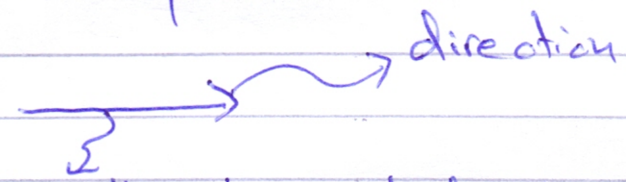
Vector Analysis

Vector: A physical quantity possessing both magnitude and direction.

e.g displacement, velocity, force etc.

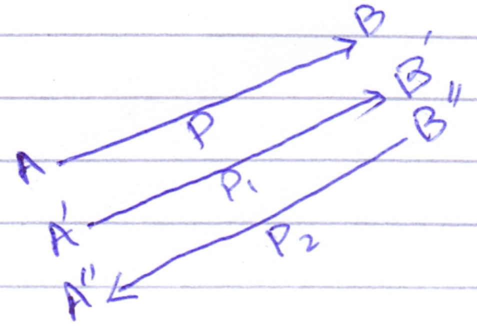
Scalars: Quantities that have magnitude but no direction are called scalars. examples are: Mass, charge, density ...

A vector is represented as



length $\propto$ to magnitude of the vector

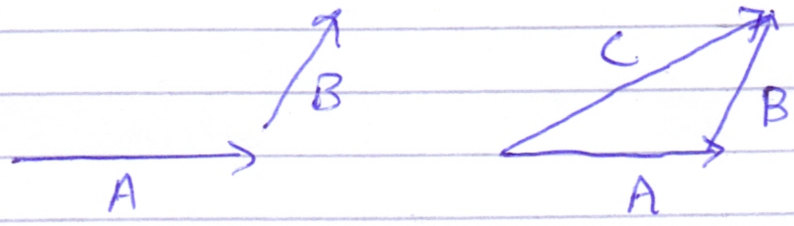
Vectors are not localized



are same

$$\vec{P} = AB = \vec{P_1} = -\vec{P_2}$$

Vector addition



$$\bar{C} = \bar{A} + \bar{B}$$

⇒ Vector addition is commutative

$$A + B = B + A$$

Addition is also associative.

$$(A+B)+C = A+(B+C)$$

$$A - B = A + (-B)$$

⇒ Multiplication by a scalar

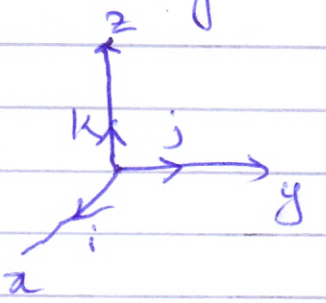
$$a(\vec{A} + \vec{B}) = a\bar{A} + a\vec{B}$$

⇒ A vector having unit magnitude is called a unit vector

$$\hat{a} = \frac{\vec{A}}{|A|} \quad \text{provided } |A| \neq 0$$

$\hat{a} \longrightarrow$ same direction as $\vec{A}$

but magnitude equal to unity

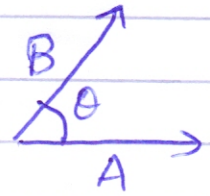


Scalar or Dot product

$$\bar{A} \cdot \bar{B} = |A||B| \cos\theta$$

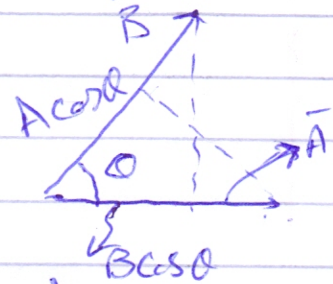
$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}$$

Commutative



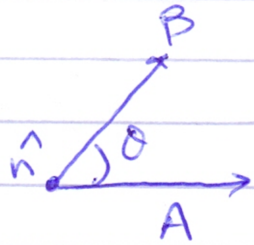
and
distributive

$$A \cdot (B + C) = \bar{A} \cdot \bar{B} + A \cdot C$$



Cross Product of two vectors

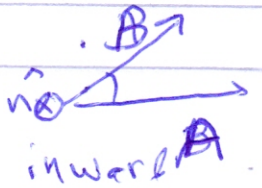
$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$



$$\bar{A} \times \bar{B} = -(\bar{B} \times \bar{A})$$

Distributive

$$A \times (B + C) = (A \times B) + (A \times C)$$



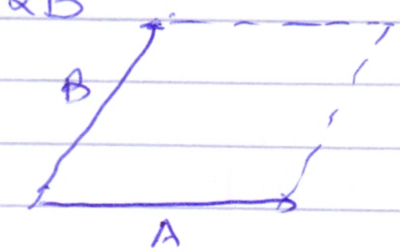
Not Commutative

$$\bar{B}\times\bar{A} = -(A\times B)$$

e.g

$$L = \vec{r}\times\vec{p}$$
$$\tau = \vec{r}\times\vec{F}$$
$$F = q(\bar{v}\times\bar{B})$$

$|A\times B|$ = the area of a parallelogram generated by $A \times B$



Scalar Triple Product

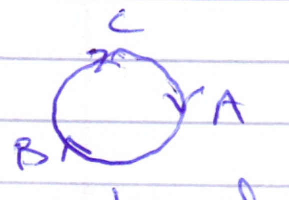
$$\bar{A}\cdot(\bar{B}\times\bar{C}) = B\cdot(\bar{C}\times\bar{A}) = C\cdot(\bar{A}\times B)$$

cyclic order.

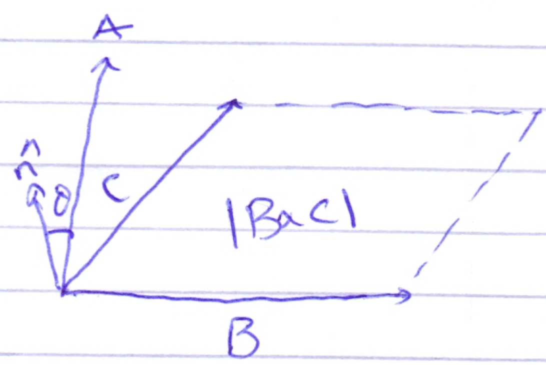
$|A\cdot(B\times C)|$ — volume of the parallele-piped generated by A, B and C.

$|\vec{B}\times\vec{C}| \rightarrow$ area of base

$|A\cos\theta| \rightarrow$ altitude



$$A\cdot(B\times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$



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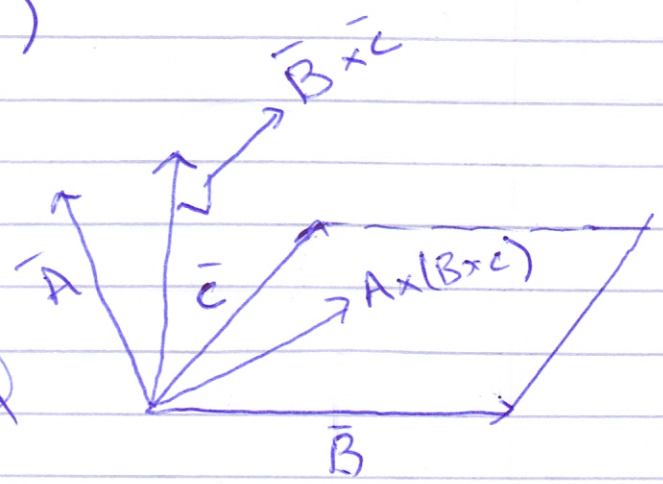
Vector Triple Product.

$$\bar{A} \times (\bar{B} \times \bar{C})$$

is $\perp$ to vector $\bar{A}$ and lies in the plane having B & C



It can be simplified using

$\bar{B}\bar{A}\bar{C} - \bar{C}\bar{A}\bar{B}$ Rule

$$\Rightarrow \bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(A \cdot B)$$

Can be solved by using Levi-Civita ϵ_{ijk} in tensor notation.

The Levi-Civita symbol ϵ_{ijk} is a tensor of rank three and is defined by

$$\epsilon_{ijk} \begin{cases} = 0 & \text{if any two labels are the same} \\ = 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ = -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

$$\epsilon_{ijk}\,\epsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$ (k is repeated index

Similarly.

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$$

δ_{ij} — kronecker's delta.

$\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$ " $i \neq j$

$$\delta_{ij} A_i = A_j$$

$$\delta_{ij} A_i B_j = A_i B_i = A_j B_j = \bar{A}\cdot\bar{B}$$

⇒ Tensor's are generalization of scalars and vectors.

→ Tensor of order (or rank) zero are scalars

→ Tensor of order one are vectors.

In 3-D space — scalar is $3^0 = 1$ comp

" vector is $3^1 = 3$ comp

In general a tensor of rank n has 3^n — compts

⑦

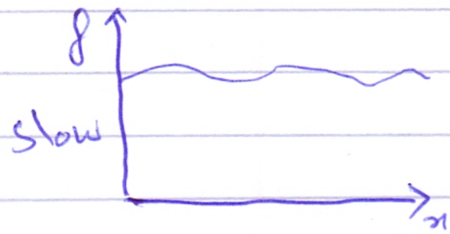
Differential Calculus

ordinary derivatives:

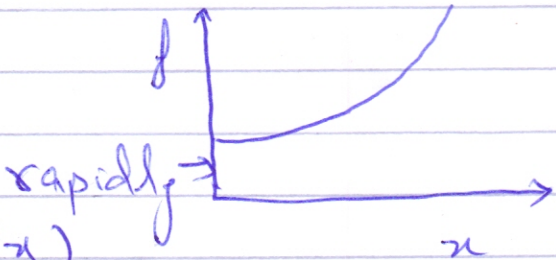
$f(x)$ — function of one variable x

$\frac{d f(x)}{dx}$ — How rapidly $f(x)$ changes with a tiny change dx in x

$\Rightarrow \; df(x) = \left(\frac{d f(x)}{dx}\right) dx$



$\Rightarrow$ if x changes by dx
then $f(x)$ ———— df



$\frac{df}{dx}$ — slope of curve $f(x)$

Operator Del.

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{k}$$

operator "Del" is not a vector in a usual sense.

An ordinary vector A can multiply in three ways.

i, Multiply with a scalar $a \Rightarrow \bar{A}a$

ii, Multiply with another vector $\vec{B}$ via dot product: $\bar{A}.\bar{B}$

iii, Multiply another vector via the cross product: $\bar{A} \times \bar{B}$

Correspondingly there are three ways the operator ∇ can act.

i, Act on a scalar function: $\nabla f \Rightarrow$ Gradient

ii, Act on a vector function $\bar{V}$; via dot product $\nabla . V \Rightarrow$ The divergence

iii, Act on a vector function $\bar{V}$: via the cross product $\bar{\nabla} \times \bar{V} \Rightarrow$ The curl.

The Gradient.

The gradient of a scalar function $f(x,y,z)$ is given by.

$$\bar{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

In tensor notation the ith component is represented as.

$$(\nabla f)_i = \nabla_i f \implies (\nabla f)_i = \frac{\partial f}{\partial x_i}$$

$$i = x, y, z$$

A small change in function f when we go small distance (dx, dy, dz) away from point (x,y,z) from theory of partial derivatives it is given by.

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}\right)dz$$

$$= \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right) \cdot \left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$\implies df = \bar{\nabla} f \cdot \vec{dl}$$

$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ is infinitesimal displacement vector and $\bar{\nabla} f$ is gradient of f which is generalization of a derivative in 3-D

$\bar{\nabla} f$ is a vector quantity.

The Divergence

$$\bar{\nabla} \cdot \bar{V} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left(V_x \hat{x} + V_y \hat{y} + V_z \hat{z} \right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Divergence of a vector $\bar{V}$ is a scalar quantity.

The curl

Curl of a vector function $\vec{V}$ is given by.

$$\bar{\nabla} \times \bar{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \longrightarrow \text{vector quantity.}$$

$$= \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

Product Rules

Like ordinary derivatives a number of general rules, such as the sum rule, multiplication by a constant, the product rule and quotient rule hold for vector derivatives

Ordinary derivatives

$$\frac{d}{dx}(\bar{f}+\bar{g}) = \frac{d\bar{f}}{dx} + \frac{d\bar{g}}{dx}$$

$$\frac{d}{dx}(kf) = k\frac{df}{dx}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

Vector Derivatives

Sum Rules.

a, $\bar{\nabla}(\bar{f}+\bar{g}) = \bar{\nabla}f + \bar{\nabla}g$

b, $\nabla\cdot(\bar{A}+\bar{B}) = \bar{\nabla}\cdot\bar{A} + \bar{\nabla}\cdot B$

c, $\bar{\nabla}\times(\bar{A}+B) = \bar{\nabla}\times\bar{A} + \bar{\nabla}\times\bar{B}$

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2, Rule for Multiplication with a Constant.

a, $\bar{\nabla}(K\bar{f}) = K\bar{\nabla}f$

b, $\bar{\nabla}\cdot(K\bar{A}) = K(\bar{\nabla}\cdot\bar{A})$

c, $\bar{\nabla}\times(K\bar{A}) = K(\bar{\nabla}\times\bar{A})$

Product Rules.

$\Rightarrow$ Two for gradient.

i, $\nabla(fg) = f\nabla g + g\nabla f$

ii, $\nabla(\bar{A}\cdot\bar{B}) = \bar{A}\times(\nabla\times B) + B\times(\nabla\times A) + (A\cdot\nabla)\bar{B}$
$+ (\bar{B}\cdot\bar{\nabla})\bar{A}$

$\Rightarrow$ Two for divergence.

i, $\nabla\cdot(f\bar{A}) = f(\bar{\nabla}\cdot\bar{A}) + \vec{A}\cdot(\bar{\nabla}f)$

ii, $\nabla\cdot(\bar{A}\times\bar{B}) = \vec{B}\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

$\Rightarrow$ Two for curls

i, $\nabla\times(f\bar{A}) = f(\bar{\nabla}\times\bar{A}) - \bar{A}\times\bar{\nabla}f$

ii, $\bar{\nabla}\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\bar{\nabla})\bar{A} - (\bar{A}\cdot\bar{\nabla})\bar{B} + \vec{A}(\bar{\nabla}\cdot\bar{B})$
$- \vec{B}(\bar{\nabla}\cdot\bar{A})$

Quotient Rule

i, $\nabla\left(\frac{f}{g}\right) = \frac{g\bar{\nabla}f - f\bar{\nabla}g}{g^2}$ f&g (scalars)

ii, $\bar{\nabla}\cdot\left(\frac{\bar{A}}{f}\right) = \frac{f(\bar{\nabla}\cdot A) - \bar{A}\cdot(\bar{\nabla}f}{f^2}$

iii, $\nabla\times\left(\frac{\bar{A}}{f}\right) = \frac{f(\bar{\nabla}\times\bar{A}) + \bar{A}\times(\bar{\nabla}f)}{f^2}$

Second Derivatives

The Gradient $\bar{\nabla}t \longrightarrow$ is vector.

i, $\bar{\nabla}\cdot(\bar{\nabla}t) = \nabla^2 t$

ii, $\nabla\times(\nabla t) = 0$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \rightarrow$ Laplacian

$\nabla^2 T$ —— scalar

$\nabla^2\bar{V} \longrightarrow$ vector

13b

The Divergence $\bar{\nabla}\cdot\bar{V} \longrightarrow$ is Scalar

i, Gradienent of divergence

$$\nabla(\bar{\nabla}\cdot\bar{V}) \neq \underbrace{(\nabla\cdot\nabla)V}_{\nabla^2 V}$$

The curl $\bar{\nabla}\times\bar{V} \longrightarrow$ is vector.

ii, $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{V}) = 0$

iii, $\nabla\times(\nabla\times V) = \underbrace{\bar{\nabla}(\bar{\nabla}\cdot\bar{V})}_{\text{gradienet of divergen.}} - \underbrace{\nabla^2 V}_{\text{Laplacian of vecto}}$

INTEGRAL CALCULUS

Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable. The fundamental theorem of calculus states.

If $F(x)$ is continous on a closed interval $[a, b]$ and if $f(x)$ is an antiderivative of $F(x)$ so that

$$F(x) = \frac{d f(x)}{dx}$$

or

$$\int F(x)\, dx = f(x)$$

$$\int_a^b F(x)\, dx = f(b) - f(a)$$

$$\int_a^b \frac{d f(x)}{dx}\, dx = f(b) - f(a)$$

⇒ The basic format of the fundamental theorem is

"The integral of a derivative over some interval is given by the value of the function at the end points (or boundaries)"

Vector Calculus

Three kinds of vector derivatives.

(i) The Gradienent

(ii) The Divergence

(iii) The Curl

⇒ **The fundamental Theorem for Gradient**

As

$$dT = (\vec{\nabla} T) \cdot \vec{dl}$$

$$\int_a^b dT = \int_a^b (\nabla T) \cdot \vec{dl} = T(b) - T(a)$$

It states:

"The line integral of a derivative (gradient) of a function ∇T over some interval is given by the value of the function at the boundary

Corollary: 1

$$\int_a^b (\vec{\nabla T}) \cdot \vec{dl} \quad \text{is independent}$$

of the path taken from

$a \to b$.

Co-rollary: 2

$$\oint \vec{\nabla} T \cdot \vec{dl} = 0 \qquad \because T(b) = T(a)$$

$\Rightarrow$ The Fundamental Theorem for divergence.

Also called Gauss's or divergence theorem.

Mathematical.

$$\int_V (\vec{\nabla}\cdot\vec{V})\, d\tau = \oint_S \vec{V}\cdot d\vec{a}$$

States:

"The integral of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surfaces that bounds the ~~surface~~. volume)

<u>Fundamental theorem for curl</u>

Also called Stokes's theorem.

$$\int_{surface} (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = \oint_{boundary\ line} \vec{V}\cdot d\vec{l}$$

In this case integral of a

18

derivative (here, the curl) over a region (here a patch of surface) is equal to the value of the function at the boundary (here the perimeter of the patch).