Lecture III:

→ As I described briefly in the beginning, I am interested on the following solution in 5D:

$$\int d^5x \sqrt{g} \Big[C_{abc}\,\sigma^a \sigma^b \sigma^c R + C_{abc}\,\sigma^a D_\mu \sigma^b D^\mu \sigma^c$$

$$C_{abc}\,\sigma^a F_{\mu\nu}{}^b F^{\mu\nu\, c} + \dots + C_{abc}\, A^a \wedge F^b \wedge F^c$$

+ Higher Derivatives:

$$\int C_a \sigma^a R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\mu\nu} + C_a A^a \,\mathrm{Tr}\, R \wedge R$$

$+ \dots$ (Susy terms)

$$\text{Couplings} = \begin{cases} C_{abc} = \displaystyle\int_{CY} \omega_a \wedge \omega_b \wedge \omega_c \\[2ex] C_{2a} = \displaystyle\int_{\omega_4^a = \text{4-cycle}} \mathrm{Tr}\, R \wedge R \end{cases}$$

→ Off-Shell BPS eqs :

Metric : $(AdS_2 \times S^1) \times S^2$

$\simeq AdS_3$
(Locally)

Euclidean time

$$ds^2_{AdS_2 \times S^1} = \frac{\ell^2}{4}\left(\frac{dr^2}{r^2-1} + (r^2-1)\,d\tau^2\right)$$

$$+ \frac{\ell^2}{4(\phi_0)^2}\left(dy + i\,\phi^0 (r-1)\,d\tau\right)$$

Take $y' = \frac{y}{\phi^0}$, then metric becomes independent of $\phi^0 \Rightarrow AdS_3$ quotient

$\equiv AdS_3/\Gamma$ w/ $|\Gamma| = \phi^0$

ICTP
www.ictp.it

→ U(1) gauge fields:

$$A^a = \frac{\phi^a}{\phi^0} dy + p^a A_{Dirac}$$

ϕ^0 = constant

→ Scalars (vector-multiplet)

$$\underset{\uparrow \text{scale}}{\ell}\, \sigma^a = p^a = \text{Magnetic charge}$$

→ All the remaining fields are covariantly constant.

→ What do we obtain under dimensional reduction to 4D:

$$ds^2_{4D} = AdS_2 \times S^2 \quad + \text{KK gauge-field}$$

$$A^0 = i\phi^0(r-1)d\tau$$

→ U(1) gauge fields:

$$A^a = \underbrace{\frac{\phi^a}{\phi^0}}_{\text{New scalar in 4D}}(dy + i\phi^0(r-1)d\tau) - i\phi^0(r-1)d\tau + p^a A_{Dirac}$$

$$\phi^a = \mathrm{Re}(X^a)\ell$$

$$X^a = \ell^{-1}\phi^a + i\sigma^a \quad (\text{Vec complex scalar})$$

→ Reducing to 4D one obtains Lagrangian w/ couplings determined by the prepotential

$$F(X) = \frac{C_{abc} X^a X^b X^c}{X^0}$$

→ ~~5D~~ Degrees of freedom:

5D

$\phi^a, \phi^0 \,/\, p^a$

4D

$q^a, q^0 \,/\, p^a$

↳ changes provide function $\phi^a = \phi^a(q,p)$

$\phi^0 = \phi^0(q,p)$

→ Dimensional reduction/truncation to 3D

$$\begin{array}{l} 5D \\ \Big\downarrow \; S^2 \ (\text{gauged by } \tilde{A}) \\ 3D \end{array} \quad \int d^3x \sqrt{g} \left(R \overbrace{+ \frac{2}{\ell^2}}^{-2\Lambda} \right) + D_{ab} \int A^a \wedge F^b + k \int \mathrm{Tr} \left(\tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A}^3 \right)$$

$\tilde{A}$: $SU(2)$ connection

$$D_{ab} = C_{abc} p^c$$

$$k = C_{abc} p^a p^b p^c$$

Note: I'm not keeping track of constants of proportionality for the CS levels

→ It is easy to check that the 5D solution is a solution of this 3D action.

→ Note ~~that~~ also that from the AdS_3/CFT_2 point of view

$$g^{(3D)}_{\mu\nu} \longleftrightarrow T_{\mu\nu} \qquad T(z)\,T \sim \frac{c}{z^4}$$

$$A^a_\mu \longleftrightarrow J^a_\mu \qquad J\,J \sim \frac{k^s}{z^2}$$

$$\tilde{A}_\mu{}^i{}_j \longleftrightarrow J_\mu{}^i{}_j \qquad J\,J \sim \frac{k}{z^2}$$

Chern-Simons Couplings map to the levels of the affine current algebras. These levels become the indices of the Jacobi-form (elliptic genus)!

——"——

$$I = \frac{1}{16\pi G}\int d^3x\,\sqrt{g}\left(R + \frac{2}{\ell^2}\right) \qquad \left[\ \frac{\ell}{G} = \frac{2c}{3}\right.$$

ℓ: size (AdS_3)

→ At the level of the EOMs (on-shell)

$$I = \frac{k}{4\pi}\int \mathrm{Tr}\left(A_L \wedge dA_L + \frac{2}{3}A_L^3\right) - \frac{k}{4\pi}\int \mathrm{Tr}\left(A_R \wedge dA_R + \frac{2}{3}A_R^3\right)$$

$$\boxed{k = \frac{c}{6} = \frac{\ell}{4G}}$$

A_L: $SL(2,\mathbb{R})_L$ $SO(2,2)$

A_R: $SL(2,\mathbb{R})_R$

Euclidean: $SL(2,\mathbb{C})$ for $SO(1,3)$

$\rightarrow$ $N=8$ theory (1 Polar term)

$$C_\mu(m) = \sum_{c=1}^{\infty} \frac{1}{c}\, Kl\left(m-\frac{\mu^2}{4},\, -\overset{\bar{m}-\frac{\nu^2}{4}}{\frac{1}{4}},\, \mu, \overset{=1}{\nu}, c\right) \times$$

$$\times \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{t^{9/2}}\, e^{2\pi\left(m-\frac{\mu^2}{4}\right)\frac{1}{ct} + \pi\left(\frac{1}{4}\right)\frac{t}{c}}$$

— " —

$$Kl\left(m-\frac{\mu^2}{4},\, -\frac{1}{4},\, \mu, \nu, c\right) =$$

$$= \sum_{\substack{0\leq d<c \\ (d,c)=1,\ ad=1 \bmod (c)}} e^{2\pi i\left(m-\frac{\mu^2}{4}\right)\frac{d}{c}}\, M\begin{pmatrix} a & b \\ c & d\end{pmatrix}_{\mu 1} e^{-\pi i \frac{1}{4}\frac{a}{c}}$$

→ Lets consider only the sadde point value for $\left(n-\frac{\mu^2}{4}\right) \gg 1$:

$$C_\mu(n) \simeq \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{0\leq d<c \\ ad=1\,\mathrm{mod}(c)}} e^{2\pi i\left(n-\frac{\mu^2}{4}\right)\frac{d}{c}} M\begin{pmatrix} a & b \\ c & d\end{pmatrix}_{\mu 1} e^{-2\pi i \frac{1}{4}\frac{a}{c}}$$

$$\times \; e^{\frac{\pi}{c}\sqrt{4n-\mu^2}}$$

$$\simeq \sum_{\substack{c \\ 0\leq d<c \\ ad=1\,\mathrm{mod}\,c}} \boxed{e^{\frac{\pi}{c}\sqrt{4n-\mu^2} + 2\pi i\left(n-\frac{\mu^2}{4}\right)\frac{d}{c} - \pi i \frac{1}{4}\frac{a}{c}}} \times M\begin{pmatrix} a & b \\ c & d\end{pmatrix}_{\mu 1}$$

→ With Localization we can go off-shell and recover the full Bessel function.

ICTP
www.ictp.it

→ Explain the logic:

$$Z(\text{Gravity}) = \sum_{AdS_2 \times S^1/\mathbb{Z}_c} e^{i S|_{\text{on-shell}}} \qquad S|_{\text{on-shell}} = CS(A_f)$$

$\simeq$ Rademacher Expansion

→ Further corrections?
we can compute a certain 1-loop correction in CS theory

$$\sum_{AdS_2 \times S^1/\mathbb{Z}_c} e^{i S|_{\text{on-shell}}} \; \mathrm{Det}(\;) \sim K^{n}$$

↓
Matches the Bessel function
↓ 1-loop from Bessel
Matches CS 1-loop
computation

$$\sim K^{(\dim H^1_A - \dim H^0_A)/2}$$
on AdS_3

www.ictp.it

→ There is a map between the metric solutions and $SL(2,\mathbb{C})$ connections. Any metric must be a quotient of AdS_3 (essentially because there are no propagating degrees of freedom in 3D)

→ Essentially Euclidean AdS_3:

$$X = l^{-1}\begin{pmatrix} Y_1 + X_1 & X_2 + iY_2 \\ X_2 - iY_2 & Y_1 - X_1 \end{pmatrix} \quad \leftarrow \text{Hermitian}$$

$$\det(X) = 1 \quad (\text{Hyperbolic space})$$

X is invariant under

$$g X g^{\dagger} \quad \text{w/ } g \in SL(2,\mathbb{C})$$

→ We can consider quotients of the form

$$X \sim \begin{pmatrix} e^{-i\pi\tau} & 0 \\ 0 & e^{i\pi\tau} \end{pmatrix} X \begin{pmatrix} e^{i\pi\bar{\tau}} & 0 \\ 0 & e^{-i\pi\bar{\tau}} \end{pmatrix}$$

τ: complex

→ In terms of metric how does it work?

start w/ global AdS_3

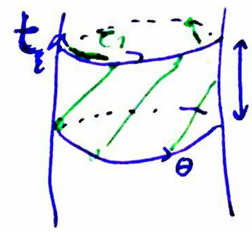
$$ds^2 = \cosh^2(\eta)\, dt_E^2 + \sinh^2(\eta)\, d\theta^2 + d\eta^2$$

$-\infty < \tau < \infty \qquad \theta \in [0, 2\pi]$

→ at the boundary $\eta \to \infty$

$$ds^2 = e^{2\eta}\left(dt_E^2 + d\theta^2\right) + d\eta^2$$

" Cylinder

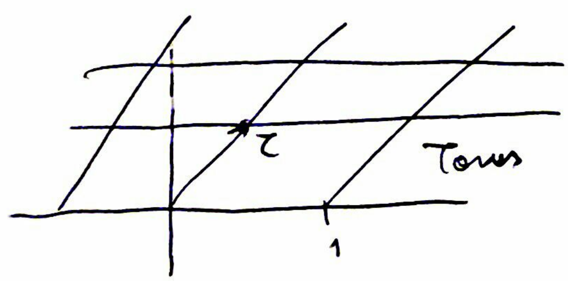


→ Identify $t_E \sim t_E + 2\pi\tau_2$ $\quad\Big|\quad \theta \sim \theta + 2\pi$

$\theta \sim \theta + 2\pi\tau_1$

This gives a complex structure to the boundary torus

$$\tau = \tau_1 + i\tau_2$$



→ The metric is known as Thermal AdS_3.

→ AdS_3 is topologically a Solid Torus $D \times S^1$

$$\underbrace{ch^2(\eta)\, dt_E^2}_{S^1} + \underbrace{sh^2(\eta)\, d\theta^2 + d\eta^2}_{D}$$

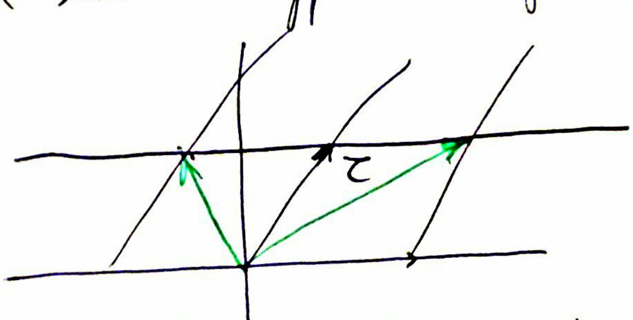
$$S^1 \times D$$

→ For thermal AdS$_3$

Cycle: $z \sim z + 2\pi\tau$ (with $z = \theta + it$) is the non-contractible (because of the Euclidean time component)

Cycle $z \sim z + 2\pi$ is the contractible (the θ direction)

→ I could use a different reparametrization



basis of cycles

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in SL(2,\mathbb{Z})} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Periodicities =

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}$$

Because we are only interested in the conformal structure, we rescale coordinates by $\frac{1}{c\tau + d}$, then periodicities $\begin{pmatrix} \frac{a\tau + b}{c\tau + d} \\ 1 \end{pmatrix}$

that is, we obtain torus w/ complex structure $= \frac{a\tau + b}{c\tau + d}$

→ Then this corresponds to the quotient

$$X \sim \begin{pmatrix} e^{-i\pi \frac{a\tau+b}{c\tau+d}} & 0 \\ 0 & e^{i\pi \frac{a\tau+b}{c\tau+d}} \end{pmatrix} X \begin{pmatrix} * & \\ & \cdot \end{pmatrix}^{\dagger}$$

→ We also see that the cycle

$$a\,C_1 + b\,C_2 \equiv \text{non-contractible}$$

$$C_1 : z \sim z + 2\pi\tau$$

$$C_2 : z \sim z + 2\pi$$

$$c\,C_1 + d\,C_2 \equiv \text{Contractible cycle}$$

Exactly like the Dehn Filling I explained Yesterday.

→ For $c \geq 1$, one can show that the metric is that of a Black Hole (BTZ) (Horizon)

→ We can generate extremal BTZ solutions by analytic continuation of $(\tau, \bar{\tau})$ that now become independent (not complex conjugate).

$$ds^2 = l^2 \left[\frac{dr^2}{r^2-1} + (r^2-1)\,d\theta^2 + R^2\left(dy + \frac{i}{R}(r-1)d\theta - d\,d\theta\right)^2 \right.$$

shift makes geometry non-singular

$$\theta \sim \theta + \frac{2\pi}{c}$$

$$\equiv M_{c,d} = AdS_2 \times S^1/\mathbb{Z}_c$$

ICTP
www.ictp.it

→ But metric only sees contractible cycle which is why we see only the (c,d) dependence.

→ We can compute the Holonomy of the $SL(2,\mathbb{C})$ connections. It is nothing more than the identification matrix:

$$\mathrm{Hol} = P\, e^{\oint_{S^1} A_L} = \begin{pmatrix} e^{-\pi i \frac{a\tau+b}{c\tau+d}} & 0 \\ 0 & e^{i\pi \frac{a\tau+b}{c\tau+d}} \end{pmatrix}$$

↳ Non-contractible cycle

$$\mathrm{Hol} = P\, e^{\oint_{S^1} A_L} = \begin{pmatrix} e^{-i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix}$$

$S^1 = \partial D$ (Contractible cycle) $= -\mathbb{1}$

(because it couples to ψ^i (fermions) in the fundamental of $SU(2)$. The parallel transport of ψ^i: $D_\mu \psi^i = 0$ ~~is nothing~~ along the contractible cycle is Holonomy.

A_R connection:

$$\mathrm{Hol}(A_R)\Big|_{S^1} = \begin{pmatrix} e^{\pi i \frac{a}{c}} & 0 \\ 0 & e^{-i\pi \frac{a}{c}} \end{pmatrix}$$

$$\mathrm{Hol}(A_R)\Big|_{S^1} = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix}$$

→ Gravitational computation I:

$$ds^2 = \frac{\ell^2}{4}\left(\frac{dr^2}{r^2-1} + (r^2-1)\,d\theta^2\right) + \frac{\ell^2}{4(\phi_0)^2}\left(dy + i\phi^0 (r-1)\,d\theta\right)^2$$

I want to compute:

$$\frac{1}{16\pi G}\int d^3x \sqrt{g}\left(R^{(3)} + \frac{2}{\ell^2}\right)$$

Now dimensional reduce to 2 dims:

$$R^{(3)} = R^{(2)} - \frac{\ell^2}{4}\frac{1}{4(\phi_0)^2} F_{\mu\nu}F^{\mu\nu}$$

$$F = dA_{KK}$$

$$\boxed{R^{(3)} = -\frac{6}{\ell^2}}$$

→ Do integral

$$I = -\frac{\pi}{8}\frac{\ell}{G}\frac{1}{\phi_0}\int_1^{r_0} dr = -\frac{\pi}{8}\frac{\ell}{G}\frac{1}{\phi_0}(r_0 - 1)$$

cutoff (r_0)

$$\frac{\ell}{G} = \frac{2c}{3} \Longrightarrow -\frac{\pi}{12} c \frac{1}{\phi^0}(r_0 - 1)$$

central charge (c)

Finite piece $\quad \mathrm{Re}\,(I) = \frac{\pi}{12}\frac{c}{\phi_0}$

$\rightarrow$ In 2D :

$$\sim \int \frac{1}{\phi^0} \sqrt{g} \left(R^{(2)} - \frac{\ell^2}{16(\phi_0)^2} F_{\mu\nu} F^{\mu\nu} \right)$$

$$A = i \phi^0 (r-1) d\theta$$

Fix ϕ^0 and consider only subleading corrections

$\rightarrow$ Introduce Bnd Wilson line because of variational principle:

$$\text{Bnd term} = \frac{c}{24} \frac{i}{(\phi_0)^2} \oint A$$

$\rightarrow$ On-shell

$$\text{Re}\left(\cdots \oint A \right) = \frac{c}{24} \frac{\phi^0}{(\phi_0)^2} \times 2\pi = \frac{\pi c}{12} \frac{1}{\phi^0}$$

$$= \text{Re}\left(I_{\text{Bulk}} \right)$$

$$\text{Total} = 2 \times \pi \frac{c}{12} \frac{1}{\phi^0} = \frac{\pi c}{6} \frac{1}{\phi_0}$$

→ Do the same computation on

$$\left(AdS_2 \times S^1\right)/\mathbb{Z}_c$$

The Bulk part gets divided by $\frac{1}{c}$ due to the orbifold:

$$\mathrm{Re}\left(I_{\text{Bulk}}\right) = \frac{\text{Central}}{12c}\,\pi\,\frac{1}{\phi_0}$$

For the Boundary term it will pick the shift on $A \to A + \frac{d}{c}$

Note phase

$$\mathrm{Re}\left(I_{\text{Bd}}\right) = \frac{\text{Central}}{12}\,\frac{\pi}{c}\,\frac{1}{\phi^0} - i\,\frac{\text{Central}}{12}\,\pi\,\frac{1}{(\phi^0)^2}\,\frac{d}{c}$$

$$\boxed{\text{Total} = \frac{\text{Central}}{6}\,\frac{\pi}{c}\,\frac{1}{\phi^0} - i\,\frac{\text{Central}}{12}\,\pi\,\frac{1}{(\phi_0)^2}\,\frac{d}{c}}$$

→ This is not the end of the story:

The $SU(2)$ connection must be turned on such that the Killing spinors survive the $\mathbb{Z}_c$ quotient!

$$P e^{\oint_{\tilde{S}^1} \tilde{A}^{(SU(2))}} = \begin{pmatrix} e^{-\pi i} & 0 \\ 0 & e^{\pi i} \end{pmatrix} \Longrightarrow$$

$$\oint_{C_1} A_R = -\frac{2\pi i}{c}\,\frac{6^3}{2} \qquad \text{such that} \qquad c\oint_{C_1} A_R + d\oint_{C_2} A_R = -2\pi i\,\frac{6^3}{2}$$

$$\oint_{C_2} A_R = 0$$

$$\Longrightarrow$$

$$\xi \sim e^{\frac{i}{2}(\overset{AdS_2}{\theta} + \overset{\phi}{\phi})}$$

$$\theta \sim \theta + \frac{2\pi}{c} \Rightarrow \phi \sim \phi - \frac{2\pi}{c}$$

Need to turn on a flat connection on S^2 such that when it integrates around θ it generate the oposite shift.

→ we can compute the Holonomy along the Non-Contractible cycle:

$$a \oint_{C_1} \tilde{A} + b \oint_{C_2} \tilde{A} = -2\pi i \frac{q}{c} \frac{6^3}{2}$$

$$CS(\tilde{A}) = 2\pi^2 \underset{\text{contractible}}{\alpha} \beta_{\to \text{non-contractible}}$$

$$\alpha = -1 \; ; \; \beta = -\frac{q}{c}$$

$$= 2\pi^2 \frac{q}{c}$$

$$\frac{K_{SU(2)}}{4\pi} CS(\tilde{A}) = \frac{2\pi}{4} \frac{q}{c} K_{SU(2)} = -\frac{iK}{4\pi} \int \mathrm{tr}(\dots)$$

$$\text{Total} = i \left(I_{\text{Bulk}} + I_{\text{Bnd}} \right) + I(SU(2)\, CS)$$

There is - sign (or $\phi^0 \to -\phi^0$)

$$= -\frac{\text{Central}}{6} \frac{\pi}{c} \frac{1}{\phi^0} + i \frac{\text{Central}}{12} \pi \frac{1}{(\phi_0)^2} \frac{d}{c} - i 2\pi \frac{K_{SU(2)}}{4} \frac{q}{c}$$

$$K_{SU(2)} = \frac{\text{Central}}{6}$$ | ϕ^0 can be fixed in terms of charges $\phi^0 = \phi^0(q, P)$

→ In the Chern-Simons Formulation:

$$S = -i\frac{K}{4\pi}\int_{SL(2,\mathbb{C})} \mathrm{tr}(A_L \wedge dA_L + \cdots) + i\frac{K}{4\pi}\int_{\overline{SL(2,\mathbb{C})}} \mathrm{tr}(A_R \wedge dA_R + \cdots)$$

$$-i\frac{K}{4\pi}\int_{SU(2)} \mathrm{tr}(\tilde{A}\wedge d\tilde{A} + \cdots)$$

& Compute on Solid Torus

→ From the identifications we know the Holonomies

$$\oint_{C_{n\text{-}c}} A_L = 2\pi i \frac{a\tau+b}{c\tau+d}\frac{\sigma^3}{2} \quad ; \quad \oint_{C_c} A_L = 2\pi i \frac{\sigma^3}{2}$$

(Non-contractible)

$$\oint_{C_{n\text{-}c}} A_R = -2\pi i \frac{a}{c}\frac{\sigma^3}{2} \quad ; \quad \oint_{C_c} A_R = -2\pi i \frac{\sigma^3}{2}$$

→ It is easy to see that

$$\frac{iK}{4\pi}\int \mathrm{tr}(A_R \wedge dA_R + \cdots) - i\frac{K}{4\pi}\int \mathrm{tr}(\tilde{A}\wedge d\tilde{A} + \cdots)$$

$$= 0$$

→ From the Hol(A_L) we can compute

$$\oint_{C_1} A_L = 2\pi i \frac{\tau}{c\tau+d}\frac{\sigma^3}{2} \quad ; \quad \cancel{\oint_{C_1} A_L = 2\pi i}$$

$$\oint_{C_2} A_L = 2\pi i \frac{1}{c\tau+d}\frac{\sigma^3}{2}$$

www.ictp.it

→ Such that

$$a \oint_{C_1} A_L + b \oint_{C_2} A_L = 2\pi i \frac{a\tau+b}{c\tau+d} \frac{\sigma^3}{2}$$

and

$$c \oint_{C_1} A_L + d \oint_{C_2} A_L = 2\pi i \frac{c\tau+d}{c\tau+d} \frac{\sigma^3}{2}$$

→ $CS(A_L)$ computes

$$-i \frac{K}{4\pi} 2\pi^2 \underset{1}{\underset{\shortparallel}{\alpha}} \beta_{= \frac{a\tau+b}{c\tau+d}} = -i\, 2\pi \frac{K}{4} \frac{a\tau+b}{c\tau+d}$$

→ We need to add in addition the Boundary term

$$-\frac{iK}{4\pi} \int \mathrm{tr}\, \underset{\text{along } C_1}{A_{L1}}\ \underset{\text{along } C_2}{A_{L2}}$$

(We are fixing component C_2 and fluctuating C_1)

$$= +\frac{K\pi}{2} \frac{\tau}{(c\tau+d)^2}$$

→ Total action w/ Boundary terms

$$S = -2\pi i \frac{K}{4} \frac{a\tau+b}{c\tau+d} + 2\pi i \frac{K}{4} \frac{\tau}{(c\tau+d)^2}$$

→ Use

$$\frac{a\tau+b}{c\tau+d} = \frac{a}{c} - \frac{1}{c(c\tau+d)} \;;\; \frac{\tau}{(c\tau+d)^2} = \frac{1}{c(c\tau+d)} - \frac{d}{c}\frac{1}{(c\tau+d)^2}$$

$$S = -2\pi i \frac{K}{4}\frac{a}{c} + 2\pi i \frac{K}{2}\frac{1}{c}\frac{1}{(c\tau+d)} - 2\pi i \frac{K}{4}\frac{d}{c}\frac{1}{(c\tau+d)^2}$$

Under the map metric ⟷ Connection

$$\boxed{\frac{1}{c\tau+d} = \text{Fixed} = -\frac{1}{\phi^0}}$$

↳ $$S = -2\pi i \frac{K}{4}\frac{a}{c} - \pi \frac{c_{\text{central}}}{6}\frac{1}{\phi^0 c} + 2\pi i \frac{K}{4}\frac{d}{c}\frac{1}{(\phi^0)^2}$$

= Gravitational Computation

→ Use EOM in 4D to determine ϕ^0! in term of the charges q_a, p^a: (Following entropy funct for example

→ In $N=8$:

$$\left(\frac{1}{\phi^0}\right)^2 = \frac{4nK-\ell^2}{K}$$

$$-\pi \frac{\overbrace{c_{\text{central}}}^{=6K}}{6}\frac{1}{\phi^0 c} = \pi\sqrt{4nK-\ell^2}$$

but $K=1$ to compare w/ $\phi_{-2,1}(\tau,z)$

ICTP

www.ictp.it

m
ℓ
κ changes

$\mathcal{N}=8$

$\longrightarrow$

$$S = -2\pi i \frac{\kappa}{4}\frac{a}{c} + \frac{\pi}{c}\sqrt{4m\kappa-\ell^2} + 2\pi i \frac{d}{c}\left(m-\frac{\ell^2}{4\kappa}\right)$$

Set $\kappa=1$ to compare w/ $\Phi_{-2,1}(\tau, z)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\, {}_{\kappa}$

$$S = -2\pi i \frac{1}{4}\frac{a}{c} + \frac{\pi}{c}\sqrt{4m-\ell^2} + 2\pi i \frac{d}{c}\left(m-\frac{\ell^2}{4}\right)$$

→ Compare S w/ the saddle point value of the Rademacher expansion. (Not including the Multiplier Matrix $M\begin{pmatrix} a & b \\ c & d\end{pmatrix}_{\mu\nu}$). The value of S agrees even at the Non-perturbative order!

→ To compute $M\begin{pmatrix} a & b \\ c & d\end{pmatrix}_{\mu\nu}$, we need to consider additional $SU(2)_L$ Chern-Simons term, and sum over Flat connections. We can, this way, reproduce exactly the full matrix $M\begin{pmatrix} a & b \\ c & d\end{pmatrix}$!!