator. This operator is ideally suited for asymptotically locally AdS backgrounds, but in order to construct the corresponding covariant expansion one must fix the dimension Δ from the beginning. Hence, contrary to the expansion in eigenfunctions of δ_γ, one must repeat the whole procedure for every different value of Δ.

As above, we start by writing the principal function as[12]

$$\mathcal{S} = \int_{\Sigma_r} d^d x \sqrt{\gamma} \mathcal{L}, \tag{5.29}$$

and formally expand $\mathcal{L}[\gamma, \varphi]$ in an expansion in eigenfunctions of the dilatation operator as

$$\mathcal{L} = \mathcal{L}_{(0)} + \mathcal{L}_{(2)} + \cdots + \tilde{\mathcal{L}}_{(d)} \log \mathrm{e}^{-2r} + \mathcal{L}_{(d)} + \cdots , \tag{5.30}$$

where

$$\delta_D \mathcal{L}_{(n)} = -n \mathcal{L}_{(n)}, \ \forall n < d, \quad \delta_D \tilde{\mathcal{L}}_{(d)} = -d \tilde{\mathcal{L}}_{(d)}. \tag{5.31}$$

A number of comments are in order here. Firstly, note that here we have defined $\mathcal{L}_{(n)}$ as eigenfunctions of δ_D, while earlier we only required $\mathcal{S}_{(\alpha_k)}$ to be eigenfunctions of the operator $\mathfrak{S}$. This implied that $\mathcal{L}_{(\alpha_k)}$ is an eigenfunction of $\mathfrak{S}$ up to a total derivative term. In order to derive (5.6), however, we argued that, since $\mathcal{L}_{(\alpha_k)}$ is defined only up to a total derivative, one can always choose the total derivatives terms in $\mathcal{L}_{(\alpha_k)}$ such that it is an eigenfunction of $\mathfrak{S}$. In (5.30) we have applied this argument already so that $\mathcal{L}_{(n)}$ are eigenfunctions of δ_D. A second comment concerns the eigenvalue of $\mathcal{L}_{(n)}$ under δ_D, and the corresponding subscript labeling $\mathcal{L}_{(n)}$. In general, these eigenvalues depend on the value of the conformal dimension Δ of the scalar operator and need not be integer. However, the terms of weight 0 and d are universal and are always there. What changes depending on the value of Δ is the intermediate terms. Finally, notice that we have included the logarithmic term already in the expansion (5.30), introducing explicit cut-off dependence. We could have proceeded instead using dimensional regularization as in the expansion in eigenfunctions of δ_γ above, but it is instructive to discuss this alternative argument as well.

In particular, the explicit cut-off dependence introduced in the expansion (5.30) implies that the term $\mathcal{L}_{(d)}$ transforms inhomogeneously under δ_D. In order to derive the action of the dilatation operator on the coefficient $\mathcal{L}_{(d)}$ we recall that the full on-shell action must not depend explicitly on the radial coordinate r, as a consequence of the diffeomorphism invariance of the bulk action. Hence, requiring that ∂_r gives asymptotically the same result as δ_D we must have

$$\partial_r \left(\sqrt{\gamma} (\tilde{\mathcal{L}}_{(d)} \log e^{-2r} + \mathcal{L}_{(d)}) \right) \sim \delta_D \left(\sqrt{\gamma} (\tilde{\mathcal{L}}_{(d)} \log e^{-2r} + \mathcal{L}_{(d)}) \right), \tag{5.32}$$

which determines, using $\delta_D \sqrt{\gamma} = d\sqrt{\gamma}$, that

$$\delta_D \mathcal{L}_{(d)} = -d \mathcal{L}_{(d)} - 2\tilde{\mathcal{L}}_{(d)}. \tag{5.33}$$

This transformation of the finite part of the on-shell action implies that $\mathcal{L}_{(d)}$ cannot be a local function of the fields γ_{ij} and φ, unless $\tilde{\mathcal{L}}_{(d)}$ vanishes identically. This is summarized in the following lemma:

Lemma 5.1 *If $\tilde{\mathcal{L}}_{(d)}$ is not identically zero, then the transformation $\delta_D \mathcal{L}_{(d)} = -d\mathcal{L}_{(d)} - 2\tilde{\mathcal{L}}_{(d)}$ implies that $\mathcal{L}_{(d)}$ cannot be a local functional of the induced fields γ_{ij} and φ.*

Proof:

What we need to show is that $\mathcal{L}_{(d)}$ cannot be a polynomial in derivatives. Suppose $\mathcal{L}_{(d)}$ is a polynomial in derivatives. Since $\mathcal{L}_{(d)}$ is scalar, derivatives must come in pairs and must be

[12] To keep in line with the original notation in [39], we define the density $\mathcal{L}$ without $\sqrt{\gamma}$ here, in contrast to the earlier definition (5.3).

contracted with an inverse metric γ^{ij}. It follows that every polynomial in derivatives can be decomposed as a finite sum of eigenfunctions of the dilatation operator, namely,

$$\mathcal{L}_{(d)} = \mathcal{F}_{(0)} + \mathcal{F}_{(1)} + \cdots + \mathcal{F}_{(N)}, \tag{5.34}$$

for some positive integer N, where $\delta_D \mathcal{F}_{(n)} = -n\mathcal{F}_{(n)}$. Hence,

$$\delta_D \mathcal{L}_{(d)} = -\left(\mathcal{F}_{(1)} + 2\mathcal{F}_{(2)} + \cdots + N\mathcal{F}_{(N)}\right) = -d(\mathcal{F}_{(0)} + \mathcal{F}_{(1)} + \cdots + \mathcal{F}_{(N)}) - 2\tilde{\mathcal{L}}_{(d)}. \tag{5.35}$$

Identifying terms of equal dilatation weight then gives

$$\mathcal{F}_{(n)} = 0,\, n \neq d, \quad 2\tilde{\mathcal{L}}_{(d)} = (n-d)\mathcal{F}_{(n)} = 0,\, n = d. \tag{5.36}$$

This implies that $\tilde{\mathcal{L}}_{(d)} = 0$, contradicting the original hypothesis. □

In fact this is no accident. As we shall see, the term $\mathcal{L}_{(d)}$ corresponds to the renormalized on-shell action, while $\tilde{\mathcal{L}}_{(d)}$ is the conformal anomaly. The fact that $\tilde{\mathcal{L}}_{(d)}$ is the conformal anomaly we will see more explicitly below when we derive the trace Ward identity. However, the fact that $\mathcal{L}_{(d)}$ corresponds to the renormalized on-shell action can be deduced directly from the dilatation weight of the various terms in the covariant expansion. Note that $\mathcal{L}_{(n)}$ with $n < d$, as well as $\tilde{\mathcal{L}}_{(d)}$ all lead to divergences as $r \to \infty$. This is because $\mathcal{L}_{(n)} \sim e^{-nr}$ as $r \to \infty$ and $\sqrt{\gamma} \sim e^{dr}$. We therefore *define* the counterterms as

$$\boxed{S_{\text{ct}} := -\int_{\Sigma_r} d^d x \sqrt{\gamma} \left(\mathcal{L}_{(0)} + \mathcal{L}_{(2)} + \cdots + \tilde{\mathcal{L}}_{(d)} \log \mathrm{e}^{-2r}\right).} \tag{5.37}$$

It follows that the renormalized on-shell action on the radial cut-off is

$$S_{\text{ren}} := S_{\text{reg}} + S_{\text{ct}} = \int_{\Sigma_r} d^d x \sqrt{\gamma} \mathcal{L}_{(d)} + \cdots, \tag{5.38}$$

where the dots stand for terms of higher dilatation weight that vanish as $r \to \infty$. By construction, S_{ren} admits a finite limit as $r \to \infty$, namely

$$\boxed{\widehat{S}_{\text{ren}} := \lim_{r\to\infty} S_{\text{ren}} = \lim_{r\to\infty} \int_{\Sigma_r} d^d x \sqrt{\gamma} \mathcal{L}_{(d)}.} \tag{5.39}$$

As we anticipated, the term $\mathcal{L}_{(d)}$, which is a non-local function of the induced fields, determines the renormalized on-shell action.

Let us now proceed to determine the divergent coefficients $\mathcal{L}_{(n)}$ with $n < d$ and $\tilde{\mathcal{L}}_{(d)}$. Since the canonical momenta are related to the on-shell action via the relations (4.12), it follows that the momenta also admit an expansion of the form

$$\pi^{ij} = \frac{\delta}{\delta\gamma_{ij}} \int_{\Sigma_r} d^d x \sqrt{\gamma} \mathcal{L} = \sqrt{\gamma} \left({\pi_{(0)}}^{ij} + {\pi_{(2)}}^{ij} + \cdots + {\tilde{\pi}_{(d)}}^{ij} \log e^{-2r} + {\pi_{(d)}}^{ij} + \cdots\right), \tag{5.40a}$$

$$\pi_\varphi = \frac{\delta}{\delta\varphi} \int_{\Sigma_r} d^d x \sqrt{\gamma} \mathcal{L} = \sqrt{\gamma}(\pi_{\varphi(d-\Delta)} + \ldots + \tilde{\pi}_{\varphi(\Delta)} \log e^{-2r} + \pi_{\varphi(\Delta)} + \ldots). \tag{5.40b}$$

Note that $\delta_D \pi^i_{j(n)} = -n\pi^i_{j(n)}$ and $\delta_D {\pi^{ij}}_{(n)} = -(n+2){\pi^{ij}}_{(n)}$. With these expansions at hand, we are ready to develop the recursive algorithm. Before we discuss the general algorithm, however, let us point out that the first two of the $\mathcal{L}_{(n)}$ coefficients can be obtained easily, without relying

on the algorithm. From the asymptotic relations (5.27) and the expressions (4.7) for the canonical momenta we deduce that

$$\pi^{ij} \sim -\frac{1}{2\kappa^2}(d-1)\sqrt{\gamma}\gamma^{ij}, \quad \pi_\varphi \sim -\frac{1}{\kappa^2}(d-\Delta)\sqrt{\gamma}\varphi, \tag{5.41}$$

and hence

$$\pi_{(0)}{}^{ij} = -\frac{1}{2\kappa^2}(d-1)\gamma^{ij}, \quad \pi_{\varphi(d-\Delta)} = -\frac{1}{\kappa^2}(d-\Delta)\varphi. \tag{5.42}$$

Integrating $\pi_{(0)}{}^{ij}$ with respect to γ_{ij} determines $\mathcal{L}_{(0)}$, whereas integrating $\pi_{(d-\Delta)}$ with respect to φ (assuming $\Delta < d$) determines $\mathcal{L}_{(2(d-\Delta))}$. Namely,

$$\mathcal{L}_{(0)} = -\frac{1}{\kappa^2}(d-1), \quad \mathcal{L}_{(2(d-\Delta))} = -\frac{1}{2\kappa^2}(d-\Delta)\varphi^2. \tag{5.43}$$

As we shall see below, these results are reproduced by the general algorithm.

The first step in the algorithm it to relate the coefficients $\mathcal{L}_{(n)}$ with $n < d$ and $\tilde{\mathcal{L}}_{(d)}$ to the corresponding canonical momenta using the identity (5.8). Since

$$\delta_D \gamma_{ij} = 2\gamma_{ij}, \quad \delta_D \varphi = -(d-\Delta)\varphi, \tag{5.44}$$

applied to the dilatation operator this identity reads

$$2\pi^i_i - (d-\Delta)\pi_\varphi \varphi = \delta_D\left(\sqrt{\gamma}\mathcal{L}\right), \tag{5.45}$$

or, inserting the expansions (5.30) and (5.40),

$$\begin{aligned} &2\sqrt{\gamma}\left(\pi_{(0)} + \pi_{(2)} + \cdots + \tilde{\pi}_{(d)} \log e^{-2r} + \pi_{(d)} + \cdots\right) \\ &-(d-\Delta)\sqrt{\gamma}\varphi(\pi_{\varphi(d-\Delta)} + \ldots + \tilde{\pi}_{\varphi(\Delta)} \log e^{-2r} + \pi_{\varphi(\Delta)} + \ldots) = \\ &\sqrt{\gamma}\left(d\mathcal{L}_{(0)} + (d-2)\mathcal{L}_{(2)} + \cdots + 0\cdot\tilde{\mathcal{L}}_{(d)} \log \mathrm{e}^{-2r} - 2\tilde{\mathcal{L}}_{(d)} + 0\cdot\mathcal{L}_{(d)} + \ldots\right). \end{aligned} \tag{5.46}$$

In order to equate terms of the same dilatation weight, i.e. to obtain the exact analogue of (5.8), we need to know the precise value of the scalar dimension Δ. However, this identity shows that the coefficients $\mathcal{L}_{(n)}$ of the on-shell action can always be expressed in terms of the coefficients in the expansion of the canonical momenta.

As an example, we can use (5.46) to determine $\mathcal{L}_{(0)}$. Provided $\Delta < d$, identifying terms of dilatation weight zero gives

$$\mathcal{L}_{(0)} = \frac{2}{d}\pi_{(0)} = \frac{2}{d}\left(-\frac{1}{2\kappa^2}d(d-1)\right) = -\frac{1}{\kappa^2}(d-1), \tag{5.47}$$

where we have used the trace of $\pi_{(0)}{}^{ij}$ given in (5.42) in the second equality. This is in agreement with the result (5.43) we found above. Similarly we deduce that

$$\tilde{\mathcal{L}}_{(d)} = -\pi_{(d)} + \frac{1}{2}(d-\Delta)\varphi\pi_{\varphi(\Delta)}. \tag{5.48}$$

As we shall see shortly, this relation is in fact the trace Ward identity. The general algorithm using the dilatation operator expansion can be summarized as follows:

The algorithm:

1. The first step is to use the identity (5.46) to express $\mathcal{L}_{(n)}$, for $n < d$, and $\tilde{\mathcal{L}}_{(d)}$, in terms of the canonical momenta by matching terms of equal dilatation weight. Note that the on-shell action, $\mathcal{L}$, depends only on the trace of π^{ij}.

2. The second step is to insert the expansions (5.40) into the Hamiltonian constraint (4.9a) and match terms of equal dilatation weight. This gives an iterative relation for the trace $\pi_{(n)}$ and $\pi_{\varphi(\Delta-d+n)}$ in terms of the momentum terms of lower dilatation weight.

3. Having determined $\pi_{(n)}$ and $\pi_{\varphi(\Delta-d+n)}$ at order n, we can use the relations we found in the first step to determine $\mathcal{L}_{(n)}$. The *full* momentum ${\pi_{(n)}}^{ij}$ - i.e. not just its trace - is then obtained via the relations (5.7).

4. Steps 2 and 3 are iterated until all local terms are determined.

5.3 An example

It is instructive to work out the counterterms explicitly in a concrete example. To this end, let us apply the dilatation operator expansion to asymptotically AdS gravity in five dimensions ($d = 4$) coupled to a scalar field, φ, dual to an operator of conformal dimension $\Delta = 3$, and with a general scalar potential. The action takes the form[13]

$$S = \int d^5x\sqrt{g}\left(-\frac{1}{2\kappa^2}R[g] + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right), \tag{5.49}$$

where

$$V(\varphi) = \kappa^{-2}V_0 + \kappa^{-1}V_1\varphi + V_2\varphi^2 + \kappa V_3\varphi^3 + \kappa^2 V_4\varphi^4 + \cdots, \tag{5.50}$$

with

$$V_0 = \Lambda = -6, \quad V_1 = 0, \quad V_2 = \frac{1}{2}m^2 = -3/2. \tag{5.51}$$

Let us now implement step by step the algorithm we described above. The first step is to use equation (5.46) to express all local terms of the expansion of the on-shell action, i.e. $\mathcal{L}_{(n)}$, $n < d$, and $\tilde{\mathcal{L}}_{(d)}$, in terms of the canonical momenta by matching terms of equal dilatation weight. For the system at hand, and dropping the total divergence term, (5.46) becomes

$$\begin{aligned} &2\left(\pi_{(0)} + \pi_{(1)} + \pi_{(2)} + \pi_{(3)} + \tilde{\pi}_{(4)}\log e^{-2r} + \pi_{(4)} + \cdots\right) \\ &-\varphi({\pi^\varphi}_{(1)} + {\pi^\varphi}_{(2)} + {\tilde{\pi}^\varphi}_{(3)}\log e^{-2r} + {\pi^\varphi}_{(3)} + \ldots) = \end{aligned} \tag{5.52}$$

$$\left(4\mathcal{L}_{(0)} + 3\mathcal{L}_{(1)} + 2\mathcal{L}_{(2)} + \mathcal{L}_{(3)} + 0\cdot\tilde{\mathcal{L}}_{(4)}\log \mathrm{e}^{-2r} - 2\tilde{\mathcal{L}}_{(4)} + 0\cdot\mathcal{L}_{(4)} + \ldots\right). \tag{5.53}$$

[13] Note that the scalar field here is rescaled by a factor of $\sqrt{2\kappa^2}$ relative to the scalar in (4.1).

Matching terms of equal dilatation weight we obtain

$$\begin{aligned}
\mathcal{L}_{(0)} &= \frac{1}{2}\pi_{(0)} = -3/\kappa^2, \\
\mathcal{L}_{(1)} &= \frac{2}{3}\pi_{(1)}, \\
\mathcal{L}_{(2)} &= \pi_{(2)} - \frac{1}{2}\varphi\pi^{\varphi}{}_{(1)} = \pi_{(2)} + \frac{1}{2}\varphi^2, \\
\mathcal{L}_{(3)} &= 2\pi_{(3)} - \varphi\pi^{\varphi}{}_{(2)}, \\
\tilde{\mathcal{L}}_{(4)} &= -\pi_{(4)} + \frac{1}{2}\varphi\pi^{\varphi}{}_{(3)},
\end{aligned} \tag{5.54}$$

as well as the constraint on the momenta

$$\tilde{\pi}_{(4)} - \frac{1}{2}\varphi\tilde{\pi}^{\varphi}{}_{(3)} = 0. \tag{5.55}$$

Note that $\mathcal{L}_{(4)}$ is not determined, but it does not contribute to the divergences of the on-shell action. As we saw in (5.39), it is the renormalized part of the on-shell action. At this point we have determined all divergent terms of the on-shell action in terms of the canonical momenta.

The second step is to insert the covariant expansions for the momenta into the Hamiltonian constraint (4.9a), which in this case takes the form

$$\mathcal{H} = \sqrt{\gamma}\left\{\frac{1}{2\kappa^2}R[\gamma] + 2\kappa^2\gamma^{-1}\left(\pi^{ij}\pi_{ij} - \frac{1}{3}\pi^2\right) + \frac{1}{2}\gamma^{-1}(\pi^{\varphi})^2 - \frac{1}{2}\gamma^{ij}\partial_i\varphi\partial_j\varphi - V(\varphi)\right\} = 0. \tag{5.56}$$

Inserting the covariant expansions for the momenta and equating terms of equal dilatation weight we obtain

$$\begin{aligned}
&2\kappa^2\left({\pi_{(0)}}^{ij}\pi_{(0)ij} - \frac{1}{3}{\pi_{(0)}}^2\right) - \kappa^{-2}V_0 = 0, \\
&4\kappa^2\left({\pi_{(0)}}^{ij}\pi_{(1)ij} - \frac{1}{3}\pi_{(0)}\pi_{(1)}\right) - \kappa^{-1}V_1\varphi = 0, \\
&\frac{1}{2\kappa^2}R[\gamma] + 2\kappa^2\left(2{\pi_{(0)}}^{ij}\pi_{(2)ij} + {\pi_{(1)}}^{ij}\pi_{(1)ij} - \frac{2}{3}\pi_{(0)}\pi_{(2)} - \frac{1}{3}{\pi_{(1)}}^2\right) + \frac{1}{2}(\pi^{\varphi})^2 - V_2\varphi^2 = 0, \\
&4\kappa^2\left({\pi_{(0)}}^{ij}\pi_{(3)ij} + {\pi_{(1)}}^{ij}\pi_{(2)ij} - \frac{1}{3}\pi_{(0)}\pi_{(3)} - \frac{1}{3}\pi_{(1)}\pi_{(2)}\right) + {\pi^{\varphi}}_{(1)}{\pi^{\varphi}}_{(2)} - \kappa V_3\varphi^3 = 0, \\
&2\kappa^2\left(2{\pi_{(0)}}^{ij}\pi_{(4)ij} + 2{\pi_{(1)}}^{ij}\pi_{(3)ij} + {\pi_{(2)}}^{ij}\pi_{(2)ij} - \frac{2}{3}\pi_{(0)}\pi_{(4)} - \frac{2}{3}\pi_{(1)}\pi_{(3)} - \frac{1}{3}{\pi_{(2)}}^2\right) \\
&+{\pi^{\varphi}}_{(1)}{\pi^{\varphi}}_{(3)} + \frac{1}{2}({\pi^{\varphi}}_{(2)})^2 - \frac{1}{2}\gamma^{ij}\partial_i\varphi\partial_j\varphi - \kappa^2 V_4\varphi^4 = 0, \\
&4\kappa^2\left({\pi_{(0)}}^{ij}\tilde{\pi}_{(4)ij} - \pi_{(0)}\tilde{\pi}_{(4)}\right) + {\pi^{\varphi}}_{(1)}\tilde{\pi}^{\varphi}{}_{(3)} = 0.
\end{aligned} \tag{5.57}$$

The first of these equations is trivially satisfied, while the second equation determines $\pi_{(1)} = 0$ and hence from above $\mathcal{L}_{(1)} = 0$. Next we must use the third step in the algorithm, namely the relations

$${\pi_{(n)}}^{ij} = \frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ij}}\int d^dx\sqrt{\gamma}\mathcal{L}_{(n)}, \quad {\tilde{\pi}_{(d)}}^{ij} = \frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ij}}\int d^dx\sqrt{\gamma}\tilde{\mathcal{L}}_{(d)}. \tag{5.58}$$

This allows us to determine the full momentum ${\pi_{(n)}}^{ij}$ from its trace $\pi_{(n)}$ for $n < d$. In particular, we conclude ${\pi_{(1)}}^{ij} = 0$. The third equation in (5.57) gives

$$\pi_{(2)} - \frac{1}{4\kappa^2}R[\gamma] - \varphi^2, \tag{5.59}$$

and hence,

$$\mathcal{L}_{(2)} = -\frac{1}{4\kappa^2}R[\gamma] - \frac{1}{2}\varphi^2. \tag{5.60}$$

It follows that

$$\pi_{(2)}{}^{ij} = \frac{1}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ij}}\int d^dx\sqrt{\gamma}\mathcal{L}_{(2)} = \frac{1}{4\kappa^2}\left(R^{ij} - \frac{1}{2}R\gamma^{ij}\right) - \frac{1}{4}\varphi^2\gamma^{ij}. \tag{5.61}$$

Continuing this recursive procedure we determine

$$\begin{aligned}
\mathcal{L}_{(3)} &= \kappa V_3\varphi^3, \\
\tilde{\mathcal{L}}_{(4)} &= \frac{1}{16\kappa^2}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right) - \frac{1}{24}R\varphi^2 - \frac{1}{4}\gamma^{ij}\partial_i\varphi\partial_j\varphi - \frac{\kappa^2}{2}\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^4, \\
\pi_{(3)}{}^{ij} &= \frac{\kappa}{2}V_3\varphi^3\gamma^{ij}, \\
\pi^{\varphi}{}_{(2)} &= 3\kappa V_3\varphi^2, \\
\tilde{\pi}^{\varphi}{}_{(3)} &= \frac{1}{12}R\varphi + \frac{1}{2}\Box_\gamma\varphi - 2\kappa^2\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^3, \\
\tilde{\pi}_{(4)}{}^{ij} &= \frac{1}{16\kappa^2}\left[-2R^{kl}R_k{}^i{}_l{}^j + \frac{1}{3}D^iD^jR - \Box_\gamma R^{ij} + \frac{2}{3}RR^{ij} + \frac{1}{2}\gamma^{ij}\left(R^{kl}R_{kl} + \frac{1}{3}\Box_\gamma R - \frac{1}{3}R^2\right)\right] \\
&\quad + \frac{1}{24}\left(R^{ij} - \frac{1}{2}R\gamma^{ij}\right)\varphi^2 - \frac{1}{24}\left(D^iD^j - \gamma^{ij}\Box_\gamma\right)\varphi^2 + \frac{1}{4}\partial^i\varphi\partial^j\varphi - \frac{1}{8}\gamma^{ij}\partial^k\varphi\partial_k\varphi \\
&\quad - \frac{\kappa^2}{4}\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^4\gamma^{ij}.
\end{aligned} \tag{5.62}$$

Note that these satisfy the identity

$$\tilde{\pi}_{(4)} - \frac{1}{2}\varphi\tilde{\pi}^{\varphi}{}_{(3)} = 0, \tag{5.63}$$

as required.

6 Renormalized one-point functions and Ward identities

We found above that the renormalized action (5.39) admits a finite limit, $\widehat{S}_{\text{ren}}$, as $r \to \infty$. The AdS/CFT dictionary identifies this with the generating functional of renormalized connected correlation functions in the dual quantum field theory. In particular, the first derivatives of the renormalized action with respect to the sources correspond to the one-point functions of the dual operators. This implies that we can identify the renormalized one-point functions with certain terms in the covariant expansion of the canonical momenta in eigenfunctions of the dilatation operator. Namely, we define

$$\langle\mathcal{T}^{ij}\rangle_{\text{ren}} = -2|\gamma|^{-1/2}\frac{\delta S_{\text{ren}}}{\delta\gamma_{ij}} = -2\pi_{(d)}{}^{ij}, \tag{6.1a}$$

$$\langle\mathcal{O}\rangle_{\text{ren}} = |\gamma|^{-1/2}\frac{\delta S_{\text{ren}}}{\delta\varphi} = \pi_{\varphi(\Delta)}. \tag{6.1b}$$

Note that these expressions are evaluated on the cut-off, i.e. they are covariant expressions of the induced metric and scalar field. Since these fields asymptotically behave as

$$\gamma_{ij} \sim e^{2r}g_{(0)ij}, \quad \varphi \sim e^{-(d-\Delta)r}\varphi_{(0)}, \tag{6.2}$$

and since $S_{\rm ren}$ has a finite limit as $r \to \infty$, it follows that we must multiply these one-point functions with a suitable factor of the radial coordinate to obtain finite values as $r \to \infty$. In particular, we define

$$\boxed{\begin{aligned} \langle \widehat{\mathcal{T}}^{ij} \rangle_{\rm ren} &:= \lim_{r\to\infty} e^{(d+2)r} \langle \mathcal{T}^{ij} \rangle_{\rm ren} = -2|g_{(0)}|^{-1/2} \frac{\delta \widehat{S}_{\rm ren}}{\delta g_{(0)ij}} = -2\widehat{\pi}_{(d)}{}^{ij}, \\ \langle \widehat{\mathcal{O}} \rangle_{\rm ren} &:= \lim_{r\to\infty} e^{\Delta r} \langle \mathcal{O} \rangle_{\rm ren} = |g_{(0)}|^{-1/2} \frac{\delta \widehat{S}_{\rm ren}}{\delta \varphi_{(0)}} = \widehat{\pi}_{\varphi(\Delta)}. \end{aligned}} \tag{6.3}$$

Using these expressions for the renormalized one-point functions we can now derive the holographic Ward identities. Inserting the expansions (5.40) into the momentum constraint (4.9) and matching terms of equal dilatation weight gives for the terms with weight d

$$-2D_j \pi_{(d)}{}^{ij} + \pi_{\varphi(\Delta)} \partial^i \varphi = 0. \tag{6.4}$$

Rescaling this with the appropriate radial factor and taking the limit $r \to \infty$ leads to the diffeomorphism Ward identity

$$\boxed{D_{(0)j} \langle \widehat{\mathcal{T}}^{ij} \rangle_{\rm ren} + \langle \widehat{\mathcal{O}} \rangle_{\rm ren} \partial^i \varphi_{(0)} = 0.} \tag{6.5}$$

Finally, in order to derive the trace Ward identity note that under an infinitesimal Weyl transformation the renormalized action transforms as

$$\delta_\sigma S_{\rm ren} = \int_{\Sigma_r} \sqrt{\gamma} (-2\tilde{\mathcal{L}}_{(d)}) \delta\sigma + \text{total derivative}. \tag{6.6}$$

This follows from the fact that such a transformation corresponds to the infinitesimal bulk diffeomorphism $r \to r + \delta\sigma(x)$. It follows that the conformal anomaly $\mathcal{A}$ is given by

$$\mathcal{A} := 2\tilde{\mathcal{L}}_{(d)}. \tag{6.7}$$

To see that this is compatible with the trace Ward identity, recall that we have shown in (5.48) that

$$-2\pi_{(d)} + (d-\Delta)\varphi \pi_{\varphi(\Delta)} = 2\tilde{\mathcal{L}}_{(d)}, \tag{6.8}$$

which, using the identifications (6.3), becomes

$$\boxed{\langle \widehat{\mathcal{T}}_i^i \rangle_{\rm ren} + (d-\Delta)\varphi_{(0)} \langle \widehat{\mathcal{O}} \rangle_{\rm ren} = \mathcal{A}.} \tag{6.9}$$

It should be emphasized that these Ward identities hold in the presence of arbitrary sources. This has important implications. Namely, even if the conformal anomaly vanishes numerically on a particular background where the sources are set to zero, the anomaly does contribute to some n-point function because the nth derivative of the anomaly with respect to the sources will not be zero even when evaluated at zero sources. The anomaly therefore is a genuine property of the quantum field theory and affects the dynamics even in flat space.

7 Fefferman-Graham asymptotic expansions

Having obtained the asymptotic solution of the HJ equation in the form of a covariant expansion in eigenfunctions of some suitable operator $\mathfrak{S}$, we can now use the first order flow equations (4.14) to construct the asymptotic Fefferman-Graham expansions for the induced fields γ_{ij} and φ. In order to integrate these expansions, however, we must pick a specific example and a specific solution of

the HJ equation. We will therefore demonstrate how this works in the example we worked out above.

Inserting the expansions (5.40) in the flow equations (4.14) we get[14]

$$\begin{aligned}
\dot{\gamma}_{ij} &= 4\kappa^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{3}\gamma_{kl}\gamma_{ij}\right)\left({\pi_{(0)}}^{ij} + {\pi_{(2)}}^{ij} + \cdots + {\tilde{\pi}_{(4)}}^{ij}\log e^{-2r} + {\pi_{(4)}}^{ij} + \cdots\right), \\
\dot{\varphi} &= \pi_{\varphi(1)} + \ldots + \tilde{\pi}_{\varphi(3)}\log e^{-2r} + \pi_{\varphi(3)} + \ldots .
\end{aligned} \tag{7.1}$$

From the expressions (5.62) above we obtain

$$\begin{aligned}
&\pi_{(0)ij} - \frac{1}{3}\pi_{(0)}\gamma_{ij} = \frac{1}{2\kappa^2}\gamma_{ij}, \\
&\pi_{(2)ij} - \frac{1}{3}\pi_{(2)}\gamma_{ij} = \frac{1}{4\kappa^2}\left(R_{ij} - \frac{1}{6}R\gamma_{ij}\right) + \frac{1}{12}\varphi^2\gamma_{ij}, \\
&\pi_{(3)ij} - \frac{1}{3}\pi_{(3)}\gamma_{ij} = -\frac{\kappa}{6}V_3\varphi^3\gamma_{ij}, \\
&\tilde{\pi}_{(4)ij} - \frac{1}{3}\tilde{\pi}_{(4)}\gamma_{ij} = \frac{1}{16\kappa^2}\left[-2R^{kl}R_{kilj} + \frac{1}{3}D_iD_jR - \Box_\gamma R_{ij} + \frac{2}{3}RR_{ij} + \frac{1}{2}\gamma_{ij}\left(R^{kl}R_{kl} + \frac{1}{3}\Box_\gamma R - \frac{1}{3}R^2\right)\right] \\
&+\frac{1}{24}\left(R_{ij} - \frac{5}{6}R\gamma_{ij}\right)\varphi^2 - \frac{1}{24}\left(D_iD_j - \gamma_{ij}\Box_\gamma\right)\varphi^2 + \frac{1}{4}\partial_i\varphi\partial_j\varphi - \frac{1}{8}\gamma_{ij}\partial^k\varphi\partial_k\varphi \\
&+\frac{\kappa^2}{12}\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^4\gamma_{ij} - \frac{1}{12}\varphi\Box_\gamma\varphi\gamma_{ij},
\end{aligned}$$

$$\begin{aligned}
&\pi_{\varphi(1)} = -\varphi, \\
&\pi_{\varphi(2)} = 3\kappa V_3\varphi^2, \\
&\tilde{\pi}_{\varphi(3)} = \frac{1}{12}R\varphi + \frac{1}{2}\Box_\gamma\varphi - 2\kappa^2\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^3.
\end{aligned} \tag{7.2}$$

Using these expressions we can integrate the flow equations (7.1) straightforwardly. There are two ways to solve these equations order by order asymptotically as $r \to \infty$. One way is to make an explicit Fefferman-Graham ansatz for the asymptotic expansions for γ_{ij} and φ and insert them in the flow equations. This will result in algebraic equations for the coefficients. A more general way that does not require prior knowledge of the form of the asymptotic expansion is expanding the induced fields formally as

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + \gamma_{ij}^{(3)} + \cdots, \quad \varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \cdots, \tag{7.3}$$

where each order is assumed to be asymptotically subleading relative to the previous one, but without assuming a specific functional form. Inserting these expansions in the flow equations results in a sequence of differential equations that can be solved order by order. To leading order we get the homogeneous equations

$$\dot{\gamma}_{ij}^{(0)} = 2\gamma_{ij}^{(0)}, \quad \dot{\varphi}^{(0)} = -\varphi^{(0)}, \tag{7.4}$$

and hence

$$\gamma_{ij}^{(0)} = e^{2r}g_{(0)ij}, \quad \varphi^{(0)} = e^{-r}\varphi_{(0)}, \tag{7.5}$$

[14] Note one needs to adjust these for the different normalization of the scalar.

where $g_{(0)ij}(x)$ and $\varphi_{(0)}(x)$ are arbitrary integration sources. At the next order for γ_{ij} we still get the same homogeneous equation

$$\dot{\gamma}^{(1)}_{ij} = 2\gamma^{(1)}_{ij}. \tag{7.6}$$

However, we have already introduced an arbitrary source at order 0 and, since $\gamma^{(1)}_{ij}$ is asymptotically subleading relative to $\gamma^{(0)}_{ij}$ by the hypothesis, we must set $\gamma^{(1)}_{ij} = 0$. At the next order we obtain the inhomogeneous equations

$$\begin{aligned}
\dot{\gamma}^{(2)}_{ij} &= 2\gamma^{(2)}_{ij} + R[g_{(0)}]_{ij} - \frac{1}{6}R[g_{(0)}]g_{(0)ij} + \frac{\kappa^2}{3}\varphi^2_{(0)}g_{(0)ij}, \\
\dot{\varphi}^{(1)} &= -\varphi^{(1)} + 3\kappa V_3 \varphi^2_{(0)} e^{-2r}.
\end{aligned} \tag{7.7}$$

Discarding the homogeneous solutions again, the inhomogeneous solutions are

$$\begin{aligned}
\gamma^{(2)}_{ij} &= -\frac{1}{2}\left(R_{ij}[g_{(0)}] - \frac{1}{6}R[g_{(0)}]g_{(0)ij} + \frac{\kappa^2}{3}\varphi^2_{(0)}g_{(0)ij}\right), \\
\varphi^{(1)} &= -3\kappa V_3 e^{-2r}\varphi^2_{(0)}.
\end{aligned} \tag{7.8}$$

At the next order for the metric we get

$$\gamma^{(3)}_{ij} = \frac{8}{9}\kappa^3 V_3 e^{-r}\varphi^3_{(0)}g_{(0)ij}, \tag{7.9}$$

while, using the following expansions of the momenta

$$\begin{aligned}
\pi_{(2)ij} - \frac{1}{3}\pi_{(2)}\gamma_{ij} &= \frac{1}{4\kappa^2}\left(R[g_{(0)}]_{ij} - \frac{1}{6}R[g_{(0)}]g_{(0)ij}\right) + \frac{1}{12}\varphi^2_{(0)}g_{(0)ij} \\
&\quad + \frac{1}{6}\varphi^{(0)}\varphi^{(1)}\gamma^{(0)}_{ij} \\
&\quad + e^{-2r}\left[\frac{1}{4\kappa^2}\left(R^k_{(i}[g_{(0)}]\gamma^{(2)}_{kj)} - R_i{}^k{}_j{}^l[g_{(0)}]\gamma^{(2)}_{kl} + D_{(0)(i}D^k_{(0)}\gamma^{(2)}_{kj)}\right.\right. \\
&\quad - \frac{1}{2}\left(\Box_{(0)}\gamma^{(2)}_{ij} + g_{(0)}{}^{kl}D_{(0)i}D_{(0)j}\gamma^{(2)}_{kl}\right) - \frac{1}{6}R[g_{(0)}]\gamma^{(2)}_{ij} \\
&\quad \left.\left. - \frac{1}{6}g_{(0)ij}\left(-R^{kl}[g_{(0)}]\gamma^{(2)}_{kl} + D^k_{(0)}D^l_{(0)}\gamma^{(2)}_{kl} - g^{kl}_{(0)}\Box_{(0)}\gamma^{(2)}_{kl}\right)\right)\right] \\
&\quad + \frac{1}{12}\left(\left[(\varphi^{(1)})^2 + 2\varphi^{(0)}\varphi^{(2)}\right]\gamma^{(0)}_{ij} + (\varphi^{(0)})^2\gamma^{(2)}_{ij}\right) + \mathcal{O}(e^{-3r}), \\
\pi_{(3)ij} - \frac{1}{3}\pi_{(3)}\gamma_{ij} &= -\frac{\kappa}{6}V_3(\varphi^{(0)})^3\gamma^{(0)}_{ij} - \frac{\kappa}{2}V_3(\varphi^{(0)})^2\varphi^{(1)}\gamma^{(0)}_{ij} + \mathcal{O}(e^{-3r}), \\
\pi_{\varphi(2)} &= 3\kappa V_3(\varphi^{(0)})^2 + 6\kappa V_3\varphi^{(0)}\varphi^{(1)} + \mathcal{O}(e^{-4r}),
\end{aligned} \tag{7.10}$$

we obtain the next order equations

$$
\begin{aligned}
&\dot{\gamma}^{(4)}_{ij} = 2\gamma^{(4)}_{ij} \\
&+(-2r)e^{-2r}\left\{\frac{1}{16\kappa^2}\left[-2R^{kl}[g_{(0)}]R_{kilj}[g_{(0)}] + \frac{1}{3}D_{(0)i}D_{(0)j}R[g_{(0)}] - \Box_{(0)}R_{ij}[g_{(0)}] + \frac{2}{3}R[g_{(0)}]R_{ij}[g_{(0)}]\right.\right. \\
&\left. +\frac{1}{2}g_{(0)ij}\left(R^{kl}[g_{(0)}]R_{kl}[g_{(0)}] + \frac{1}{3}\Box_{(0)}R[g_{(0)}] - \frac{1}{3}R^2[g_{(0)}]\right)\right] + \frac{1}{24}\left(R_{ij}[g_{(0)}] - \frac{5}{6}R[g_{(0)}]g_{(0)ij}\right)\varphi^2_{(0)} \\
&-\frac{1}{24}\left(D_{(0)i}D_{(0)j} - g_{(0)ij}\Box_{(0)}\right)\varphi^2_{(0)} + \frac{1}{4}\partial_i\varphi_{(0)}\partial_j\varphi_{(0)} - \frac{1}{8}g_{(0)ij}\partial^k\varphi_{(0)}\partial_k\varphi_{(0)} \\
&\left. +\frac{\kappa^2}{12}\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^4_{(0)}g_{(0)ij} + \frac{1}{6}\varphi_{(0)}\widetilde{\varphi}_{(2)}g_{(0)ij} - \frac{1}{12}\varphi_{(0)}\Box_{(0)}\varphi_{(0)}g_{(0)ij}\right\} \\
&+e^{-2r}\left\{\widehat{\pi}_{(4)ij} - \frac{1}{3}g_{(0)ij}\widehat{\pi}_{(4)} + \frac{1}{4\kappa^2}\left(R^k_{(i}[g_{(0)}]\gamma^{(2)}_{kj)} - R_i{}^k{}_j{}^l[g_{(0)}]\gamma^{(2)}_{kl} + D_{(0)(i}D^k_{(0)}\gamma^{(2)}_{kj)}\right.\right. \\
&-\frac{1}{2}\left(\Box_{(0)}\gamma^{(2)}_{ij} + g_{(0)}{}^{kl}D_{(0)i}D_{(0)j}\gamma^{(2)}_{kl}\right) - \frac{1}{6}R[g_{(0)}]\gamma^{(2)}_{ij} \\
&\left. -\frac{1}{6}g_{(0)ij}\left(-R^{kl}[g_{(0)}]\gamma^{(2)}_{kl} + D^k_{(0)}D^l_{(0)}\gamma^{(2)}_{kl} - g^{kl}_{(0)}\Box_{(0)}\gamma^{(2)}_{kl}\right)\right) \\
&\left. +\frac{1}{12}\left(\left[9\kappa^2V_3^2\varphi^4_{(0)} + 2\varphi_{(0)}\widehat{\varphi}_{(2)}\right]g_{(0)ij} + \varphi^2_{(0)}\gamma^{(2)}_{ij}\right) + \frac{3\kappa^2}{2}V_3^2\varphi^4_{(0)}g_{(0)ij}\right\}, \qquad (7.11)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\varphi}^{(2)} &= -\varphi^{(2)} \\
&\quad +e^{-3r}(-2r)\left[\frac{1}{12}R[g_{(0)}]\varphi_{(0)} + \frac{1}{2}\Box_{(0)}\varphi_{(0)} - 2\kappa^2\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^3_{(0)}\right] \\
&\quad +e^{-3r}\left(\widehat{\pi}_{\varphi(3)} - 18\kappa^2V_3^2\varphi^3_{(0)}\right). \qquad (7.12)
\end{aligned}
$$

The inhomogeneous solutions of these equations take the form

$$
\gamma^{(4)}_{ij} = e^{-2r}\left(-2rh_{(4)ij} + g_{(4)ij}\right), \quad \varphi^{(2)} = e^{-3r}\left(-2r\widetilde{\varphi}_{(2)} + \widehat{\varphi}_{(2)}\right), \qquad (7.13)
$$

where

$$
\begin{aligned}
h_{(4)ij} &= -\kappa^2\left\{\frac{1}{16\kappa^2}\left[-2R^{kl}[g_{(0)}]R_{kilj}[g_{(0)}] + \frac{1}{3}D_{(0)i}D_{(0)j}R[g_{(0)}] - \Box_{(0)}R_{ij}[g_{(0)}] + \frac{2}{3}R[g_{(0)}]R_{ij}[g_{(0)}]\right.\right. \\
&\quad \left. +\frac{1}{2}g_{(0)ij}\left(R^{kl}[g_{(0)}]R_{kl}[g_{(0)}] + \frac{1}{3}\Box_{(0)}R[g_{(0)}] - \frac{1}{3}R^2[g_{(0)}]\right)\right] + \frac{1}{24}\left(R_{ij}[g_{(0)}] - \frac{5}{6}R[g_{(0)}]g_{(0)ij}\right)\varphi^2_{(0)} \\
&\quad -\frac{1}{24}\left(D_{(0)i}D_{(0)j} - g_{(0)ij}\Box_{(0)}\right)\varphi^2_{(0)} + \frac{1}{4}\partial_i\varphi_{(0)}\partial_j\varphi_{(0)} - \frac{1}{8}g_{(0)ij}\partial^k\varphi_{(0)}\partial_k\varphi_{(0)} \\
&\quad \left. +\frac{\kappa^2}{12}\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi^4_{(0)}g_{(0)ij} + \frac{1}{6}\varphi_{(0)}\widetilde{\varphi}_{(2)}g_{(0)ij} - \frac{1}{12}\varphi_{(0)}\Box_{(0)}\varphi_{(0)}g_{(0)ij}\right\}, \qquad (7.14)
\end{aligned}
$$

$$
\begin{aligned}
g_{(4)ij} &= -\kappa^2\left\{\widehat{\pi}_{(4)ij} - \frac{1}{3}g_{(0)ij}\widehat{\pi}_{(4)} + \frac{1}{4\kappa^2}\left(R^k_{(i}[g_{(0)}]\gamma^{(2)}_{kj)} - R_i{}^k{}_j{}^l[g_{(0)}]\gamma^{(2)}_{kl} + D_{(0)(i}D^k_{(0)}\gamma^{(2)}_{kj)}\right.\right. \\
&\quad -\frac{1}{2}\left(\Box_{(0)}\gamma^{(2)}_{ij} + g_{(0)}{}^{kl}D_{(0)i}D_{(0)j}\gamma^{(2)}_{kl}\right) - \frac{1}{6}R[g_{(0)}]\gamma^{(2)}_{ij} \\
&\quad \left. -\frac{1}{6}g_{(0)ij}\left(-R^{kl}[g_{(0)}]\gamma^{(2)}_{kl} + D^k_{(0)}D^l_{(0)}\gamma^{(2)}_{kl} - g^{kl}_{(0)}\Box_{(0)}\gamma^{(2)}_{kl}\right)\right) \\
&\quad \left. +\frac{1}{12}\left(\left[9\kappa^2V_3^2\varphi^4_{(0)} + 2\varphi_{(0)}\widehat{\varphi}_{(2)}\right]g_{(0)ij} + \varphi^2_{(0)}\gamma^{(2)}_{ij}\right) + \frac{3\kappa^2}{2}V_3^2\varphi^4_{(0)}g_{(0)ij}\right\} - \frac{1}{2}h_{(4)ij} \qquad (7.15)
\end{aligned}
$$

$$\widetilde{\varphi}_{(2)} = -\frac{1}{3}\left[\frac{1}{12}R[g_{(0)}]\varphi_{(0)} + \frac{1}{2}\Box_{(0)}\varphi_{(0)} - 2\kappa^2\left(V_4 - \frac{9}{2}V_3^2 + \frac{1}{6}\right)\varphi_{(0)}^3\right], \tag{7.16}$$

and

$$\widehat{\varphi}_{(2)} = -\frac{1}{3}\left(\widehat{\pi}_{\varphi(3)} - 18\kappa^2 V_3^2\varphi_{(0)}^3 - \frac{2}{3}\widetilde{\varphi}_{(2)}\right). \tag{7.17}$$

This completes the computation since the coefficients $\widehat{\pi}_{(4)ij}$ and $\widehat{\pi}_{(3)}$ have been identified above with the renormalized one-point functions. In particular, taking the trace of the expression for $g_{(4)ij}$ relates the trace of $\widehat{\pi}_{(4)ij}$ with the trace of $g_{(4)ij}$. Inserting this back in the expression for $g_{(4)ij}$ one obtains the renormalized stress tensor $\widehat{\pi}_{(4)ij}$ in terms of $g_{(4)ij}$, its trace, and lower order terms that are explicitly expressed in terms of the sources.

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Appendix

A ADM identities

A few identities relating to the ADM decomposition (4.2) of the metric are collected in this appendix. In particular, in matrix form, the metric (4.2) and its inverse are

$$g = \begin{pmatrix} N^2 + N_k N^k & N_i \\ N_i & \gamma_{ij} \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1/N^2 & -N^i/N^2 \\ -N^i/N^2 & \gamma^{ij} + N^i N^j/N^2 \end{pmatrix}, \tag{A.1}$$

where the indices $i = 1, \ldots, d$ are raised and lowered respectively with γ^{ij} and γ_{ij}. Moreover, the Christoffel symbols $\Gamma^\rho_{\mu\nu}[g]$ can be decomposed into the following components in terms of N, N_i and γ_{ij}:

$$\begin{aligned}
\Gamma^r_{rr} &= N^{-1}\left(\dot{N} + N^i\partial_i N - N^i N^j K_{ij}\right), \\
\Gamma^r_{ri} &= N^{-1}\left(\partial_i N - N^j K_{ij}\right), \\
\Gamma^r_{ij} &= -N^{-1}K_{ij}, \\
\Gamma^i_{rr} &= -N^{-1}N^i\dot{N} - ND^iN - N^{-1}N^iN^j\partial_j N + \dot{N}^i + N^jD_jN^i + 2NN^jK^i_j + N^{-1}N^iN^kN^lK_{kl}, \\
\Gamma^i_{rj} &= -N^{-1}N^i\partial_j N + D_jN^i + N^{-1}N^iN^kK_{kj} + NK^i_j, \\
\Gamma^k_{ij} &= \Gamma^k_{ij}[\gamma] + N^{-1}N^kK_{ij}.
\end{aligned} \tag{A.2}$$

B Hamilton-Jacobi primer

In this appendix we collect a few essential facts about HJ theory in classical mechanics. For an in-depth account of HJ theory we refer the interested reader to [24, 40]. A more abstract exposition can be found in [41].

Let $\mathcal{Q}$ be the configuration space of a point particle described by the action[15]

$$S = \int^t dt' L(q, \dot{q}; t), \tag{B.1}$$

where q^α are coordinates on $\mathcal{Q}$. In the Hamiltonian formalism the generalized coordinates q^α and the canonical momenta

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}, \tag{B.2}$$

are independent variables parameterizing the phase space, which is isomorphic to the cotangent bundle $T^*\mathcal{Q}$ of the configuration space $\mathcal{Q}$. The cotangent bundle is a symplectic manifold with a canonical closed 2-form (symplectic form)

$$\Omega = dp_\alpha \wedge dq^\alpha. \tag{B.3}$$

Since Ω is closed, it can be *locally* expressed as

$$\Omega = d\Theta, \tag{B.4}$$

where

$$\Theta = p_\alpha dq^\alpha, \tag{B.5}$$

is known as the canonical 1-form, or pre-symplectic form. The Hamiltonian, given by the Legendre transform of the Lagrangian,

$$H(p, q; t) = p_\alpha \dot{q}^\alpha - L, \tag{B.6}$$

is a map $H : T^*\mathcal{Q} \longrightarrow \mathbb{R}$ and governs the time evolution of the dynamical system through Hamilton's equations

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}. \tag{B.7}$$

At this point it is instructive to distinguish two cases, depending on whether the Hamiltonian depends explicitly on time t or not.

Time-independent systems

A section, s, of the cotangent bundle is a map $s : \mathcal{Q} \longrightarrow T^*\mathcal{Q}$, providing a 1-form over each point $q \in \mathcal{Q}$. A closed section of $T^*\mathcal{Q}$ is locally exact and so it can be written as $s = d\mathcal{W}$ for some function $\mathcal{W}(q)$ on $\mathcal{Q}$. Under the isomorphism between phase space and the cotangent bundle this means that locally

$$p_\alpha = \frac{\partial \mathcal{W}(q)}{\partial q^\alpha}. \tag{B.8}$$

Moreover,

$$\Theta \circ s = d\mathcal{W}, \quad \Omega \circ s = 0. \tag{B.9}$$

These results hold for any closed section s of $T^*\mathcal{Q}$. The HJ theorem relates certain closed sections, s, of the cotangent bundle to solutions of Hamilton's equations (B.7). In particular,

$$\begin{aligned} d\left(H \circ s\right) &= \left(\frac{\partial H}{\partial q^\alpha} + \frac{\partial^2 \mathcal{W}}{\partial q^\beta \partial q^\alpha}\frac{\partial H}{\partial p_\beta}\right) dq^\alpha \\ &= \left(\frac{\partial H}{\partial q^\alpha} + \dot{p}^\alpha\right) dq^\alpha + \frac{\partial^2 \mathcal{W}}{\partial q^\beta \partial q^\alpha}\left(\frac{\partial H}{\partial p_\beta} - \dot{q}^\beta\right) dq^\alpha, \end{aligned} \tag{B.10}$$

which implies that the following two statements are equivalent (see Theorem 2.1 in [42]):

[15] In this appendix a dot $\dot{}$ denotes a derivative with respect to time t.

(i) If $\sigma : \mathbb{R} \to \mathcal{Q}$ satisfies the first of Hamilton's equations in (B.7), then $s \circ \sigma$ satisfies the second Hamilton equation.

(ii) $d\,(H \circ s) = 0$.

Hence, a closed section $s = d\mathcal{W}$ of the cotangent bundle that satisfies the (time-independent) HJ equation

$$H \circ s = H\left(\frac{\partial \mathcal{W}}{\partial q^\alpha}, q^\beta\right) = E, \tag{B.11}$$

where E is some constant, provides a solution of Hamilton's equations.

Time-dependent systems

In order to accommodate systems with a Hamiltonian that explicitly depends on time we extend the configuration space by including time as a generalized coordinate so that $\mathcal{Q}_{\text{ext}} = \mathcal{Q} \times \mathbb{R}$ is now the extended configuration space. Phase space is accordingly extended by including $-H$ as the canonical momentum conjugate to t. This extended phase space is isomorphic to the cotangent bundle $T^*\mathcal{Q}_{\text{ext}}$, which carries the canonical symplectic form

$$\Omega_{\text{ext}} = d\Theta_{\text{ext}} = dp_\alpha \wedge dq^\alpha - dH \wedge dt. \tag{B.12}$$

Moreover, to Hamilton's equations we can now append the equation

$$\dot{H} = \frac{\partial H}{\partial t}. \tag{B.13}$$

A closed section of $T^*\mathcal{Q}_{\text{ext}}$ can be locally written as $s = d\mathcal{S}$ for some function on $\mathcal{Q}_{\text{ext}}$, and consequently

$$p_\alpha = \frac{\partial \mathcal{S}(q;t)}{\partial q^\alpha}, \quad -H = \frac{\partial \mathcal{S}(q;t)}{\partial t}, \tag{B.14}$$

which imply that

$$\Theta_{\text{ext}} \circ s = d\mathcal{S}, \quad \Omega_{\text{ext}} \circ s = 0. \tag{B.15}$$

It follows that

$$\begin{aligned} 0 = d\left(H \circ s + \frac{\partial \mathcal{S}}{\partial t}\right) &= \left[\frac{\partial H}{\partial q^\alpha} + \dot{q}^\beta \frac{\partial^2 \mathcal{S}}{\partial q^\beta \partial q^\alpha} + \frac{\partial^2 \mathcal{S}}{\partial t \partial q^\alpha} + \frac{\partial^2 \mathcal{S}}{\partial q^\beta \partial q^\alpha}\left(\frac{\partial H}{\partial p_\beta} - \dot{q}^\beta\right)\right] dq^\alpha \\ &\quad + \left[\frac{\partial H}{\partial t} + \dot{q}^\alpha \frac{\partial^2 \mathcal{S}}{\partial t \partial q^\alpha} + \frac{\partial^2 \mathcal{S}}{\partial t^2} + \frac{\partial^2 \mathcal{S}}{\partial q^\beta \partial t}\left(\frac{\partial H}{\partial p_\beta} - \dot{q}^\beta\right)\right] dt \\ &= \left[\frac{\partial H}{\partial q^\alpha} + \dot{p}^\alpha + \frac{\partial^2 \mathcal{S}}{\partial q^\beta \partial q^\alpha}\left(\frac{\partial H}{\partial p_\beta} - \dot{q}^\beta\right)\right] dq^\alpha \\ &\quad + \left[\frac{\partial H}{\partial t} - \dot{H} + \frac{\partial^2 \mathcal{S}}{\partial q^\beta \partial t}\left(\frac{\partial H}{\partial p_\beta} - \dot{q}^\beta\right)\right] dt, \end{aligned} \tag{B.16}$$

which allows us to generalize the HJ theorem to time-dependent Hamiltonians. Namely, a closed section $s = d\mathcal{S}$ of $T^*\mathcal{Q}_{\text{ext}}$ that satisfies the HJ equation

$$H \circ s + \frac{\partial \mathcal{S}}{\partial t} = H\left(\frac{\partial \mathcal{S}}{\partial q^\alpha}, q^\beta; t\right) + \frac{\partial \mathcal{S}}{\partial t} = 0, \tag{B.17}$$

provides a solution to Hamilton's equations.

A few comments are in order at this point. Firstly, note that the HJ formalism for time-dependent Hamiltonians reduces to that for time-independent Hamiltonians upon setting

$$\mathcal{S}(q;t) = \mathcal{W}(q) - Et. \tag{B.18}$$

The function $\mathcal{S}(q;t)$ is known as Hamilton's principal function, while $\mathcal{W}(q)$ is called the characteristic function. Secondly, the expressions (4.12) for the canonical momenta and the Hamiltonian should be familiar from quantum mechanics. Indeed, Hamilton's principal function $\mathcal{S}(q;t)$ is related to the WKB wavefunction by

$$\psi_{WKB}(q;t) \sim e^{i\mathcal{S}(q;t)/\hbar}, \tag{B.19}$$

and so the expressions (4.12) are respectively the coordinate representation of the momentum operator and the identification of the Hamiltonian with the time evolution operator.

Finally, Hamilton's principal function $\mathcal{S}(q;t)$, defined as a solution of the HJ equation (B.17), is closely related to the on-shell action. To elucidate the relation, consider the action (B.1) on the semi-infinite line $(-\infty, t]$. A general variation of the action (B.1) gives

$$\delta S = \int^t dt' \left(\frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha \right) = \int^t dt' \left(\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt'} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) \right) \delta q + \frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \Big|_t . \tag{B.20}$$

To ensure that the variational principle implies the equations of motion we need to impose the boundary condition $\delta q^\alpha = 0$ at $t' = t$. The on-shell action therefore becomes a function of the fixed but arbitrary boundary condition $q^\alpha(t)$, namely $S_{\text{on-shell}}(q;t)$, while

$$p_\alpha|_t = \frac{\partial L}{\partial \dot{q}^\alpha}\Big|_t = \frac{\partial S_{\text{on-shell}}}{\partial q^\alpha}. \tag{B.21}$$

Moreover,

$$\dot{S}_{\text{on-shell}} = L = \frac{\partial S_{\text{on-shell}}}{\partial t} + \frac{\partial S_{\text{on-shell}}}{\partial q^\alpha} \dot{q}^\alpha, \tag{B.22}$$

and so $S_{\text{on-shell}}$ satisfies the HJ equation (B.17):

$$0 = p_\alpha \dot{q}^\alpha - L + \frac{\partial S_{\text{on-shell}}}{\partial t} = H\left(\frac{\partial S_{\text{on-shell}}}{\partial q^\alpha}, q^\beta; t \right) + \frac{\partial S_{\text{on-shell}}}{\partial t}. \tag{B.23}$$

We therefore conclude that the on-shell action as a function of the arbitrary but fixed boundary condition $q(t)$, $S_{\text{on-shell}}(q;t)$, can be identified with Hamilton's principal function $\mathcal{S}(q;t)$. The fact that the on-shell action is a solution of the HJ equation is the fundamental reason for the critical role that HJ theory has in holographic renormalization.

References

[1] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, hep-th/9711200. [Adv. Theor. Math. Phys.2,231(1998)].

[2] E. Witten, *Anti-de Sitter space and holography*, *Adv.Theor.Math.Phys.* **2** (1998) 253–291, hep-th/9802150.

[3] S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys.Lett.* **B428** (1998) 105–114, hep-th/9802109.

[4] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023, hep-th/9806087.

[5] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413–428, hep-th/9902121.

[6] J. de Boer, E. P. Verlinde, and H. L. Verlinde, *On the holographic renormalization group*, *JHEP* **08** (2000) 003, hep-th/9912012.

[7] P. Kraus, F. Larsen, and R. Siebelink, *The gravitational action in asymptotically AdS and flat space-times*, *Nucl.Phys.* **B563** (1999) 259–278, hep-th/9906127.

[8] S. de Haro, S. N. Solodukhin, and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595–622, hep-th/0002230.

[9] M. Bianchi, D. Z. Freedman, and K. Skenderis, *How to go with an RG flow*, *JHEP* **08** (2001) 041, hep-th/0105276.

[10] M. Bianchi, D. Z. Freedman, and K. Skenderis, *Holographic renormalization*, *Nucl. Phys.* **B631** (2002) 159–194, hep-th/0112119.

[11] D. Martelli and W. Mueck, *Holographic renormalization and Ward identities with the Hamilton-Jacobi method*, *Nucl. Phys.* **B654** (2003) 248–276, hep-th/0205061.

[12] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849–5876, hep-th/0209067.

[13] I. Papadimitriou and K. Skenderis, *AdS / CFT correspondence and geometry*, in *AdS/CFT correspondence: Einstein metrics and their conformal boundaries. Proceedings, 73rd Meeting of Theoretical Physicists and Mathematicians, Strasbourg, France, September 11-13, 2003*, pp. 73–101, 2004. hep-th/0404176.

[14] O. Aharony, N. Seiberg, and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115, 1305.0318.

[15] H. Osborn, *Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories*, *Nucl. Phys.* **B363** (1991) 486–526.

[16] B. P. Dolan, *Symplectic geometry and Hamiltonian flow of the renormalization group equation*, *Int. J. Mod. Phys.* **A10** (1995) 2703–2732, hep-th/9406061.

[17] I. Papadimitriou, *Holographic renormalization as a canonical transformation*, *JHEP* **11** (2010) 014, 1007.4592.

[18] D. Z. Freedman, K. Johnson, and J. I. Latorre, *Differential regularization and renormalization: A New method of calculation in quantum field theory*, *Nucl. Phys.* **B371** (1992) 353–414.

[19] I. Papadimitriou, *Holographic renormalization made simple: An example*, *Subnucl. Ser.* **41** (2005) 508–514.

[20] G. W. Gibbons and S. W. Hawking, *Action integrals and partition functions in quantum gravity*, *Physical Review D* **15** (may, 1977) 2752–2756.

[21] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004, hep-th/0505190.

[22] R. Arnowitt, S. Deser, and C. W. Misner, *Canonical variables for general relativity*, *Phys. Rev.* **117** (1960) 1595–1602.

[23] I. Papadimitriou, *Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT*, *JHEP* **05** (2007) 075, hep-th/0703152.

[24] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, . Princeton, USA: Univ. Pr. (1992) 520 p.

[25] B. C. van Rees, *Holographic renormalization for irrelevant operators and multi-trace counterterms*, *Physics* (Feb., 2011) 42, 1102.2239.

[26] B. C. van Rees, *Irrelevant deformations and the holographic Callan-Symanzik equation*, *JHEP* **10** (2011) 067, 1105.5396.

[27] K. A. Intriligator, *Maximally supersymmetric RG flows and AdS duality*, *Nucl. Phys.* **B580** (2000) 99–120, hep-th/9909082.

[28] K. Skenderis and M. Taylor, *Kaluza-Klein holography*, *JHEP* **05** (2006) 057, hep-th/0603016.

[29] K. Skenderis and M. Taylor, *Holographic Coulomb branch vevs*, *JHEP* **08** (2006) 001, hep-th/0604169.

[30] Y. Korovin, K. Skenderis, and M. Taylor, *Lifshitz as a deformation of Anti-de Sitter*, *JHEP* **08** (2013) 026, 1304.7776.

[31] W. Chemissany and I. Papadimitriou, *Lifshitz holography: The whole shebang*, *JHEP* **01** (2015) 052, 1408.0795.

[32] A. O'Bannon, I. Papadimitriou, and J. Probst, *A Holographic Two-Impurity Kondo Model*, 1510.08123.

[33] S. F. Ross, *Holography for asymptotically locally Lifshitz spacetimes*, *Class. Quant. Grav.* **28** (2011) 215019, 1107.4451.

[34] T. Griffin, P. Horava, and C. M. Melby-Thompson, *Conformal Lifshitz Gravity from Holography*, *JHEP* **05** (2012) 010, 1112.5660.

[35] I. Papadimitriou, *Holographic Renormalization of general dilaton-axion gravity*, *JHEP* **1108** (2011) 119, 1106.4826.

[36] W. Chemissany and I. Papadimitriou, *Generalized dilatation operator method for non-relativistic holography*, *Phys. Lett.* **B737** (2014) 272–276, 1405.3965.

[37] I. Kanitscheider, K. Skenderis, and M. Taylor, *Precision holography for non-conformal branes*, *JHEP* **09** (2008) 094, 0807.3324.

[38] P. Breitenlohner and D. Z. Freedman, *Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity*, *Phys. Lett.* **B115** (1982) 197.

[39] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, *JHEP* **0410** (2004) 075, [hep-th/0407071].

[40] H. J. Rothe and K. D. Rother, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*. World Scientific, 2010.

[41] A. R. and J. Marsden, *Foundations of Mechanics (2nd edition)*. Benjamin-Cumming, Reading, 1978.

[42] M. de Leon, J. C. Marrero, and D. M. de Diego, *A Geometric Hamilton-Jacobi Theory for Classical Field Theories*, *ArXiv e-prints* (Jan., 2008) [0801.1181].