# Linear Systems

$$s(x) \longrightarrow \boxed{\text{system}} \longrightarrow r(x)$$

$$s(x) \xrightarrow{\square} r(x)$$

Linear

$$s_1(x) + s_2(x) \xrightarrow{\square} r_1(x) + r_2(x)$$

shift invariant

$$s(x - x_0) \xrightarrow{\square} r(x - x_0)$$

consider LSI system

if $s(x) \xrightarrow{\square} r(x)$, then $s'(x) \xrightarrow{\square} ?$

$$s'(x) = \lim_{\Delta x \to 0} \frac{s(x+\Delta x)}{\Delta x} - \frac{s(x)}{\Delta x} \xrightarrow{\square} \lim_{\Delta x \to 0} \frac{r(x+\Delta x)}{\Delta x} - \frac{r(x)}{\Delta x}$$

$$= r'(x)$$

$$\underline{s'(x) \xrightarrow{\square} r'(x)}$$

$$\underbrace{s'(x) - i\nu 2\pi s(x)}_{\text{if this} = 0} \xrightarrow{\square} \underbrace{r'(x) - i2\pi\nu\, r(x)}_{\text{Then this} = 0}$$

$$s(x) = s_0 e^{i2\pi\nu x} \xrightarrow{\square} r(x) = r_0\, e^{i2\pi\nu x}$$

$$e^{i2\pi\nu x} \xrightarrow{\square} H(\nu)\, e^{i2\pi\nu x}$$

In general

$$s(x) = \int_{-\infty}^{\infty} \tilde{s}(\nu)\, e^{i2\pi\nu x}\, d\nu \rightarrow \int_{-\infty} \underbrace{\tilde{s}(\nu)\, H(\nu)}_{\tilde{r}(\nu)}\, e^{i2\pi\nu x}\, d\nu = r(x)$$

$$\underline{\tilde{r}(\nu) = \tilde{s}(\nu)\, H(\nu)}$$

$H(\nu)$ = transfer function.

$$\begin{tikzcd} s(x) \arrow[r, "\square"] \arrow[d, "\hat{f}"] & r(x) \\ \tilde{s}(\nu) \arrow[r] & \tilde{r}(\nu) = \tilde{s}(\nu)\, H(\nu) \arrow[u, "\hat{f}^{-1}"'] \end{tikzcd}$$

On the other hand:

$$s(x) = \int s(x')\, \delta(x-x')\, dx' \xrightarrow{\square} r(x) = \int s(x')\, G(x-x')\, dx' = s * G\,(x)$$

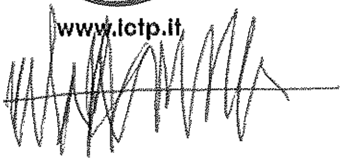
where

$\delta(x) \xrightarrow{\square} G(x)$ impulse response

So the effect of a LSI system is always a convolution, which in Fourier space is a product. Therefore $\underline{\hat{f}_{x\to\nu}\, G(x) = H(\nu)}$

The system acts as a blur, or equivalently, as a Fourier filter.

Examples:
- Low pass
- High pass
- DC removal
- derivatives (including "fractional")
- combination of images

Noise $n(x)$  random.

· Fourier transform $\hat{f}_{x\to\nu}\, n(x) = \tilde{n}(\nu)$

often, we do not know $\tilde{n}$, but we do know roughly how it behaves

$$\langle |\tilde{n}(\nu)|^2 \rangle = S_n(\nu)$$ power spectrum of the noise.

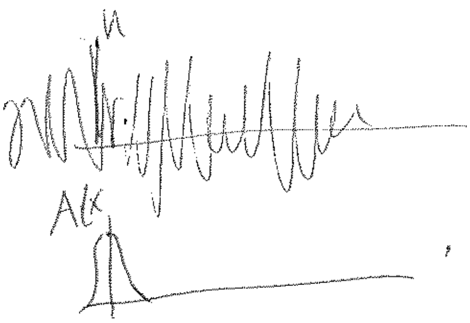
average over many "realizations"

note

$$\hat{f}^{-1}_{\nu\to x} S_n(\nu) = \int S_n(\nu)\, e^{i2\pi\nu x}\, d\nu = \Big\langle \int \underbrace{\tilde{n}^*(\nu)}_{\left[\int n(x') e^{-i2\pi\nu x'} dx'\right]^*} \tilde{n}(\nu)\, e^{i2\pi\nu x}\, d\nu \Big\rangle$$

$$= \Big\langle \iint n^*(x')\, \hat{n}(\nu)\, e^{i2\pi\nu(x+x')}\, d\nu\, dx' \Big\rangle$$

$$= \Big\langle \int n^*(x') \underbrace{\int \hat{n}(\nu)\, e^{i2\pi\nu(x+x')}\, d\nu}_{n(x+x')}\, dx' \Big\rangle = \underbrace{\Big\langle \int n^*(x')\, n(x+x')\, dx' \Big\rangle}_{\text{autocorrelation of noise } A_n(x)}$$




$$\hat{f}_{x\to\nu}\, A_n(x) = S_n(\nu)$$

Wiener-Khinchin theorem.

Examples if $A_n(x) \approx \delta(x)$, $S_n(\nu) \approx$ constant <u>white noise</u>

If $S_n(\nu) \propto \frac{1}{\nu^\alpha}$, $0<\alpha<2$ usually $\alpha \approx 1$.

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## Noise filtering & deconvolution

Consider a noisy system, where

$$\underbrace{r(x)}_{\text{response}} = \underbrace{G}_{\text{impulse response}} * \underbrace{s}_{\text{signal}}(x) + \underbrace{n(x)}_{\text{noise}}$$

we measure $r(x)$, we know $G(x)$ (system).

we do not know $n(x)$ but know its statistics $A_n(x)$.

we want to estimate $s(x)$

In Fourier space:

$$\tilde{r}(\nu) = H(\nu)\tilde{S}(\nu) + \tilde{n}(\nu)$$

Propose estimate $\hat{\tilde{S}}(\nu) = \underbrace{W(\nu)}_{\text{filter.}}\tilde{r}(\nu)$

The total error is

$$\varepsilon = \int |\hat{\tilde{S}}(\nu) - \tilde{S}(\nu)|^2 d\nu = \int |WH\tilde{s} + W\tilde{n} - \tilde{s}|^2 d\nu$$

$$= \int |(WH-1)\tilde{s} + W\tilde{n}|^2 d\nu$$

$$\mathcal{E} = \int |(WH-1)|^2 |\tilde{S}|^2 d\nu + \int |W|^2 |\tilde{n}|^2 d\nu + \int \left[ W^* \tilde{n}^* (WH-1)\tilde{S} + c.c. \right] d\nu$$

take average over possible realizations of $\tilde{n}$, and use $\langle \tilde{n} \rangle = 0$, $\langle |\tilde{n}|^2 \rangle = S_n(\nu)$ (noise power spectrum).

$$\langle \mathcal{E} \rangle = \int (W^*H^*-1)(WH-1)|\tilde{S}|^2 d\nu + \int W^* W S_n(\nu) d\nu + 0$$

Now find $W(\nu)$ that minimizes error.

use variational calculus:

$$\frac{\delta}{\delta W^*(\nu)} \int W^*(Wf+A)\, d\nu = Wf + A$$

Force $\dfrac{\delta \langle \mathcal{E} \rangle}{\delta W^*(\nu)} = 0$

$$H^*(WH-1)|\tilde{S}|^2 + W S_n = 0$$

so $W\left[|H|^2 |\tilde{S}|^2 + S_n\right] = H^* |\tilde{S}|^2$

$$\boxed{W = \frac{H^*}{|H|^2 + S_n/|\tilde{S}|^2}}$$ Wiener filter