①

Maxwell's Equations

→ Set of four equations
→ Describes the behavior of Both the electric and Magnetic fields as well as their interaction with.

Maxwell's four eqn's express.
→ How electric charges produce electric field → Gauss's law.
→ The absence of magnetic monopoles. (some books call it Gauss's law in magnetism)
→ How currents and changing electric fields produces magnetic fields ⇒ Ampere's law with Maxwell's correction "

→ How changing magnetic fields produce electric fields (Faraday's law of induction)

Electrodynamics before Maxwell's Correction.

(i) $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ — Gauss's law

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$ Faraday's Law

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ Ampere's law.

Both Fields in terms of scalar potential V and vector potential $\vec{A}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

&

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

(3)

Apply divergence to eqn (iii)

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{E}) = \bar{\nabla}\cdot\left(-\frac{\partial \bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\cdot B) \quad (v)$$

LHS is zero $\because$ divergence of a curl is zero.

RHS is zero $\because$ $\vec{\nabla}\cdot\vec{B} = 0$

Now apply divergence to eqn (iv),

$$\underbrace{\bar{\nabla}\cdot(\nabla\times B)}_{\text{zero}} = \mu_0 \underbrace{(\bar{\nabla}\cdot\bar{J})}_{\nabla\cdot J = 0} \quad (vi)$$

zero

$\nabla\cdot J = 0$ for magnetostatic i.e for steady currents

$\Rightarrow$ Beyond magnetostatic Ampere's law cannot be right In electrodynamics from equation of continuity (conservation of charge)

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \quad (vii)$$

$\Rightarrow$ eqn (iv) is not valid for non-steady currents or in electrodynamics.

1

Maxwell's correction to Ampere's law.

From Gauss's law

$$\vec{\nabla}\cdot\epsilon_0\vec{E} = \rho$$

Take time derivative

$$\frac{\partial}{\partial t}\left(\bar{\nabla}\cdot\epsilon_0\bar{E}\right) = \frac{\partial\rho}{\partial t}$$

$$\Rightarrow \frac{\partial\rho}{\partial t} = \bar{\nabla}\cdot\epsilon_0\frac{\partial\bar{E}}{\partial t}$$

using this in equation we get.

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \mu_0(\bar{\nabla}\cdot\bar{J}) + \mu_0\nabla\cdot\left(\epsilon_0\frac{\partial E}{\partial t}\right)$$

$$= \mu_0\left(-\frac{\partial\rho}{\partial t}\right) + \mu_0\nabla\cdot\left(\epsilon_0\frac{\partial E}{\partial t}\right)$$

$$= \mu_0\left(-\frac{\partial\rho}{\partial t}\right) + \mu_0\left(\frac{\partial\rho}{\partial t}\right)$$

$$0 = 0$$

$\Rightarrow$ Ampere's law and the conservation of charge eqn. suggest that there are actually two sources of magnetic field.

5

⇒ The current density and displacement current

$$\frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon_0 E) \rightarrow \text{Displacement Current.}$$

⇒ Ampere's law with Maxwells Correction.

$$\bar{\nabla} \times \vec{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

⇒ Maxwells fixed Ampere's law on pure theoretical arguments

⇒ In Maxwells time there was no experiment reason to doubt that Ampere's law was of wider validity.

→ Maxwells called extra term ~~as~~ the displacement current.

$$J_d \equiv \epsilon_0 \frac{\partial E}{\partial t}$$

it has nothing to do with current except that it adds to J in Amp. law

Paradox of charging Capcitor.

→ If the capacitor plates are very close togethe — the Electric field between them is

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{Q}{A}$$

Q — charge on the plate

A — is plate's area.

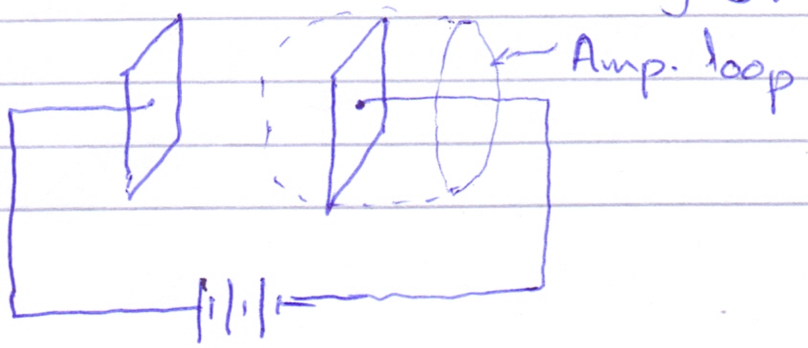
$$\Rightarrow \quad \frac{\partial E}{dt} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

From eqn.

$$\bar{\nabla} \times \bar{B} = \mu_0 J_{enc} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

in integral form.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial E}{\partial t}\cdot da$$



7

For flat surface $E = 0$ and $I_{enc} = I$

For balloon-shaped surface $\rightarrow I_{enc} = 0$

but $\int \frac{\partial E}{\partial t} \cdot da = \frac{I}{\epsilon_0}$

we get same answer in both cases

or for either surface.

→ In first case it comes from the genuine current and ~~it~~ in the second from the displacement current.

General Form of Maxwells Equations

Differential Form		Integral Form
i) $\vec{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0}$	Divergence theorem →	$\oint \vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0}\int_V \rho\, d\tau$
ii) $\vec{\nabla}\cdot\vec{B} = 0$	"	$\oint \vec{B}\cdot\vec{da} = 0$
iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\bar{B}}{\partial t}$	Stoke's theorm →	$\oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{da}$
iv) $\bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}$		$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc} + \mu_0\epsilon_0\frac{d}{dt}\int \vec{E}\cdot\vec{da}$

Together with the Force law.

$$\vec{F} = q\bar{E} + q(\bar{v}\times\bar{B}) \quad \text{Lorentz force}$$

⇒ It tells how fields affect charges.

and Equation of Continuity.

$$\nabla\cdot J = -\frac{\partial\rho}{\partial t}$$

Summarize the entire theoretical content of classical electrodynamics

Maxwell's Equations:

→ Eqns. i2, & iii, in this form reinforce the notion that

⇒ Electric fields can be produced either by charges (ρ), or by changing M.F $\left(\frac{\partial B}{\partial t}\right)$

And

⇒ Magnetic fields can be produced either by currents (J) or by changing E.F $\left(\partial E/\partial t\right)$

⇒ This is misleading because $\left(\frac{\partial E}{\partial t}\right)$ & $\left(\frac{\partial B}{\partial t}\right)$ are themselves due to charges and currents

Logical approach can be

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\bar{\nabla} \times \bar{E} + \frac{\partial B}{\partial t} = 0$$

$$\bar{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 \bar{J}$$

⇒ All E.M. fields are ultimatelly attributable to charges and currents

Maxwell's Equations in Vacuum.
No free charge & Current i.e ($\rho = J = 0$)

i. $\bar{\nabla} \cdot \vec{\bar{E}} = 0$

ii. $\bar{\nabla} \cdot \bar{B} = 0$

iii. $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

iv. $\bar{\nabla} \times \bar{B} = \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$

Vacuum is a linear, homogeneous, ~~and~~ isotropic and dispersionless medium.

<u>Linear.</u>

<u>Homogeneous medium:</u> has the same properties at every point $\Rightarrow$ uniform without irregularities.
<u>Isotropic:</u> Having identical values of a property in all directions.
<u>Dispersionless medium</u>
In which all frequencies travels with same velocity. Vacuum is dispersionless for EM waves.

From Gauss's law in dielectric

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_t}{\epsilon_0} = \frac{1}{\epsilon_0}(\rho_f + \rho_b)$$

where $\rho_f \rightarrow$ Free charges

$\rho_b \rightarrow$ bound volume charges.

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho_f - \bar{\nabla}\cdot\bar{P}\right)$$

where $\rho_b = -\bar{\nabla}\cdot\bar{P}$

and $\sigma_b = \vec{P}\cdot\hat{n} \longrightarrow$ Surface bound charges.

$$\vec{\nabla}\cdot\epsilon_0 E + \bar{\nabla}\cdot P = \frac{\rho_f}{\cancel{\epsilon_0}}$$

$$\vec{\nabla}\cdot(\epsilon_0 E + P) = \frac{\rho_f}{\cancel{\epsilon_0}}$$

Define $D \equiv \epsilon_0 E + P$ Electric displacement vector.

$$\Rightarrow \quad \vec{\nabla}\cdot\bar{D} = \rho_f$$

$$\oint D\cdot da = Q_{f\,enc}$$

Using $\vec{P} = \epsilon_0 \chi_e E$

$$\Rightarrow \quad \vec{D} = \epsilon_0 E + \epsilon_0 \chi_e E$$

$$= \epsilon_0 (1 + \chi_e) E$$

12

Ampere's law in Magnetized Materials.

$$\nabla \times B = \mu_0 J$$

The effect of magnetization is to establish bound currents

$\vec{J}_b = \bar{\nabla} \times \bar{M}$. within the material

and $\vec{K}_b = \vec{M} \times \hat{n}$ on the surface.

$$\Rightarrow \quad J = \vec{J}_f + \vec{J}_b$$

free current is due to wire connected to a battery

The bound current is there because of magnetization.

$$\frac{1}{\mu_0}(\bar{\nabla} \times \bar{B}) = \bar{J} = \bar{J}_f + \bar{J}_b = \bar{J}_f + (\bar{\nabla} \times \bar{M})$$

$$\bar{\nabla} \times \left(\frac{B}{\mu_0} - M \right) = \bar{J}_f \quad ; \quad H = \frac{1}{\mu_0} B - M$$

$$\nabla \times H = \bar{J}_f \quad ; \quad \oint H \cdot d\vec{l} = I_{f\,enc}$$

As $\quad M = \chi_m H$

$$\Rightarrow \quad \vec{B} = \mu_0 H + \mu_0 \chi_m H = \mu_0 (1 + \chi_m) H$$

$$= \mu H .$$

Maxwell's Equations inside Matter.

⇒ Eqns. are modified for polarized and magnetized materials.

For linear materials

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

The fields $\vec{E}$ & $\vec{B}$ are related with $\vec{D}$ & $\vec{H}$ by.

$$\vec{D} = \epsilon_0 E + P \Longrightarrow (1+\chi_e)\epsilon_0 E = \epsilon E$$

where $\frac{\epsilon}{\epsilon_0} = \epsilon_r = (1+\chi_e) \rightarrow$ Dielectric Constant.

and

$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = (1+\chi_m)\mu_0 H = \mu H$$

$\chi_e \longrightarrow$ electric susceptibility of material

$\chi_m \longrightarrow$ magnetic " " "

In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current, J_p.

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

Polarization current density is due to linear motion of charge when the Electric polarization changes.

$\Rightarrow$ Total charge density is

$$\rho_t = \rho_f + \rho_b$$

Total current density is

$$J_t = J_f + J_b + J_p$$

and

$$J_D = \epsilon_0 \frac{\partial E}{\partial t}$$

Maxwell's Equations Inside Matt q.

i) $\vec{\nabla}\cdot\vec{E} = \frac{\rho_t}{\epsilon_0} \longrightarrow \vec{\nabla}\cdot\vec{D} = \rho_f$ where $\bar{D} = \epsilon_0\bar{E} + \bar{P}$
& $\vec{D} = \epsilon\bar{E}$

ii) $\bar{\nabla}\cdot\bar{B} = 0$

iii) $\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$

(iv)
$$\vec{\nabla}\times\vec{B} = \mu_0 J_f + \mu_0 J_b + \mu_0 J_p + \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}$$
$$\bar{\nabla}\times\frac{\vec{B}}{\mu_0} = \bar{J}_f + \bar{\nabla}\times\bar{M} + \frac{\partial\vec{P}}{\partial t} + \epsilon_0\frac{\partial\bar{E}}{\partial t}$$
$$\bar{\nabla}\times\frac{\bar{B}}{\mu_0} = \bar{J}_f + \bar{\nabla}\times\bar{M} + \frac{\partial}{\partial t}(\epsilon_0 E + P)$$
$$\nabla\times\left(\frac{\bar{B}}{\mu_0} - \bar{M}\right) = \bar{J}_f + \frac{\partial D}{\partial t}$$
$$\bar{\nabla}\times\bar{H} = \bar{J}_f + \frac{\partial D}{\partial t}$$
; $H = \frac{B}{\mu_0} - M$
& $B = \mu H$

In non-dispersive, isotropic media ϵ & μ are time independent ~~scalars~~ and space independent.

$\bar{\nabla}\cdot\epsilon\bar{E} = \rho$

$\bar{\nabla}\cdot\mu H = 0$

$\bar{\nabla}\times\bar{E} = -\mu\frac{\partial H}{\partial t}$

$\bar{\nabla}\times\bar{H} = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}$