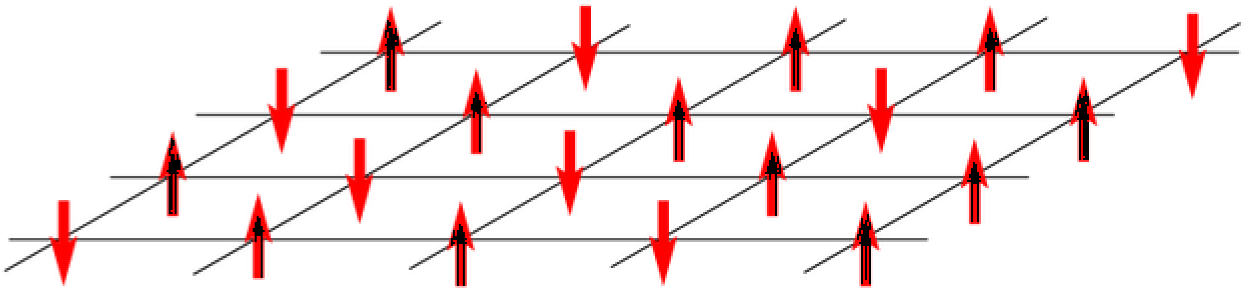
Statistical Mechanics of Two Dimensional Critical Curves
Ising Model and Percolation Problem

Shahin Rouhani
Physics Department
Sharif University of Technology
Tehran, Iran.

Scale invariance in the critical Ising model

- Let us illustrate some of previous ideas by the example of 2d Ising model.



2d Ising model

The two dimensional Ising model on a square lattice is defined by the Hamiltonian:

$$H = -J \sum_{\langle ij \rangle} \sigma_i \, \sigma_j - h \sum_i \sigma_i$$

Spins $\sigma_i = \pm 1$ sit on the nodes of a square lattice referred to by the compound index $i = (i_x, i_y)$.

2d Ising model

The Ising model (for *h=0*) is invariant under the action of the group $\mathbb{Z}_2$:

$$\sigma_i \to -\sigma_i$$

The order parameter is the mean magnetization:

$$M = \frac{1}{N}\sum_i \langle \sigma_i \rangle$$

N is the number of nodes in the lattice. It is clear that for high enough temperatures M vanishes due to the $\mathbb{Z}_2$ symmetry.

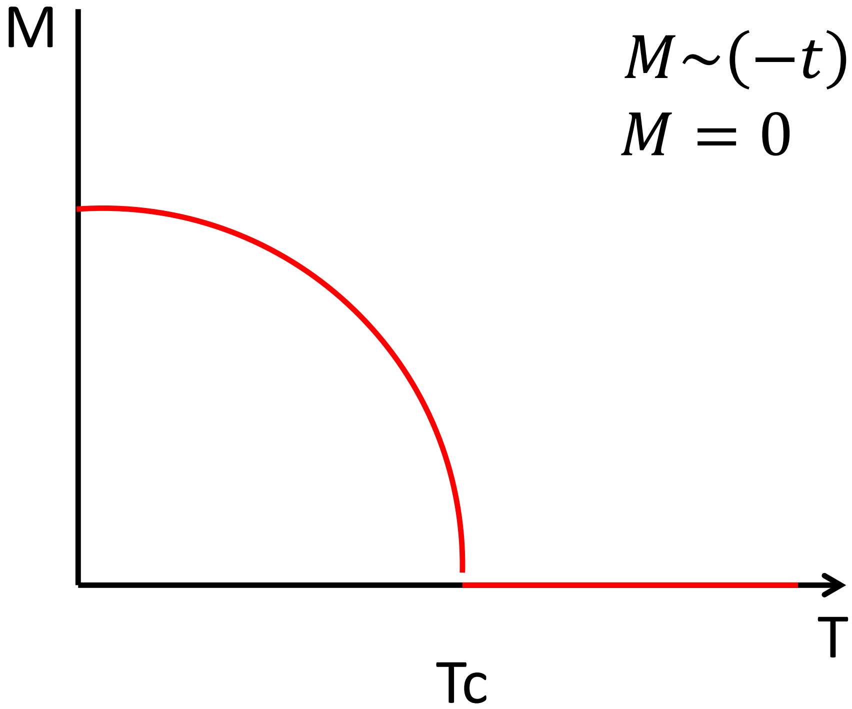
But for h=0 temperatures below T_c Magnetization is non zero.

$$\beta_c J = 1/2\ log(1+\sqrt{2}) \approx 2.269$$

2d Ising model
Magnetization

$$M = \frac{1}{N} \sum_i \sigma_i$$

$$M \sim (-t)^{\beta} \;, T < T_c$$
$$M = 0 \qquad , T > T_c$$



Ergodicity breaking in 2d Ising model

At low temperatures the symmetry breaks, and M can be nonzero. This is because the averaging of M is over half of the phase space

The lowest energy state is one in which all spins are aligned:

$$\varphi_+ = \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \cdots$$

All the excite sates are built on top of this ground state.

But there is another ground state in which all spins are aligned too, but point down:

$$\varphi_- = \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \cdots$$

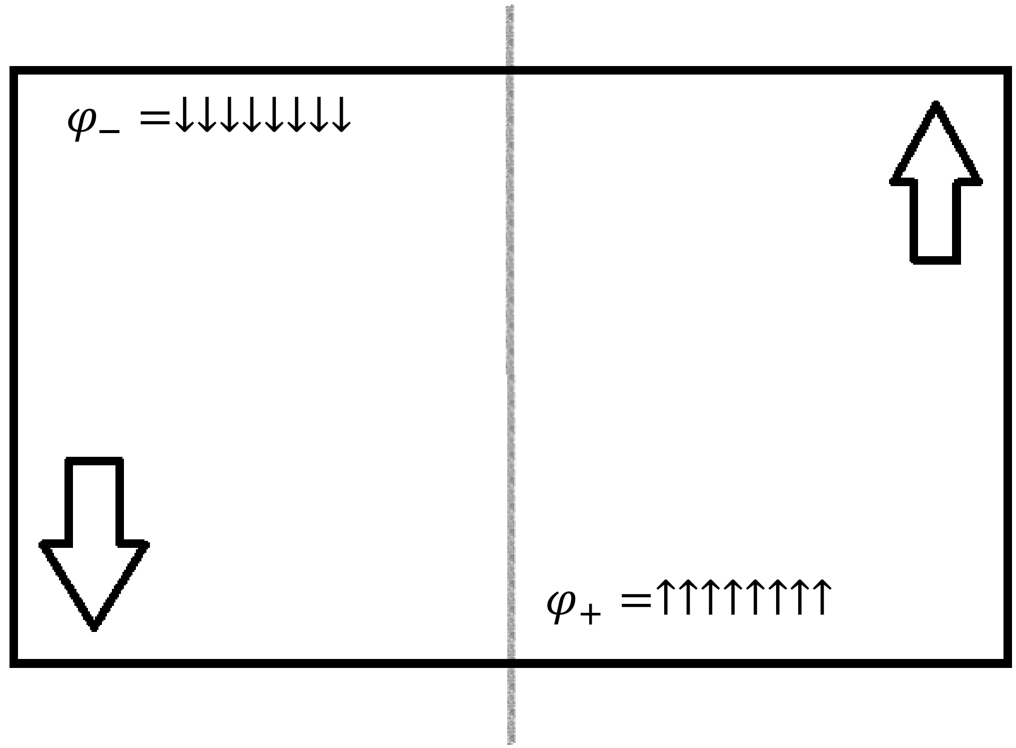
2d Ising model

Clearly these two states are connected by the action of the group $\mathbb{Z}_2$.This is a classical case of SSB, the system has to choose one of the two points as its ground state say φ_+. Now the dynamics of the system will create a phase space around φ_+:

$$\Omega_+ = \lim_{t\to\infty} T_t \varphi_+$$

Ergodicity breaking in 2d Ising model

$M = \frac{1}{N} \sum_{\sigma_i \in \varphi_+} \sigma_i \neq 0$



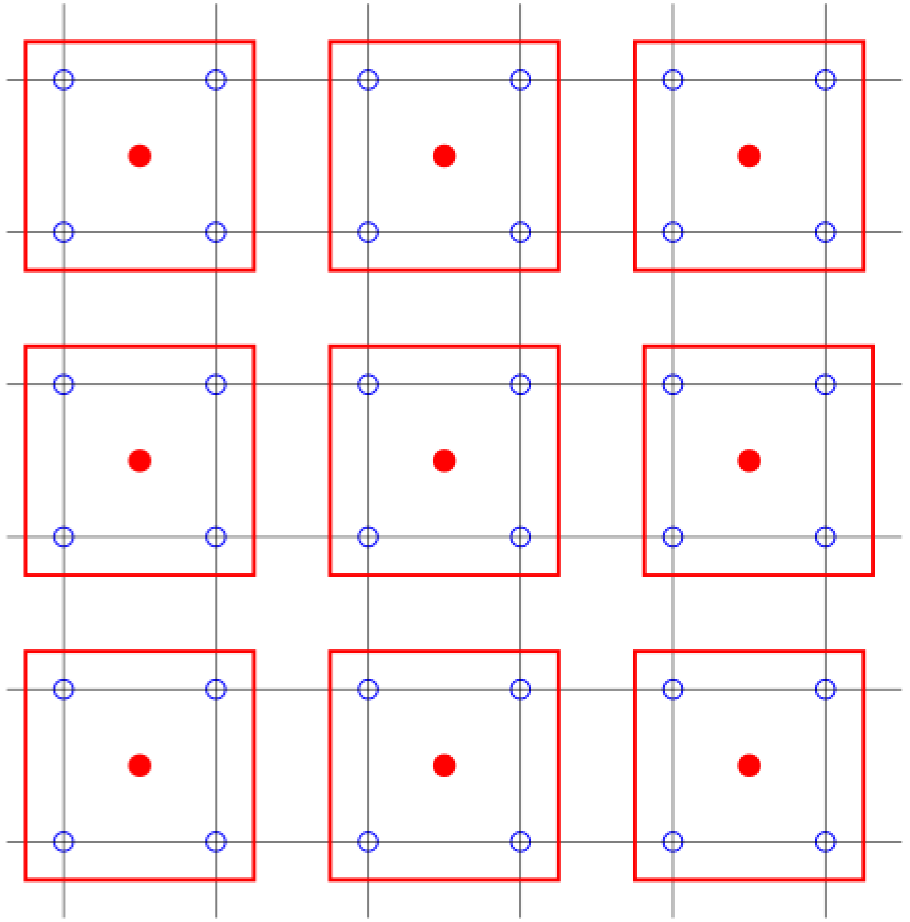
The configuration space breaks into two parts each with a ground state of aligned spins and excitations ($T<T_c$)

Critical Exponents Ising Model

exponent	d=2	3	4
α	0	0.11008	0
β	1/8	0.326419	½
γ	7/4	1.237075	1
δ	15	4.78984	3
η	1/4	0.036298	0
ν	1	0.629971	1/2

RG

Block spin renormalization happens by
summing group of spins over the cell (here blue) and replacing them into the center of the cell (here red).
The lattice spacing increases (here doubles) .
Interactions in the Hamiltonian become more complex but we hope that near the fixed point only the relevant interactions survive ie the shape of the interaction does not change



The actual process of explicitly constructing a useful renormalization group is not trivial.
Michael Fisher

RG
Niemeijer–van Leeuwen Cumulant Approximation

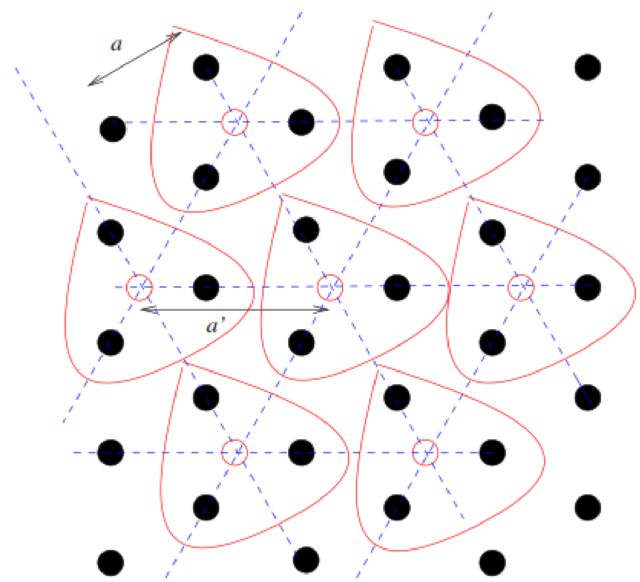
The easiest way to see the effect of block summation is over a triangular lattice for Ising model in 2d. We take the following steps:

1-The lattice is divided, as shown in figure, into triangular plaquettes. A spin variable S_I is associated to each plaquette by majority rule: $s_I = sign[s1 + s2 + s3]$

2-The number of plaquettes is N/3 and the new lattice spacing is $a' = a/\sqrt{3}$.

3-The Hamiltonian will have interactions among spins of the same plaquette and spins belonging to two neighboring plaquettes

4-$H = \sum_I h_1(I) + \sum_{<IJ>} h_2(I,J)$

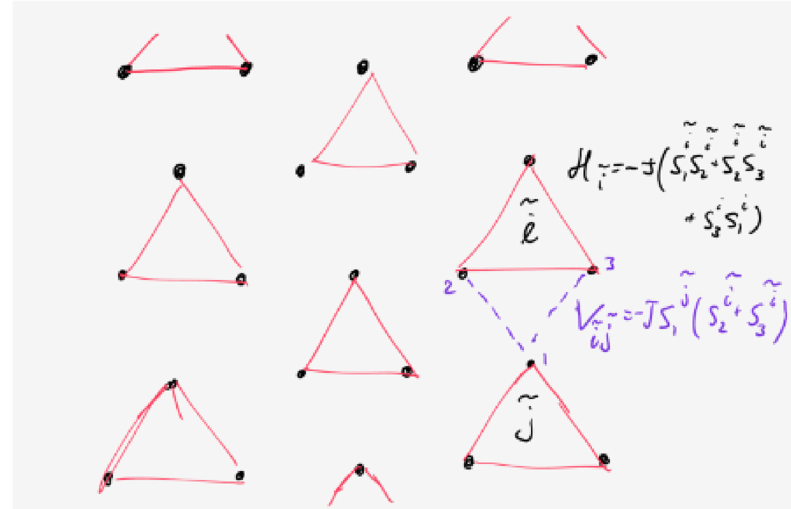


5-Now the partition function should be re-written as a sum over all S_I spins:
$Z = \sum_{S_I} \sum_{s_i} e^{-\beta H[s_i]}$.

6- Let $Z' = \sum_{s_i} e^{-\beta h_1[s_i]}$, $Z = \sum_{S_I} Z' \langle e^{-\beta h2} \rangle_1$,
where $\langle \blacksquare \rangle_1 = \frac{1}{Z'} \sum \blacksquare\, e^{-\beta h_1[s_i]}$

7-now show $\langle s_i \rangle_1 = S_I \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$

RG



$$Z = \sum_{S_I} e^{K' \sum_{\langle IJ \rangle} S_I S_J}$$

- $K' = 2K\left(\frac{e^{3k}+e^{-K}}{e^{3K}+3e^{-K}}\right), K^* = 0.335..,$
- $\frac{\partial K'}{\partial K}\,|_{K^*} \cong 1.264 \sim \sqrt{3}^{y_t}$

y_t=0.883

$$\nu = \frac{1}{y_t} = 1.13$$

Scale Invariance

The spin-spin correlation function becomes :

$$\langle \sigma(i)\sigma(j) \rangle \sim e^{-|i-j|/\xi}$$

and the correlation length ξ is given by

$$\xi \sim t^{-1}$$

t= reduced temperature

Scale Invariance

Ear the critical point $t \to 0$ the correlation length diverges:

$$\xi \to \infty$$

Hence the spin-spin correlation function for the 2d Ising model becomes :

$$\langle \sigma(i)\sigma(j) \rangle \sim |i - j|^{-1/4}$$

Conformal Field Theory

The action for 2d Ising model is:

$$\int (\psi\bar{\partial}\,\psi + \overline{\psi}\partial\overline{\psi}\,)$$

where ψ is a fermionic field

Conformal Field Theory

This corresponds to the smallest CFT in the minimal series and the field , has a scaling dimension of ½, leading to the propagator:

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{z-w} \qquad ,$$

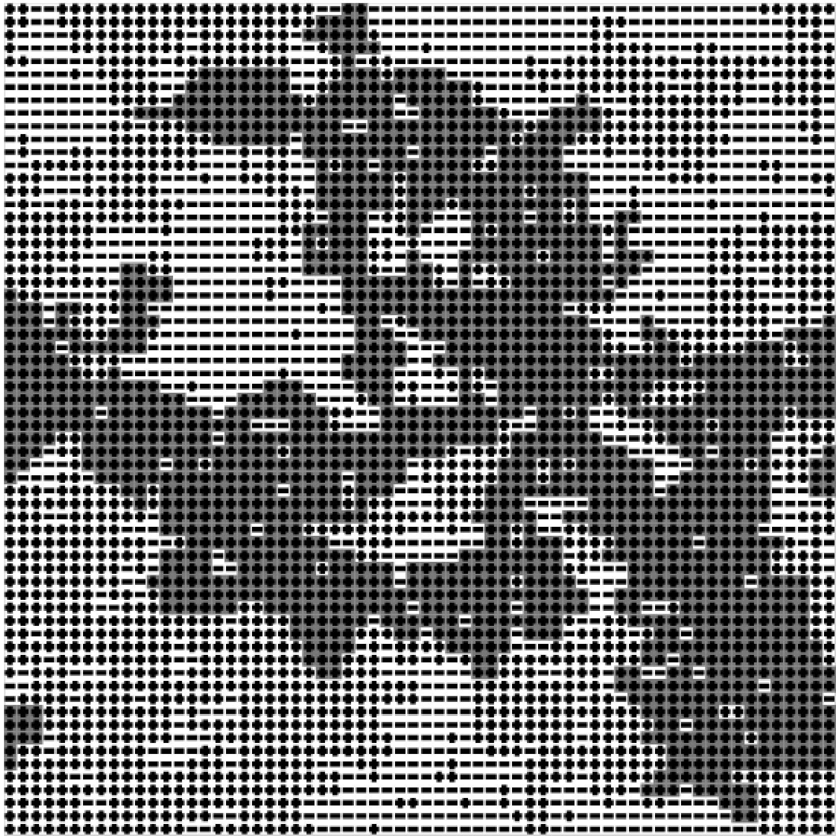
This is the energy operator, where as the spin operator s has scaling dimension 1/8

$$\langle \sigma(z)\sigma(w) \rangle = \frac{1}{(z-w)^{1/4}}$$

Therefore we have critical exponent η=1/4 .

Fractal Dimension of the boundary of spin clusters

- The boundary of a spin cluster near criticality is a fractal.
- What is its’ fractal Dimension?
- Cluster is defined as the set of connected like spins
- Which boundary?



Percolation

What is the probability that water (any liquid) can gradually filter through (percolate) soil or rock.

Or how long will it take to form a sinkhole.

The Red Lake sinkhole in Croatia.
credit Wikipedia

Percolation problem

- On a given graph G, bonds (or sites) are turned on with probability p. What is the threshold Pc beyond which a global cluster can be seen ?

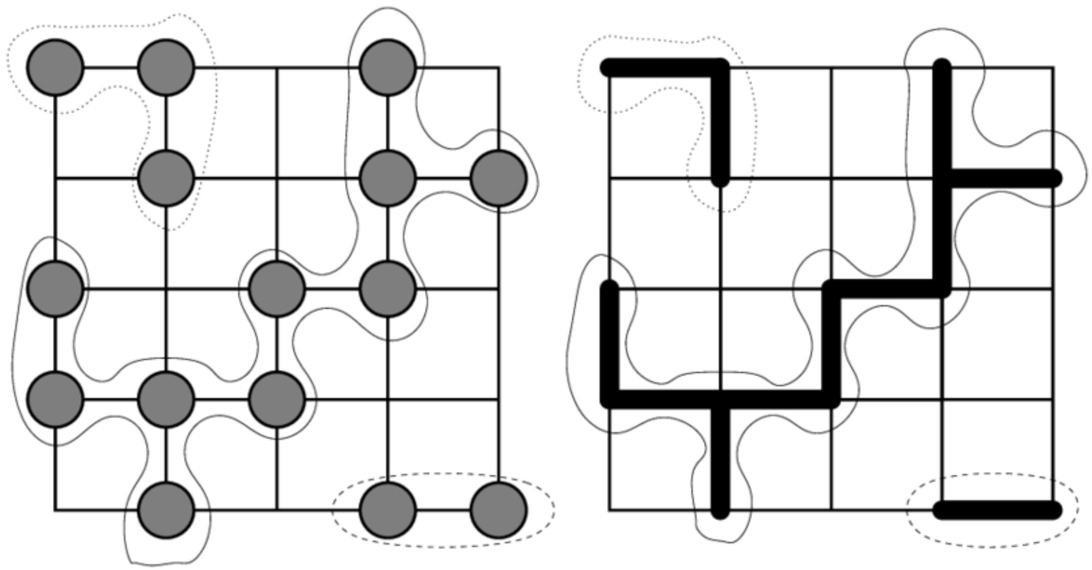


Image credit;
Rudolf Andreas Roemer,
Research gate

Bond Percolation

Three snap shots for three different parameters, respectively (top to bottom)
sub-critical, critical and super-critical.

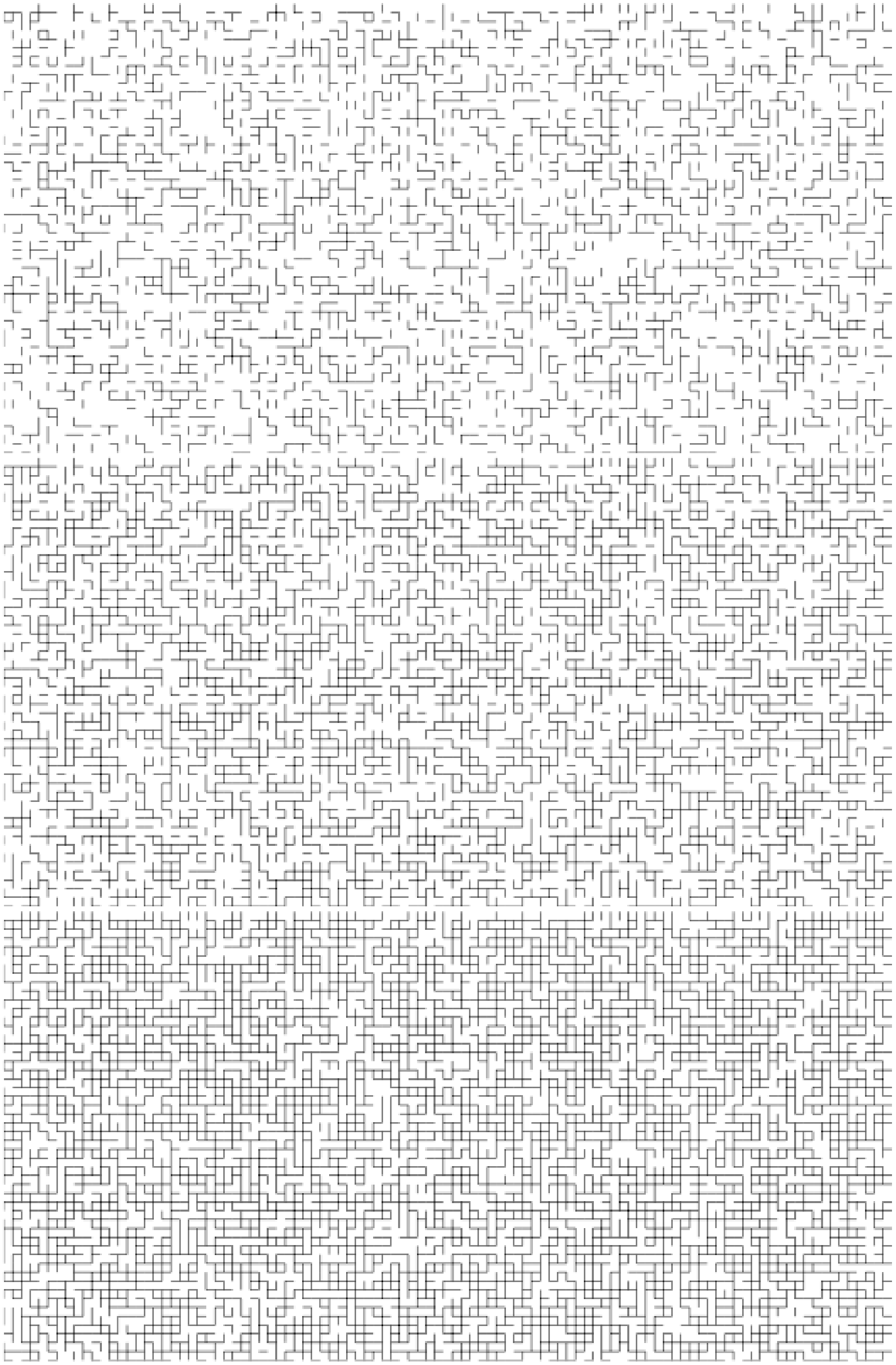


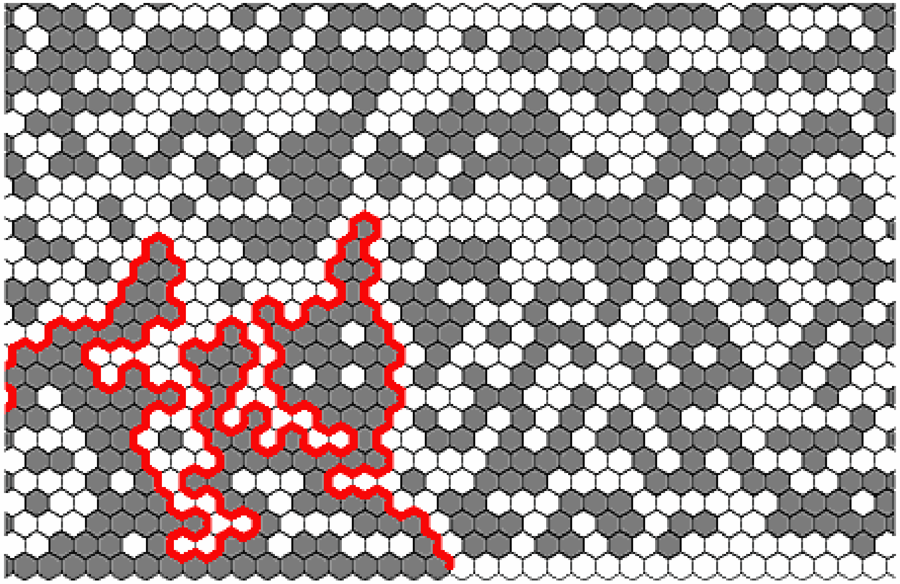
Image from:

Critical point and duality
in planar lattice models
Vincent Be
ara Hugo Duminil-Copin

Percolation problem

Is the description of the behavior of connected clusters in a random graph.

Bond percolation on a hexagonal lattice. The red line is the boundary between filled and empty hexagons. Boundary condition is set such that the path starts at origin.



1d percolation

- It is easy to accept that in 1d just one type of lattice can exist and critical p_c is 1.

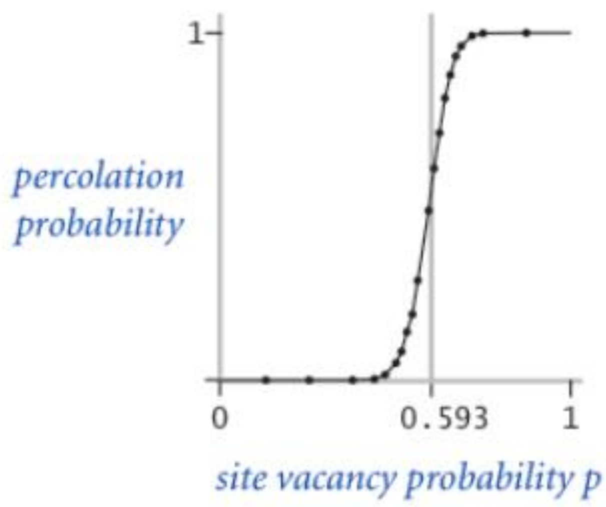
2d percolation

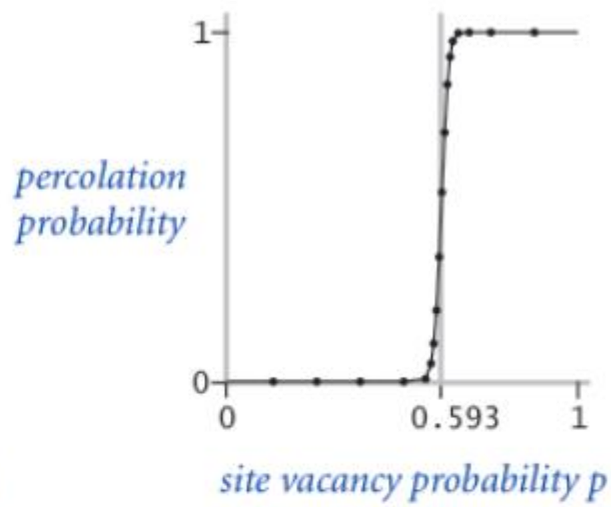
Lattice type	Coordination number	Site percolation	Bond percolation
1d	2	1	1
2d Honeycomb	3	0.69..	1-2sin(p/18)=0.65..
2d Square	4	0.59..	0.5
2d Triangular	6	0.5	2sin(p/18)=0.34..
3d Simple cubic	6	0.31..	0.25..

Kim Christensen, ""Percolation theory.," Imperial College London, , London , 40, 2002

Order parameter

is the mass of the largest cluster P_∞





Pc is found by Monte Carlo simulation 20x20 and 100x100 lattices. github

Exponents for 2d standard percolation.

Quantity	Behavior near criticality	Value for standard percolation in 2d
Mean cluster number per site	$\lvert p-p_c\rvert^{2-\alpha}$	-2/3
Percolation strength (Probability of finding an infinite cluster)	$P_\infty(p-p_c)\sim \lvert p-p_c\rvert^{\beta}$	5/36
Mean cluster size	$\chi(p-p_c)\sim\lvert p-p_c\rvert^{-\gamma}$	43/18
Probability of an “on” site belonging to a cluster of size s at critical probability	$\omega_s(p_c)\sim s^{-\frac{\delta}{\delta+1}}$	91/5
Probability of two sites distance r apart lying on the same cluster same	$G(r)\sim r^{2-d-\eta}$	5/24
Correlation length	$\xi(p-p_c)\sim\lvert p-p_c\rvert^{-\nu}$	4/3
Cluster Moments ratio	$\lvert p-p_c\rvert^{-\Delta}$	5/4

fractal dimensions

D_h = Hull fractal dimension

D_E = External perimeter fractal dimension

D_h=7/4 , D_E =4/3.

$$(D_h - 1)(D_E - 1) = \frac{1}{4}$$

Next lecture: Schramm-Loewner Evolution