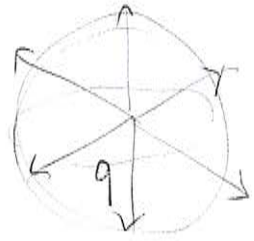
Lecture 1

S. Pascario

Electrostatic. Gauss

$$\mathbb{E} = \frac{q}{r^2}\hat{r}$$



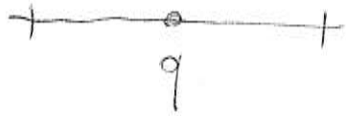
3D

$$\underset{\text{flux}}{\Phi(\mathbb{E})} = \int_\Sigma \mathbb{E}\cdot d\mathbb{S} \underset{\text{spherical symmetry}}{=} \frac{q}{R^2}4\pi R^2 = \underset{S_3}{4\pi Q_{INT}}$$

Why does this work? $\longrightarrow \frac{1}{\cancel{R^2}}\cancel{R^2}$

Let's take this as a guiding principle in $\neq$ dimensions

E.g. 1D



$$\Sigma = \{\text{2 points}\}$$

$$\Phi(\mathbb{E}) = E + E = \underset{S_1}{2q} \longrightarrow E = q = \text{const}$$

E.g. 2D

$$\Phi(\mathbb{E}) = \int_\Sigma \mathbb{E}\cdot \text{"}dS\text{"} = \int_\gamma \mathbb{E}\cdot d\ell = \frac{q}{R}2\pi R = \underset{S_2}{2\pi Q_{INT}}$$

perp. γ !

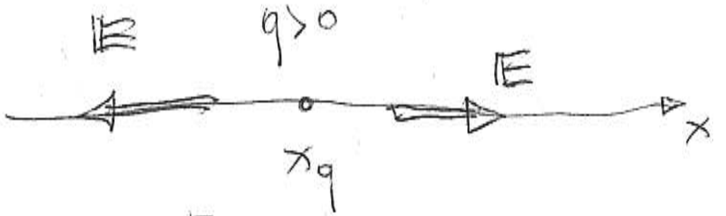
E.g. nD

$$\phi = \int_\Sigma \mathbb{E}\cdot d\mathbb{S} = \frac{q}{R^{n-1}} S_n R^{n-1} = q\, S_n$$

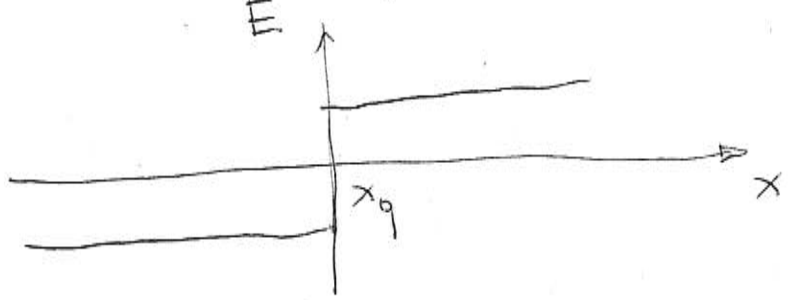
$$S_n = n\pi^{n/2} \Big/ \Gamma\left(1+\frac{n}{2}\right)$$

let's go back to 1D

(WARNING: "real" D=1 vs "effective" D=1 ← dimensional reduction)

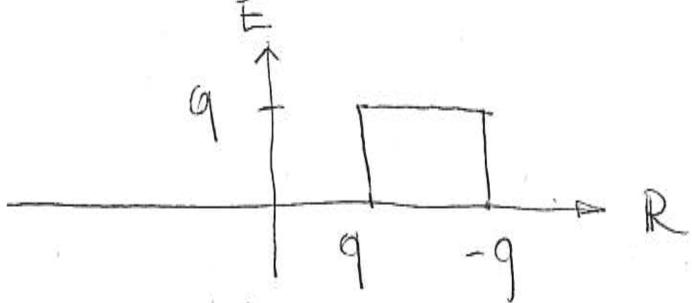


$$\mathbb{E} = E\,\hat{x}$$

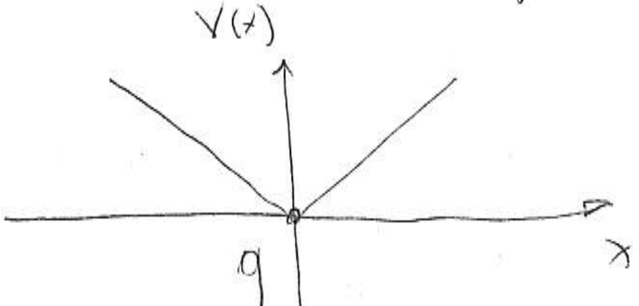


$$E = q \quad (x > x_q)$$

$$E = q\,\epsilon(x - x_q)$$



← ex: charges $q, -q$



$V = q|x|$ (potential)

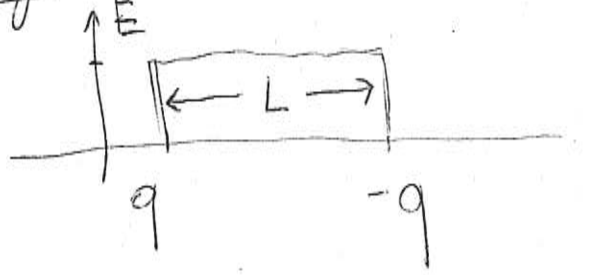
$F = qq'$ (force)

$E = q_1 + q_2 - q_1' - q_2'$

in general

Energy $U_E \sim E^2 L$

reminder 3D
$\int (E^2 + B^2)\, dV$



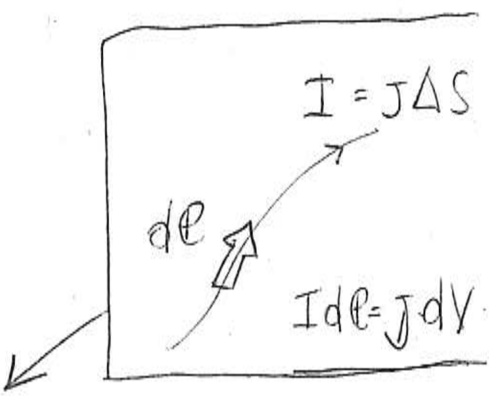
no waves → no propagation
(no speed of light)

Reminder 3D

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J} \end{cases}$$

Maxwell

$\int \mathbf{E} \cdot d\mathbf{S} = 4\pi Q_{int}$ **Gauss**



$\mathbf{E} = \frac{q}{r^2}\hat{\mathbf{r}}$ **point charge**

$\mathbf{B} = q\frac{\frac{\mathbf{v}}{c} \times \mathbf{r}}{r^3}$ **moving charge**

$\mathbf{B} = \frac{1}{c} I \frac{d\ell \times \mathbf{r}}{r^3} = \frac{1}{c}\int \frac{\mathbf{J} \times \mathbf{r}}{r^3}\, dV$ **current**

Lect. 1

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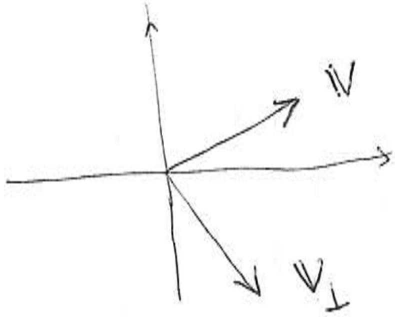
2D

let: $\mathbb{V} = (V_1, V_2)$

$\mathbb{V}_\perp = (V_2, -V_1)$ $\quad (\mathbb{V}_\perp)_\perp = \mathbb{V}$

$\mathbb{V}_\perp^2 = \mathbb{V}^2$, $\quad \mathbb{V}\cdot\mathbb{V}_\perp = 0 = \mathbb{V}_\perp\cdot\mathbb{V}$

$\mathbb{U}\cdot\mathbb{V}_\perp = -\mathbb{U}_\perp\mathbb{V} = U_1V_2 - V_2U_1 = "(\mathbb{U}\times\mathbb{V})_3"$



2-fold ambiguity

$(3D \to \infty)$

$$\mathbb{E} = \frac{q}{r}\hat{\mathbb{r}}$$

$$\phi = \int \mathbb{E}\cdot d\ell = 2\pi Q_{INT} = 2\pi\int \rho \, d\text{Area}$$

normal to line

Lorentz

$$\mathbb{F}_{q'} = q'\left(\mathbb{E} + \frac{\mathbb{V}'_\perp}{c}\mathbb{B}\right)$$

$$\mathbb{B} = q\frac{\mathbb{V}\cdot\mathbb{r}_\perp}{cr^3} \quad \left(= \int \frac{\mathbb{J}\cdot\mathbb{r}_\perp}{cr^3}\, d\text{Area}\right)$$

$$\nabla\cdot\mathbb{J} + \frac{\partial\rho}{\partial t} = 0$$

scalar (pseudo)

Ampère $\quad \nabla_\perp \mathbb{B} = \frac{2\pi}{c}\mathbb{J}$

$$\begin{cases} \nabla \cdot E = 2\pi\rho \\ \nabla_\perp E = \frac{1}{c}\frac{\partial B}{\partial t} \\ \nabla_\perp B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{2\pi}{c} J \end{cases}$$

Maxwell
(<u>NO</u> $\nabla \cdot B = 0$)

$$\Box B = \frac{2\pi}{c} \nabla_\perp \cdot J$$

$$\Box E = 2\pi \nabla\rho + \frac{2\pi}{c^2}\frac{\partial J}{\partial t}$$

!

<u>waves</u>

<u>potentials</u>

$$B = \nabla \cdot A_\perp$$

$$E = -\nabla V - \frac{1}{c}\frac{\partial A}{\partial t}$$

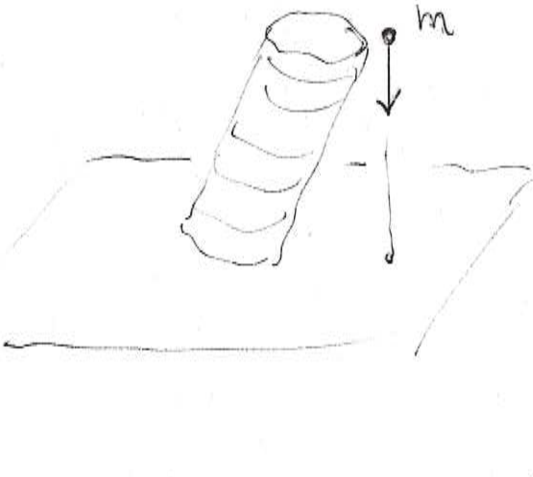
<u>NOT</u> much simpler than 3D

(Unlike in lecture 1, 1D Electrodynamics),
where one has to accept the formulation,
we shall now ask questions (parameters?
mass? charge? Lorentz?)

(Q) simulation ← meaning?

e.g. weather forecast (*) ← difficult

simpler example:



$\Big\}\, t_f$

Compute t_f ← from H, L

Eqs. of motion

↓ compute or numerics

$t_f = \dots$

contain parameters: $f(\cdot, \cdot, \dots, v, u)$

$v = \begin{cases} < 40 \text{ ys} \\ > 40 \text{ ys} \end{cases}$ $u = \begin{cases} \text{Male} \\ \text{Female} \end{cases}$

$v \leq 40$, $u = F, M$ are irrelevant

which variables are important/relevant?

more complex the problem
more crucial the issue

(*) weather forecast → butterfly in Brasil has an influence on weather in Trieste?
→ strike at airport in Frankfurt
→ volcano eruption in Sicily

we <u>hope</u> that some variables are irrelevant

x———x

A good simulator must define relevant variables
↳ to capture interesting aspects
↓ ?
by 1D do we capture 3D ?

E.g. dynamics of pair-creation
↓ e^+e^- formation
shall / can we neglect spin ?

x———x

discretize → space
→ time
→ E field
?

——→ with all the necessary care, let us now define our model.

x———x

computer vs simulator

Dirac eq.

$$(i\not{\partial}-m)\psi = (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \sim \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix} \quad 3+1$$

$4\ \gamma^\mu$'s e.g. $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Pauli

$$\mathcal{L}_D = \underbrace{\bar{\psi}}_{\psi^\dagger\gamma^0}(i\not{\partial}-m)\psi$$

Euler-Lagrange $\rightarrow$ Dirac eq. $\psi = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \\ \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \end{pmatrix}$

1D

Same eq., but we need only γ_0 and γ^1,

as $g^{\mu\nu} = \begin{pmatrix} + & \\ & - \end{pmatrix}$

E.g. $\gamma^0 = \sigma_x$ $\gamma^1 = -i\sigma_y$

$(\gamma_5 = i\gamma_0\gamma^1 \sim \sigma_z)$ $\psi = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$

matter

antimatter

$$\mathcal{L} = \psi^\dagger\gamma^0\left[\gamma^\mu(i\partial_\mu + g A_\mu) - m\right]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$-q = e^-$ charge $(A_0, A_1) = (\varphi, A)$

minimal coupling:

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu = \partial_\mu - igA_\mu$$

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_{em} + \underbrace{g A_\mu \bar{\psi} \gamma^\mu \psi}_{-q A_\mu J^\mu}$$

quantise?

$$\{\psi(t,x), \psi^+(t,x')\} = \delta(x-x')$$

$$[E(t,x), A(t,x')] = i\,\delta(x-x')$$

$\uparrow$ $\varphi = 0$ canonical gauge

$$H = \int dx \left[\psi^+ \gamma^0 \left[-\gamma^1 (i\partial_1 + gA) + m \right] \psi + E^2/2 \right]$$

Gauss law $G(x) = 0$

$$G(x) = \partial_1 E(x) + g\,\psi^+(x)\psi(x) = 0$$

$\uparrow$ operator sense? <u>NO</u>

but $[G(x), G(x')] = 0$

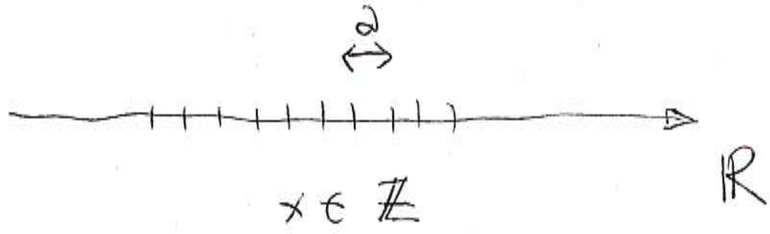
$[G(x), H] = 0$ $\longrightarrow$ therefore let us impose

$$G(x)\,|\psi\rangle = 0$$

$\uparrow$ physical states

$$G(x) \approx 0 \quad \forall x$$

Discretization



Gauge Group $U(1)$

$(*)$ $[E, A] = i$ let $U = e^{-i\eta A}$

so that $[E, U] = \eta U$, $\eta \in \mathbb{R}$

$$\underset{V}{\underset{\shortparallel}{e^{i\xi E}}}\ \underset{U}{\underset{\shortparallel}{e^{-i\eta A}}} = e^{i\eta\xi} e^{-i\eta A} e^{i\xi E} \qquad \eta, \xi \in \mathbb{R}$$

$$\boxed{V\,U = e^{i\eta\xi}\, U V}$$

group commutator (instead of $(*)$)

but we can now let η, ξ <u>discrete</u>

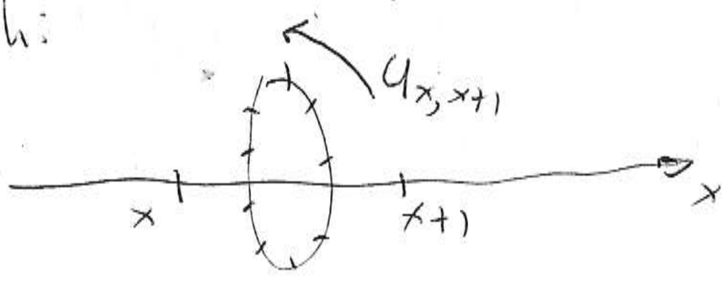
$$U_{x,x+1} = e^{-i\sqrt{\frac{2\pi}{n}} A_{x,x+1}}$$

$$V_{x,x+1} = e^{-i\sqrt{\frac{2\pi}{n}} E_{x,x+1}}$$

$$U^l V^k = e^{i\frac{2\pi}{n} kl} V^k U^l \qquad \text{in } (x, x+1) \text{ link}$$

$k, l \in \mathbb{Z}_n$

one ends up with:



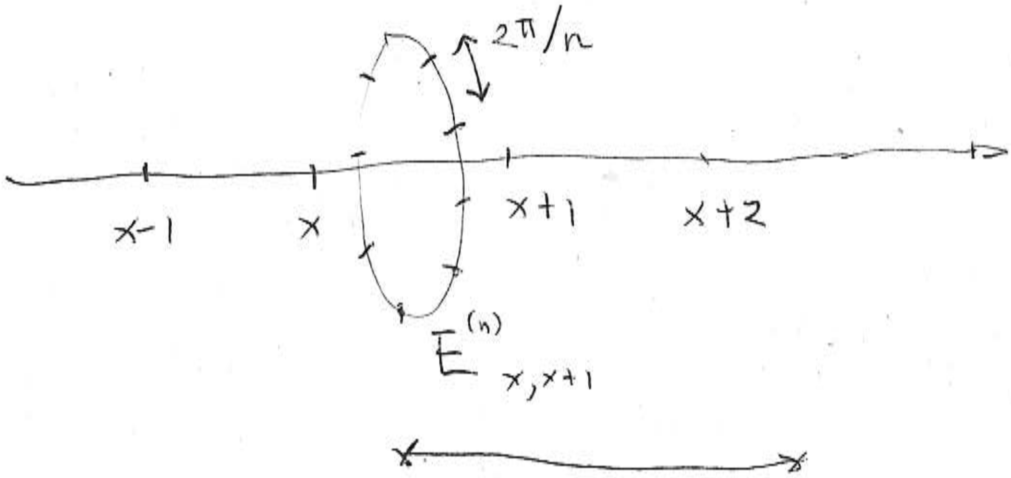
Lecture 3 S. Paganini

Lattice $\mathbb{Z}_n$ QED model (1D)

$$H = -\frac{1}{2a} \sum_x \psi_x^\dagger U_{x,x+1} \psi_{x+1} + \text{H.c.}$$

$$+ m \sum_x (-)^x \psi_x^\dagger \psi_x + \frac{g^2 a}{2} \sum_x E_{x,x+1}^2$$

staggered (fermion doubling)



Questions from the students:

1) in which sense can Dirac's eq. be considered classical $\longrightarrow$ answered privately

2) what happens if one breaks gauge invariance $\longrightarrow$ answered privately (partially)

3) γ's in 2+1 $\longrightarrow$ graphene

4) is there a universal recipe to write H?

NO - It depends on energy scales, etc.

ex. $E_n = n^2 \pi^2 \hbar^2 / 2mL^2$ (box)

Important points:

- Q. simulators vs Q. computers
- dynamics of pair creation $(e^+ e^-)$
- role of spin

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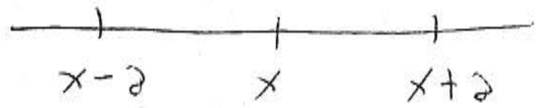
check (answer to question by colleague)



$$\psi_x^\dagger \, U_{x+a} \, \psi_{x+a}$$

$$U = e^{-i\eta A} = e^{-i\eta \int_x^{x+a} A(x)\,dx} \simeq 1 - i\eta A(x) a$$

$$\psi(x+a) \simeq \psi(x) + a\partial_x \psi = (1 + a\partial_x)\psi$$

$$\psi_x^\dagger U_{x+a} \psi_x \sim \psi^\dagger (1 - i\eta A a)(1 + a\partial_x)\psi$$

$$= \psi^\dagger (1 + a\partial_x - i\eta a A)\psi + O(a^2)$$

$$\overset{a=1}{=} \psi^\dagger (1 + \partial_x - i\eta A)\psi \quad \underline{\text{OK}} \quad \text{(see literature)}$$

quantisation:

$$\{\psi_x, \psi^+_{x'}\} = \delta_{x,x'}$$

$$[E_{x,x+1}, U_{x',x'+1}] = \delta_{xx'} U_{x,x+1}$$

$\hookrightarrow$ group $UV = VU \times$ phase

Gauge $\leftarrow$ local phase Transformation

$$\psi_x \to \psi_x e^{i\alpha_x}$$

$$U_{x,x+1} \to e^{i\alpha_x} U_{x,x+1} e^{-i\alpha_{x+1}}$$

H invariant

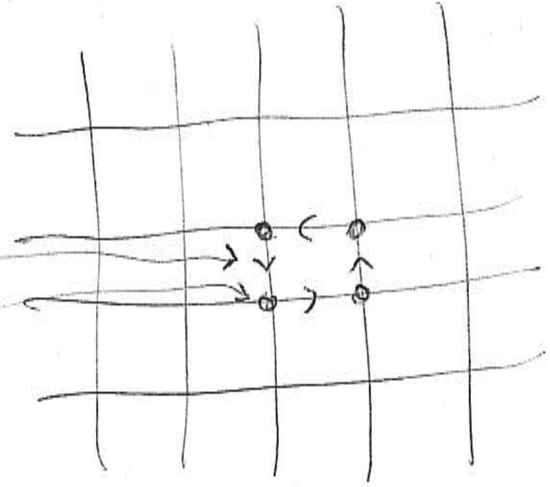
$$G_x = \psi^+_x \psi_x + \frac{1}{2}\left[(-)^x - 1\right] - \left[E_{x,x+1} - E_{x-1,x}\right]$$

(Gauss)

2D-picture:

gauge field on links

matter on sites



(1D is analogous)

$G_x |\phi\rangle = 0 \quad \forall x$ $\longrightarrow$ $\boxed{\text{physical states}}$

$\updownarrow$

$\prod_x e^{-i\alpha_x G_x} |\phi\rangle = |\phi\rangle$

Repeat: $G_x \approx 0$ (not as operator)

if $\left[G(x), G(x') \right] = 0$

$\left[G(x), H \right] = 0$ $\longrightarrow$ conserved quantity

all can be simultaneously diagonalized

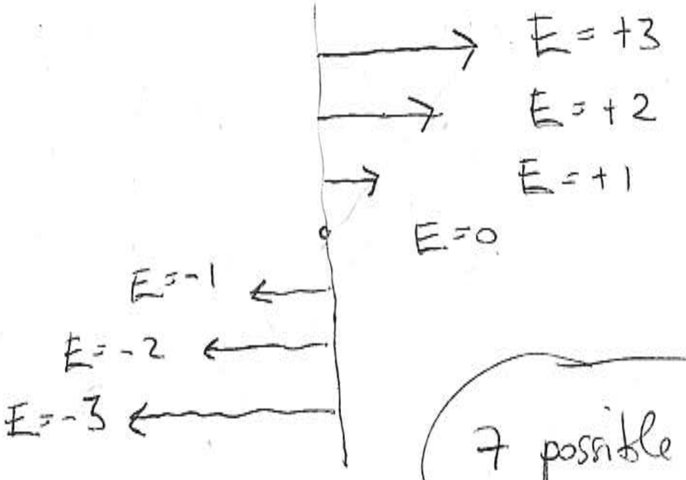
$\longrightarrow G(x)|\phi\rangle = 0$ initial condition

$\downarrow$

will be preserved

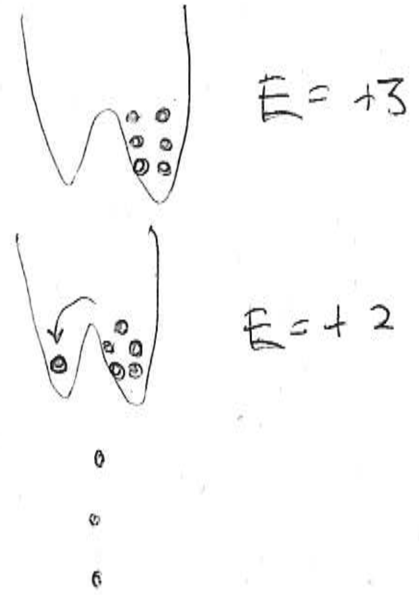
[answer to question 2) [partial]]

$\boxed{\text{show}}$ simulator E field



7 possible values for E

"physics"



simulator (fully QUANTUM)

$\boxed{\text{show}}$ Laguerre – Gauss beams

↳ Amico + Cataliotti PRL 95 (2005)

$\boxed{\text{show}}$ states (physical states, $G|\phi\rangle = 0$)

→ ~~Notarnicola et al J Ph A 48 (2015)~~

→ Ercolessi et al ~~Ercolessi et al~~ PRD 98 (2018)

$\boxed{\text{show}}$ dynamics → string breaking

- Pichler et al. PRX 6 (2016)
- unpublished