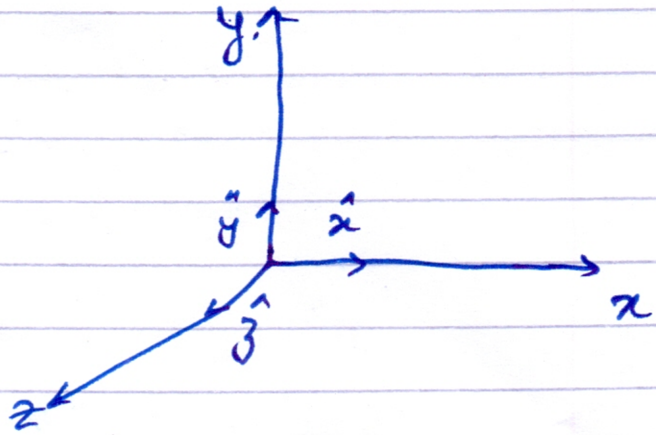
Cartesian Co-ordinates

Vector $\bar{A}$



$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

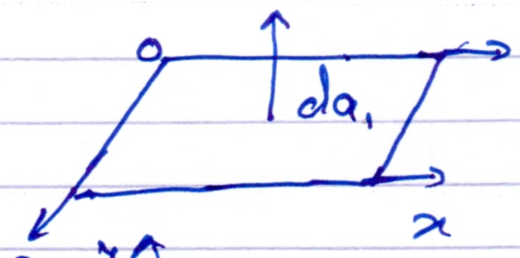
Element of length in x-direction

$\overrightarrow{dl_1}$ $\qquad \overrightarrow{dl_1} = dx\,\hat{x}$

in y-direction $\qquad \overrightarrow{dl_2} = dy\,\hat{y}$

" z- " $\qquad \overrightarrow{dl_3} = dz\,\hat{z}$

Element of Area

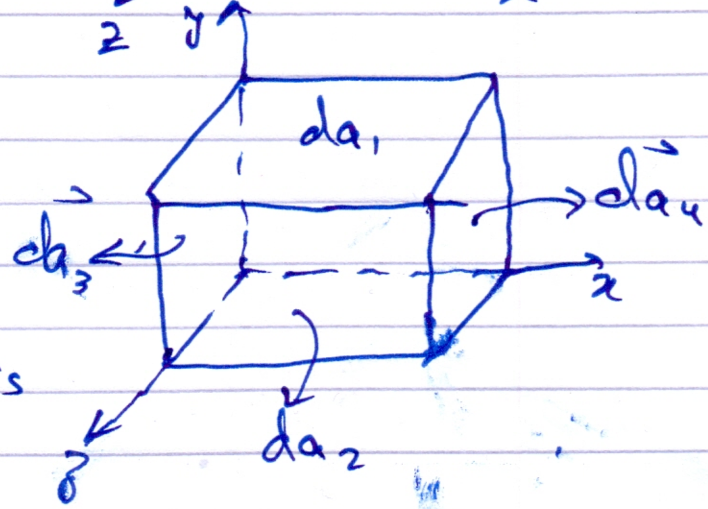


$\overrightarrow{da_1} = dx\,dz\,\hat{y}$

$\overrightarrow{da_2} = dx\,dy\,\hat{z}$

$\overrightarrow{da_3} = dy\,dz\,(-\hat{x})$

$\overrightarrow{da_4} = dy\,dz\,(\hat{x})$



Element of Volume is

$d\tau = dx\,dy\,dz$

Gradient

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Divergence

$$\bar{\nabla} \cdot \bar{V} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(V_x\hat{x} + V_y\hat{y} + V_z\hat{z}\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Divergence of a vector is a scalar quantity.

Curl

Curl of a vector function $\vec{V}$ is given by.

$$\bar{\nabla} \times \bar{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Relation with Cartesian Co-ordinates

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

$\phi = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

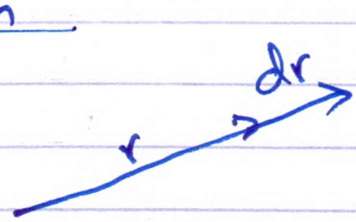
Vector $\bar{A}$ can be written as.

$\bar{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

A_r — radial compt.
A_θ — polar "
A_ϕ — azimuthal "

Elements of length

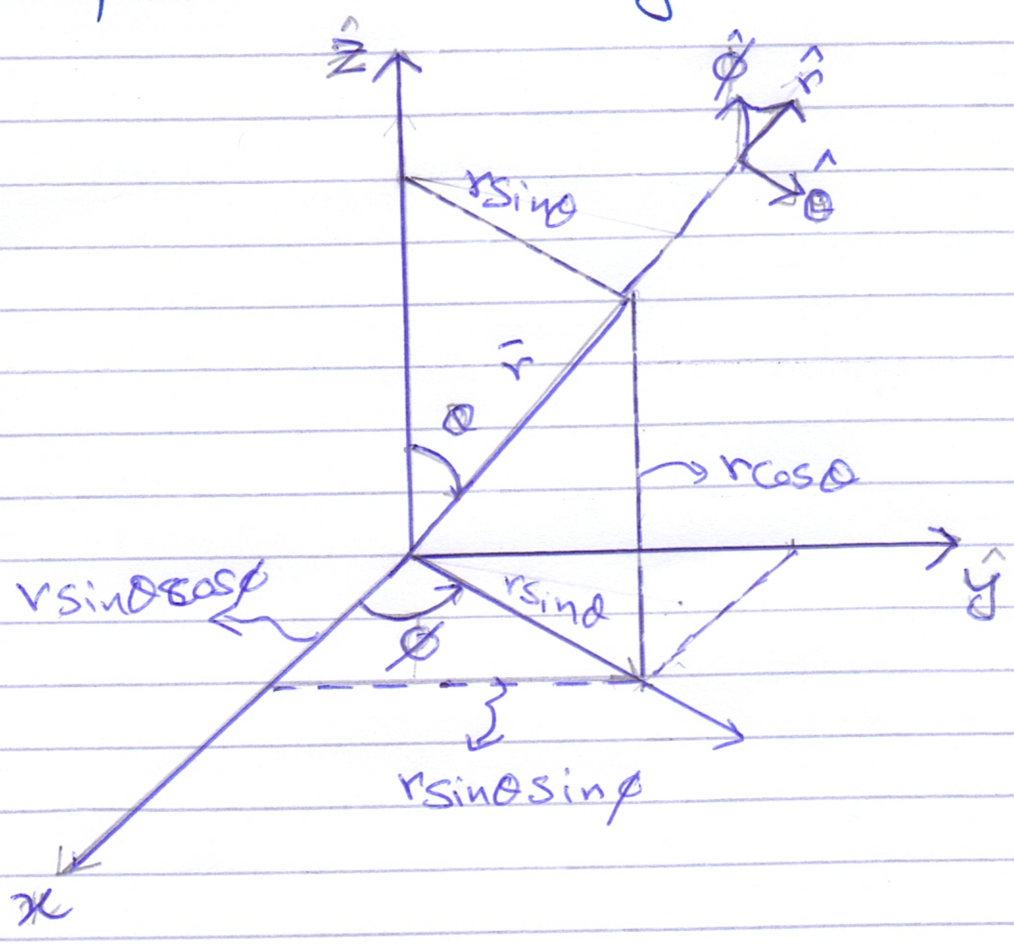
In $\hat{r}$-direction



$\vec{dl_r} = \vec{dr}$

Spherical Polar Co-ordinates

The spherical polar co-ordinates (r, θ, ϕ) of a point P are defined as



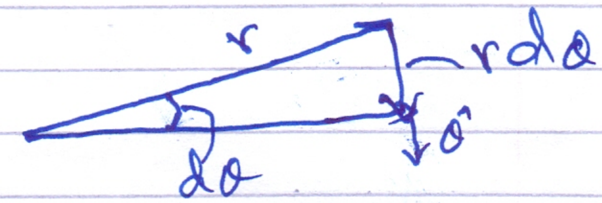
$\bar{r}$ — distance from origin

θ — the angle down from z-axis
Polar angle

ϕ — the around the x-axis
Azimuthal angle.

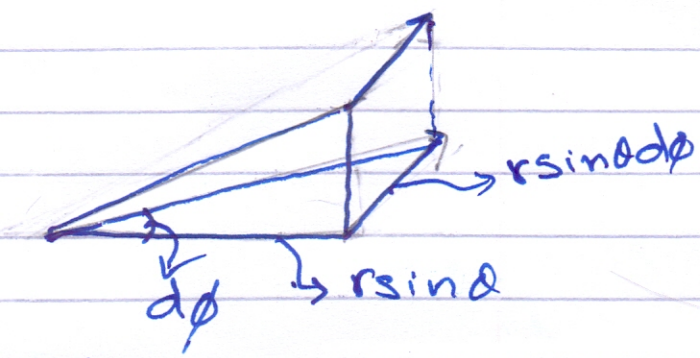
An element of length in θ-directi.

$\vec{dl_\theta} = r\,d\theta\,\hat{\theta}$



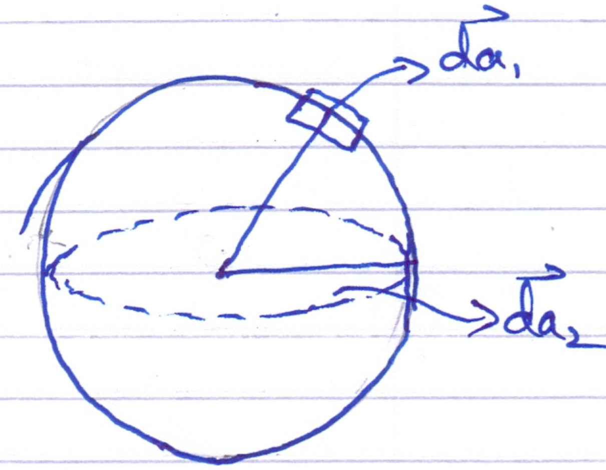
In ϕ - direction.

$\vec{dl_\phi} = r\sin\theta\,d\phi\,\hat{\phi}$



$\Rightarrow \quad \vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

No general expression for element of area $\vec{da}$



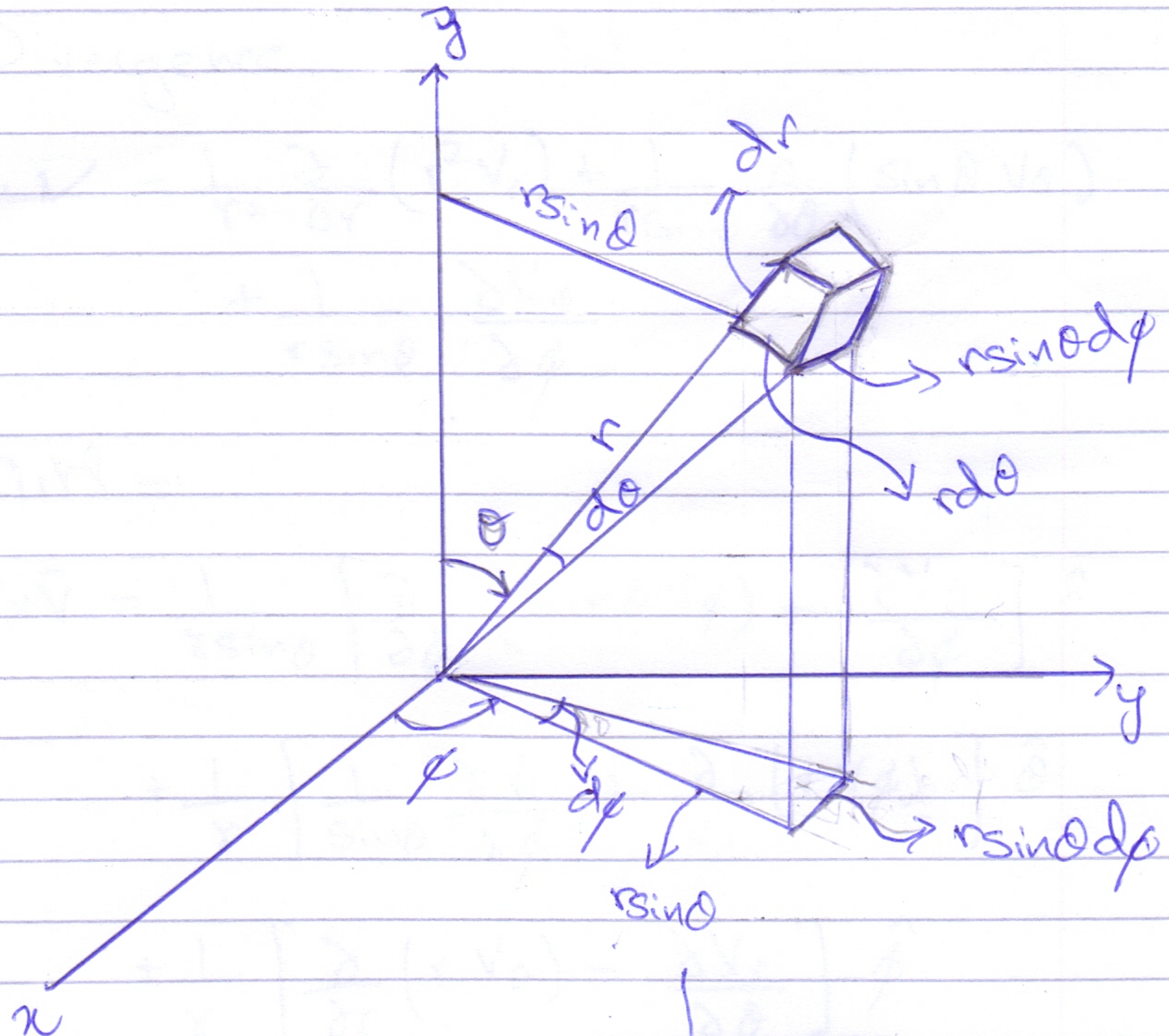
$\vec{da_1} = dl_\theta\,dl_\phi\,\hat{r}$
$= r^2 \sin\theta\,d\theta\,d\phi\,\hat{r}$

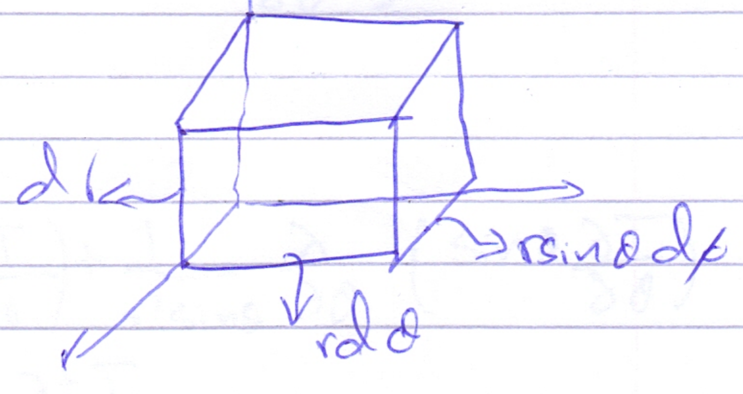
If surface lies in x-y plane

$\vec{da_2} = dl_r\,dl_\phi\,\hat{\theta} = r\sin\theta\,dr\,d\phi\,\hat{\theta}$

Volume element is

$$d\tau = dl_r \, dl_\theta \, dl_\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi$$





Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

Divergence

$$\nabla \cdot V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

Curl =

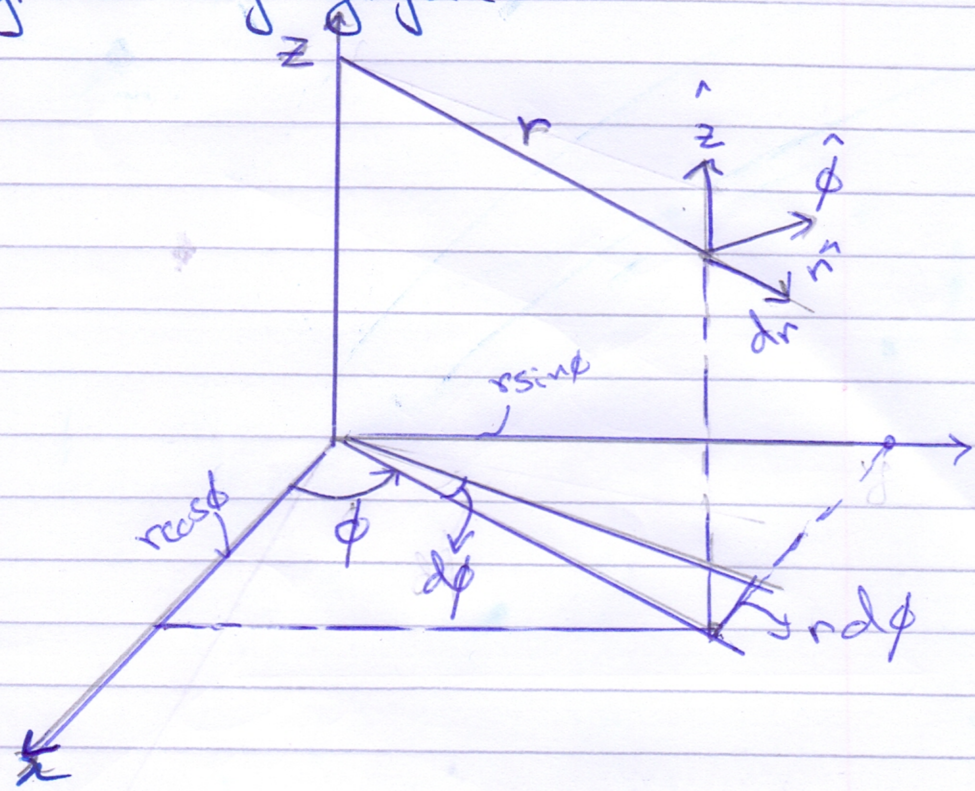
$$\bar{\nabla} \times \bar{V} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, V_\phi) - \frac{\partial V_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial}{\partial r}(r V_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r V_\theta) - \frac{\partial V_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial 6}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Cylindrical Co-ordinates

Cylindrical co-ordinates (r, ϕ, z) for a pt. P are defined by figure.



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

elements of length arc

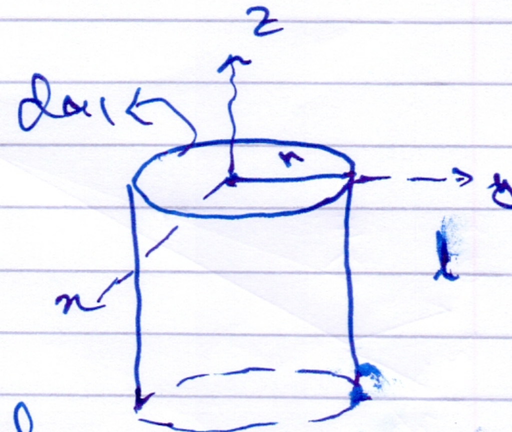
$$\vec{dl}_r = dr\,\hat{r} \qquad \hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\vec{dl}_\phi = r\,d\phi\,\hat{\phi} \qquad \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\vec{dl}_z = dz\,\hat{z} \qquad \hat{z} = \hat{z}$$

$$\vec{dl} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

$r \to 0 \to \infty$

$\phi \to 0 \to 2\pi$

$z \to -\infty \to \infty$



For flat surface

$\vec{da_1} = r\,d\phi\,dr\,\hat{z}$

$\vec{da_2} = r\,d\phi\,dz\,\hat{r}$ (curved surface)

Volume element

$$d\tau = r\,dr\,d\phi\,dz$$

Volume of a cylinder of radius R and height l is

$$V = \int_V d\tau = \int_0^R r\,dr \int_0^{2\pi} d\phi \int_0^l dz = \frac{R^2}{2}(2\pi)(L)$$

$$V = \pi R^2 L$$

The vector derivatives in cylinderical co-ordinates

Gradient:

$$\bar{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

Divergence

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

$$\text{Curl} = \bar{\nabla}\times\bar{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\hat{\phi}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\phi) - \frac{\partial V_r}{\partial \phi}\right]\hat{z}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Tensor Notation:

A vector $\vec{x}$

$$\vec{x} = (x, y, z) = x_i \qquad i = 1, 2, 3$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z$$

$$= \sum_i A_i B_i = A_i B_i$$

repeated indices are summed

Einstein summation convention

$$\Rightarrow \bar{A} \cdot \bar{B} = A_i B_i$$

Cross-Product

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Einstein summation convention

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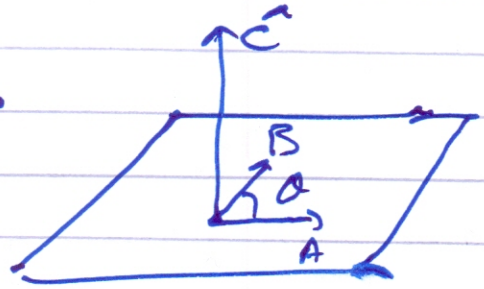
Cross-Product

$$\vec{A} \times \vec{B} = |\bar{A}||\bar{B}| \sin\theta \, \hat{n}$$

$\hat{n}$ — a unit vector $\perp$ to the plane of A & B



$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}(A_y B_z - A_z B_y) - j(A_x B_z - A_z B_x)$$
$$+ \hat{k}(A_x B_y - A_y B_x)$$

Can be written as.

$$(\vec{A} \times \vec{B})_x = (\bar{A} \times \bar{B})_i = (A \times B)_1 = A_2 B_3 - A_3 B_2$$

$(\vec{A} \times \vec{B})_{\hat{i}} = \epsilon_{ijk} A_j B_k$ (Tensor form of a vector product)

ϵ_{ijk} — Levi-Civita symbol.

$$\vec{A} \times \vec{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk}\, \hat{i}\, A_j B_k = \cancel{\sum_j}$$

$$(\bar{A} \times \bar{B})_i = \sum_{jk} \epsilon_{ijk} A_j B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_j B_k$$
$$= \epsilon_{ijk} A_j B_k$$

Totally anti-symmetric tensor.

$\epsilon_{123} = 1$ for cyclic permutation

$\Rightarrow \quad \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

For anti-cyclic order

$\epsilon_{jik} = \epsilon_{kji} = \epsilon_{ikj} = -1$

$\epsilon_{jjk} = \epsilon_{iji} = \cdots = 0$

Kronecker Delta.

$\delta_{ij} = \delta_{ji}$ = Symmetric in two indices.

$$\delta_{ij} = 1 \quad i = j$$
$$= 0 \quad i \neq j$$

$$\sum_i C_i \delta_{ij} = C_i \delta_{ij} = C_j$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

ϵ_{ijk} — a tensor of rank three.

As $\dfrac{\partial x}{\partial y} = 0$ & $\dfrac{\partial x}{\partial x} = 1$

$$\Rightarrow \delta_{ij} = \frac{\partial x_i}{\partial x_j}$$

Vector triple Product

$$\bar{A} \times (\bar{B} \times \bar{C}) = B(\bar{A} \cdot \bar{C}) - C(\bar{A} \cdot \bar{B})$$

called BAC - CAB Rule.

Proof:

$$[\bar{A} \times (\bar{B} \times \bar{C})]_i = \epsilon_{ijk} A_j (\bar{B} \times \bar{C})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m$$

$$= \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m$$

As $\epsilon_{klm} = \epsilon_{lmk} \rightarrow$ cyclic permutat.

$$[\bar{A} \times (\bar{B} \times \bar{C})]_i = \epsilon_{ijk} \epsilon_{lmk} A_j B_l C_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j B_l C_m$$

$$= \underbrace{\delta_{il}}_{i=l} \underbrace{\delta_{jm}}_{j=m} A_j B_l C_m - \underbrace{\delta_{im}}_{i=m} \underbrace{\delta_{jl}}_{j=l} A_j B_l C_m$$

$$= A_j B_i C_j - A_j B_j C_i$$

$$= B_i (A_j C_j) - C_i (A_j B_j)$$

$$= [\bar{B}(\bar{A} \cdot \bar{C}) - C(A \cdot B)]_i$$

$$\Rightarrow A \times (\bar{B} \times \bar{C}) = BAC - CAB$$

Gradient: (In Tensor form)

As

$$\bar{\nabla}\phi(x,y,z) = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$(\bar{\nabla}\phi)_i = \frac{\partial\phi}{\partial x_i} \text{ (ith compt)} = \sum_{i=1}^{3}\hat{x}_i\frac{\partial\phi}{\partial x_i}$$

Divergence:

$$\bar{\nabla}\cdot\bar{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z = \sum_i \frac{\partial A_i}{\partial x_i} = \frac{\partial A_i}{\partial x_i}$$

Curl..

$$\bar{\nabla}\times\hat{A} = \sum \hat{x}_i\,\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}$$

$$(\bar{\nabla}\times\bar{A})_i = \epsilon_{ijk}\frac{\partial A_k}{\partial x_j}$$

1, $\bar{\nabla} \times \bar{\nabla}\phi = 0$

Prove

$$(\bar{\nabla} \times \bar{\nabla}\phi)_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\bar{\nabla}\phi)_k$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k} = -\epsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial \phi}{\partial x_j}$$

using property of antisymmertic tensor

$$= -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k}$$ (In partial differentials order of indices doesnot matter)

$$\Rightarrow \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi = -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k}$$

$$\Rightarrow 2\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k} = 0$$

$$\Rightarrow (\bar{\nabla} \times \bar{\nabla}\phi)_i = 0 \Rightarrow \bar{\nabla} \times \bar{\nabla}\phi = 0$$

2, $\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = \frac{\partial}{\partial x_i} \cdot (\bar{\nabla} \times \bar{A})_i = \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} A_k = -\epsilon_{jik} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} A_k$$

using property of antisymmertic tensor

$$= -\epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} A_k$$ (In partial differentials order of indices doesnot matter)

$$= -\frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k$$

$$\Rightarrow \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k = -\frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k \Rightarrow \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k = 0$$

$$\Rightarrow \bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0$$

3, $\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\bar{\nabla} \cdot A) - \nabla^2 A$

$$(\bar{\nabla} \times (\bar{\nabla} \times \bar{A}))_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times A)_k$$

$$= \epsilon_{ijk} \epsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} A_m$$

As $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

$$(\nabla \times (\bar{\nabla} \times \bar{A}))_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} A_m$$

$$= \underbrace{\delta_{il}}_{i=l} \delta_{jm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} A_m - \underbrace{\delta_{im}}_{i=m} \delta_{jl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} A_m$$

$$= \delta_{jm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} A_m - \delta_{jl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} A_i$$

$$= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} A_i$$

$$= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} A_j - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} A_i$$

$$= \nabla_i (\bar{\nabla} \cdot \bar{A}) - (\bar{\nabla} \cdot \bar{\nabla}) A_i$$

$$= [\nabla (\bar{\nabla} \cdot A)]_i - \nabla^2 A_i$$