

# Statistical Mechanics of Two Dimensional Critical Curves

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By

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## 1. Motivation

### 1.1. Scale free random paths in 2d

### 1.2. Overview

### 1.3. Critical phenomena

Order/Disorder transition

Spontaneous symmetry breaking

Order parameter

Ergodic theorem

Critical exponents

Universality

Scale invariance

The Renormalization group

Conformal invariance

Fractals

### 1.4. Two dimensional Ising model

Order parameter

Ergodicity breaking

Critical exponents

RG

Scale and conformal invariance

Fractal dimension of cluster boundaries

### 1.5. Percolation

1d percolation

Continuous critical phenomena

Order Parameter

Fractal structure

Conformal invariance

## 2. Schramm-Loewner Evolution (SLE)

### 2.1. Loewner evolution

## 2.2. Some simple Loewner curves

The slit map

Other paths

## 2.3. Stochastic paths

Brownian motion and its characterizing properties

Conformal invariance of Brownian motion

## 2.4. Schramm-Loewner Evolution

## 2.5. Properties of SLE

Conformal invariance

Domain Markov

## 2.6. Phases of SLE

## 2.7. Scaling invariance

## 2.8. Locality

## 2.9. Restriction

## 2.10. Duality

## 2.1. SLE as a stochastic process and CFT connection

SLE/ CFT correspondence

## 2.2. Variants of SLE

Radial SLE

SLE( $\kappa, \rho$ )

## 3. Calculations with SLE

### 3.1. Left-passage probability

### 3.2. Winding number and radial SLE

### 3.3. Cardy's formula for crossing probability

### 3.4. Fractal dimension of SLE traces

### 3.1. Discretize the SLE path

### 3.2. Some examples of SLE

O(n) model

Percolation

Harmonic explorer

q-state Potts model

Random Cluster Model

## 4. Growth and roughness of surfaces

Correlation length

### 4.1. Growth Models

Random deposition model

Ballistic Model

Eden Model  
Diffusion Limited Aggregate (DLA)  
4.2. Continuum growth models  
Edwards-Wilkinson model  
Kardar-Parisi-Zhang equation  
4.3. Sand pile model  
Uniform Spanning Trees  
4.4. Level sets of random surfaces  
Scaling relations  
Loop correlation exponent  
Connection with SLE  
4.5. Some examples of level sets  
Gaussian free field (GFF)  
Percolation  
KPZ surface  
WO3 surface  
Sand pile model  
Watersheds  
Watersheds are SLE traces  
5. Loop models  
5.1. Ising model at high temperature  
5.2.  $O(n)$  model  
5.3. FK clusters and geometric exponents  
5.4. Brownian loop soup  
5.5. Conformal loop ensemble (CLE)  
5.6. Double dimer model  
The dimer model  
Double dimer model

### General References:

Specific references have been given in the main text of my lecture notes but for students who like to read reviews on the topics covered in these lectures the following references are suggested:

### **Critical Phenomena:**

- Sornette, D. Critical phenomena in natural sciences: chaos, fractals, self-organization and disorder: concepts and tools. Springer Science & Business Media, 2006.
- Stanley, H. Eugene. "Scaling, universality, and renormalization: Three pillars of modern critical phenomena." *Reviews of modern physics* 71.2 (1999): S358.

### **Conformal Field Theory**

- Schellekens, A. N. "Introduction to conformal field theory." *Fortschritte der Physik/Progress of Physics* 44.8 (1996): 605-705.
- Francesco, Philippe, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Springer Science & Business Media, 2012.

### **The renormalization group**

- Goldenfeld, Nigel. *Lectures on phase transitions and the renormalization group*. CRC Press, 2018.

### **SLE**

- 2D growth processes: SLE and Loewner chains, Michel Bauer and Denis Bernard, arxiv:0602049v1
- SLE for Theoretical Physicists , John Cardy, arxiv: 0503313

### **Surface growth**

- Barabási, A-L., and Harry Eugene Stanley. *Fractal concepts in surface growth*. Cambridge university press, 1995.
- Family, Fereydoon, and Tamás Vicsek. *Dynamics of fractal surfaces*. World Scientific, 1991.

### **Loop models and conformal loop ensembles**

- Duplantier, Bertrand. "Two-dimensional fractal geometry, critical phenomena and conformal invariance." *Physics reports* 184.2-4 (1989): 229-257.
- Gruzberg, Ilya A. "Stochastic geometry of critical curves, Schramm–Loewner evolutions and conformal field theory." *Journal of Physics A: Mathematical and General* 39.41 (2006): 12601.