General versus projective measurements

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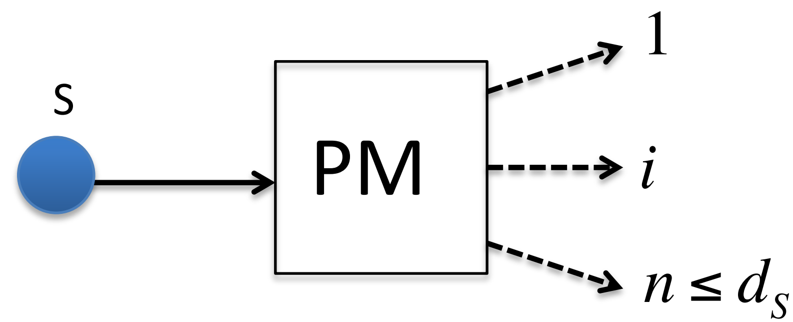
General vs projective measurements

Textbook quantum measurements in a space of dimension *d* are defined by $n \leq d$ orthogonal projectors summing up to the identity.



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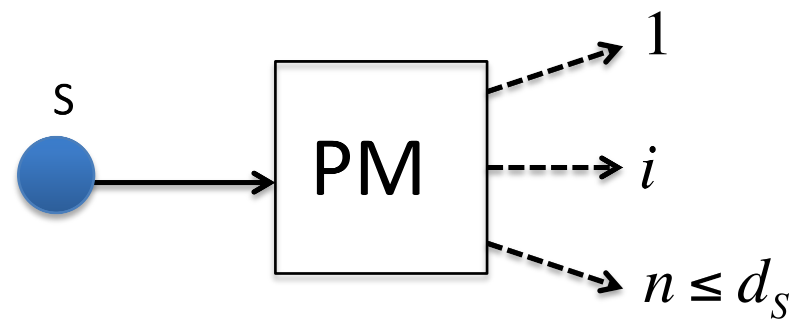

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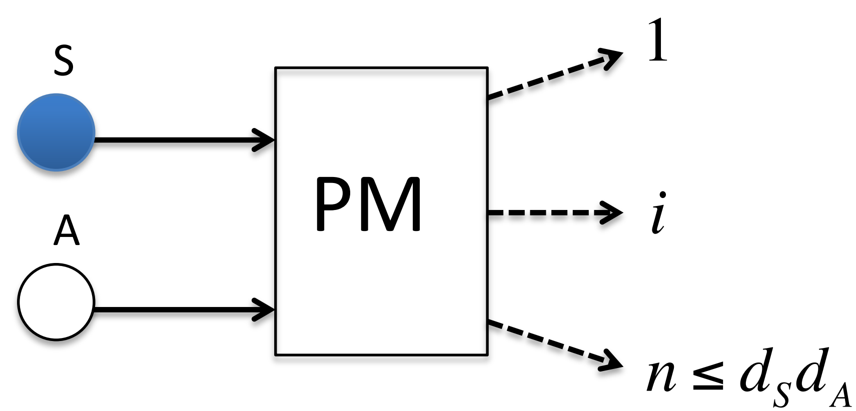

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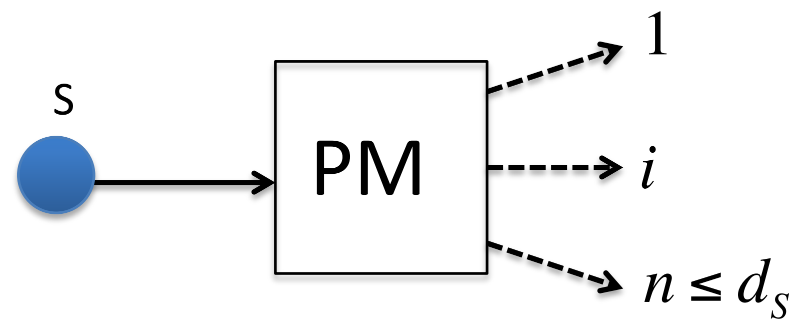

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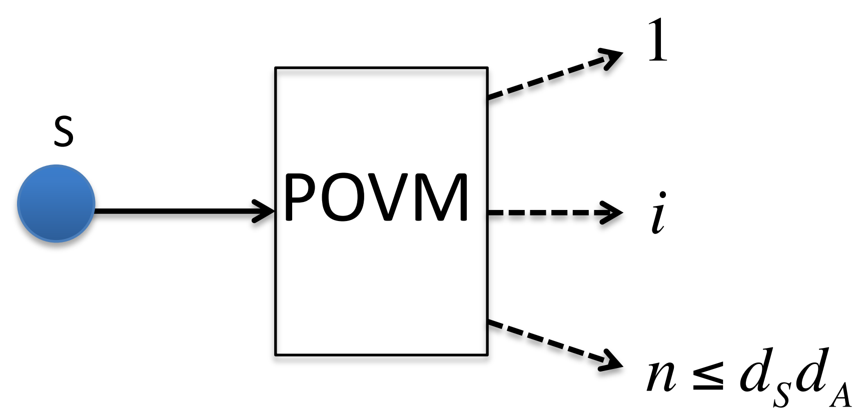

This general measurement is defined by a Positive-Operator Valued Measure (POVM):

$$M_i \ge 0 \qquad \sum_{i=1}^{n} M_i = 1_S$$

General vs projective measurements

General measurements provide an advantage over projective in many quantum information applications:

- quantum tomography
- unambiguous discrimination of quantum states
- state estimation
- quantum cryptography
- information acquisition from a quantum source
- Bell inequalities (*)

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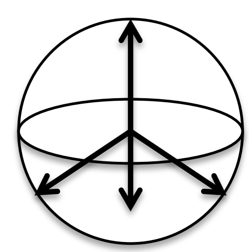
Example: tetrahedron POVM.

$$M_i = \frac{1}{2}\left|\hat{n}_i\right\rangle\left\langle\hat{n}_i\right|$$



POVMs and nonlocality

Open question: do POVM's help for nonlocality detection?

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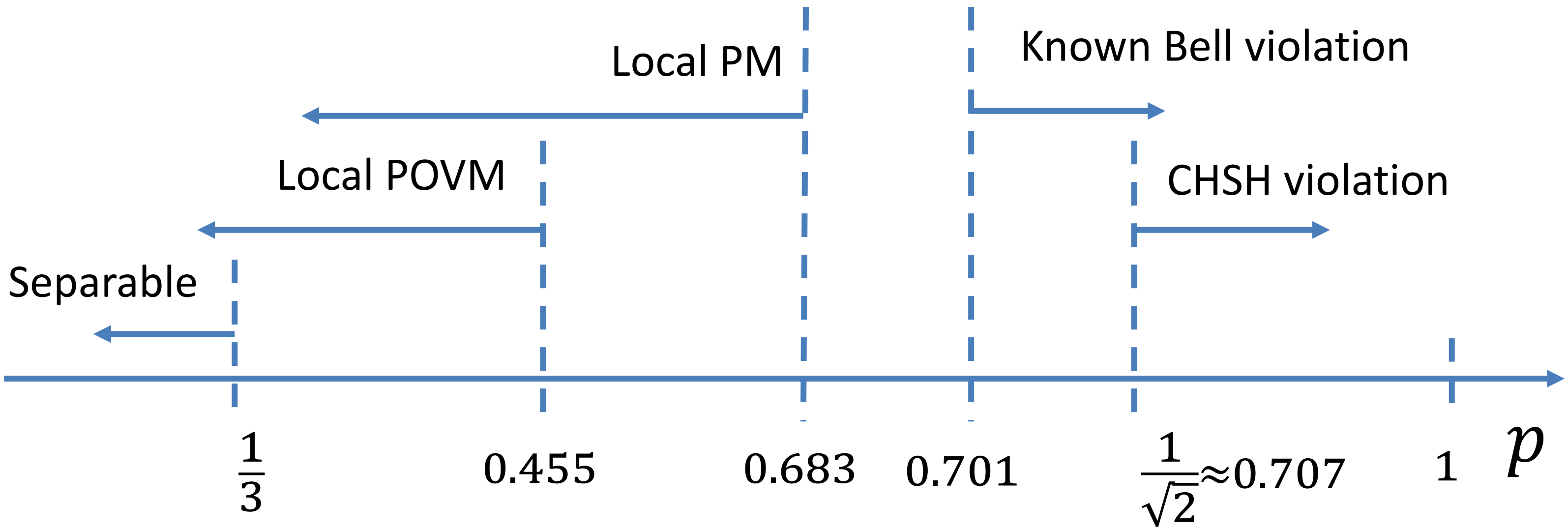
$$\rho(p) = p|\Phi\rangle\langle\Phi| + (1-p)\frac{1}{4}$$



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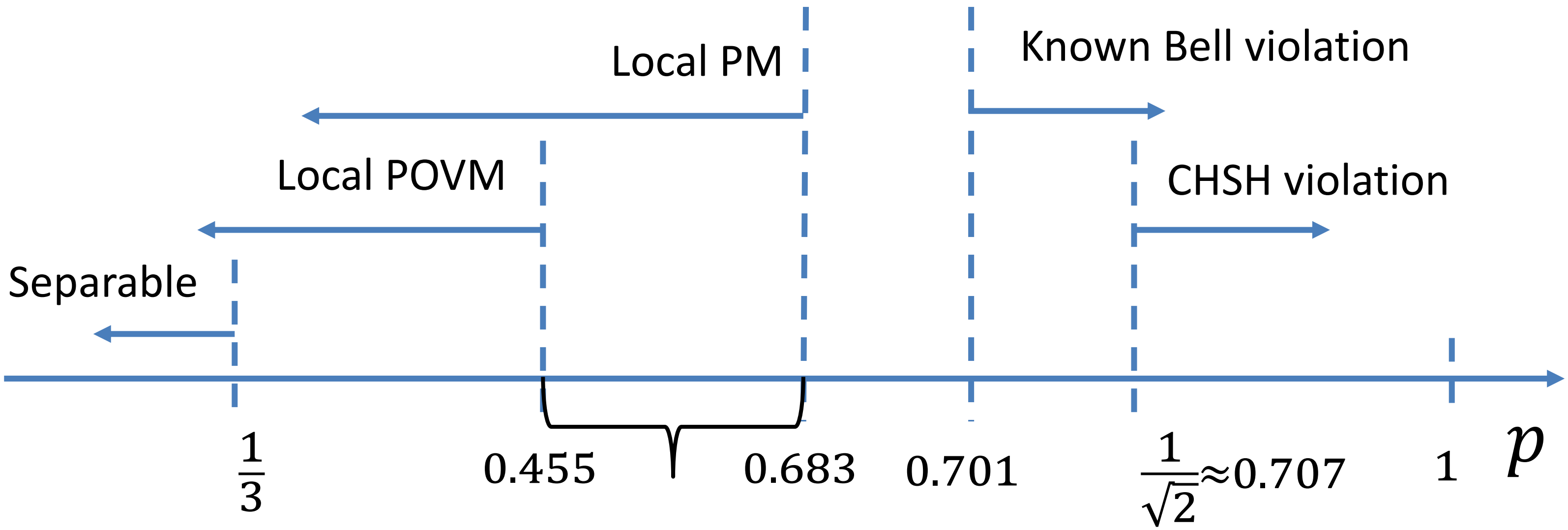
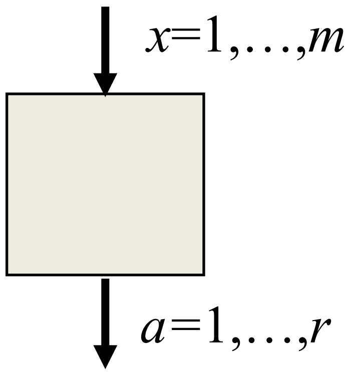
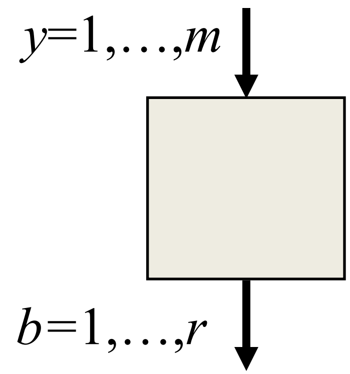


Is there a state that: (i) has a local model for projective measurements but
(ii) violates a Bell inequality when using POVM's?

Bell certified randomness



$$p(a,b|x,y)$$



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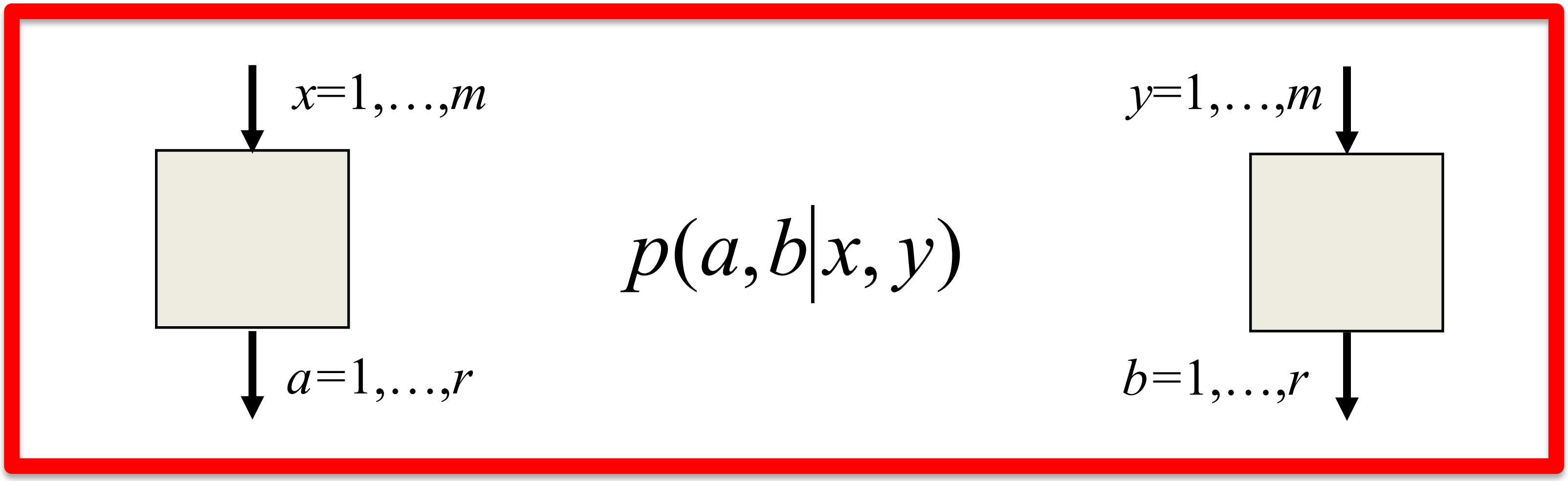
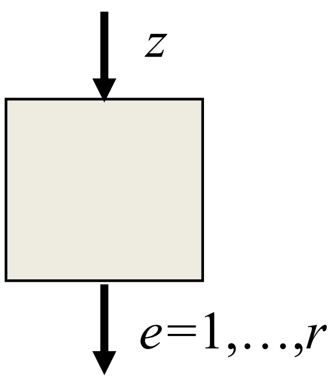
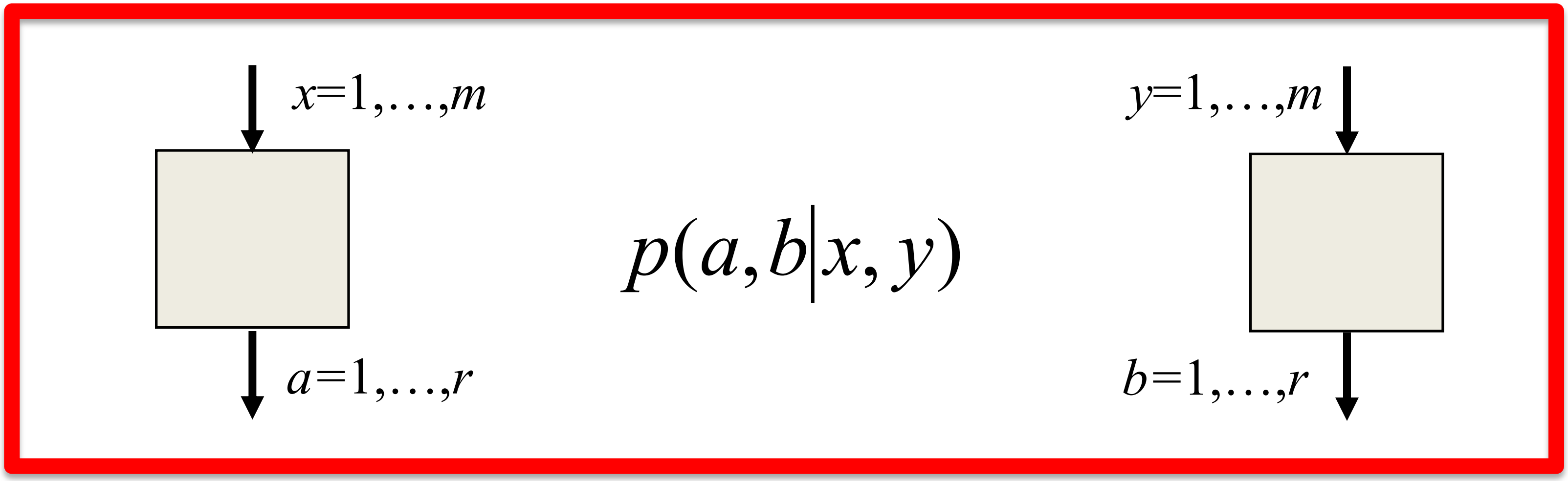
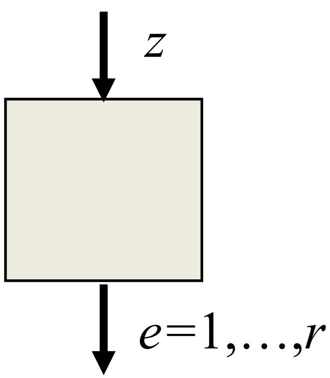




1. An external party, possibly an eavesdropper, has a system correlated to the user's systems.
2. She makes a measurement and according to the result guesses the result of one (or the two) boxes for one (or any) possible measurement.
3. The guess can be seen as the final measurement output by the eavesdropper.

Bell certified randomness



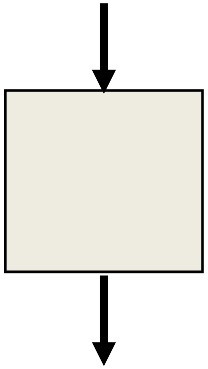


If someone, e.g. the enemy, is able to predict the outputs ➔ local model ➔ No Bell violation.

It is possible to bound the randomness of the outputs from the Bell inequality violation.

Bell certified randomness

In quantum theory, the only way of getting the maximal violation of the CHSH inequality is by measuring a two-qubit maximally entangled state.



$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \underset{C}{\leq} 2$$

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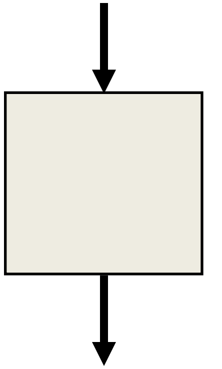

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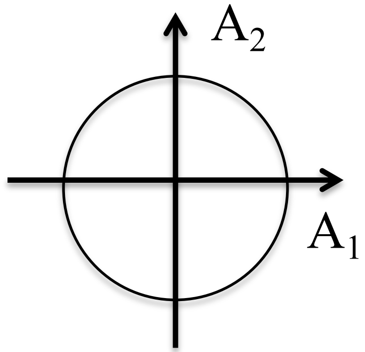

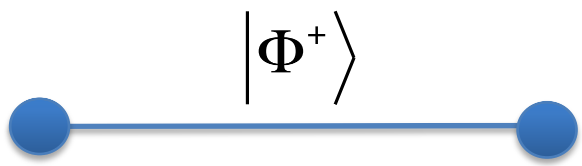

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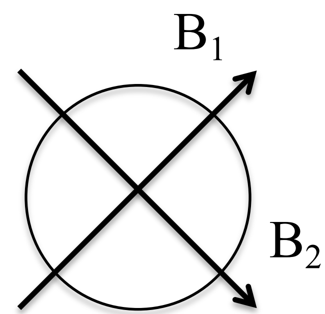

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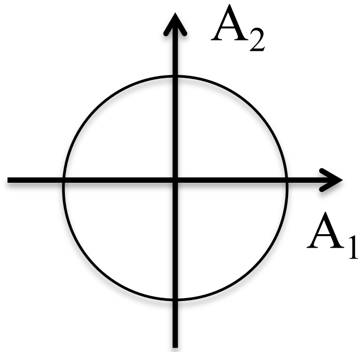

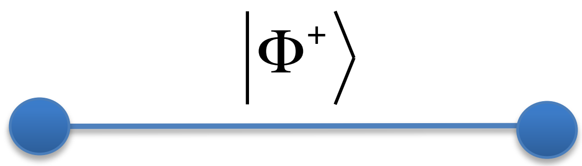

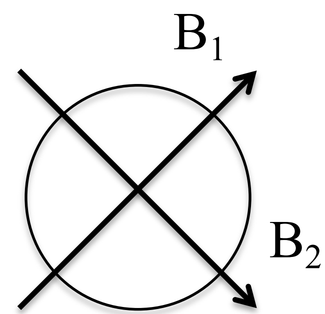

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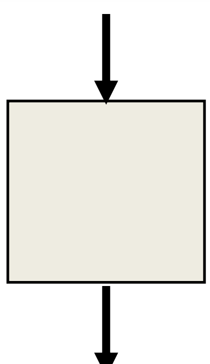

$$|\Psi\rangle_{ABE} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \otimes |E\rangle$$

Projective measurements and randomness

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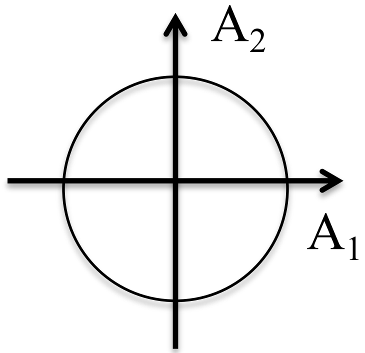

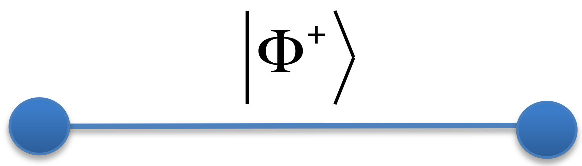

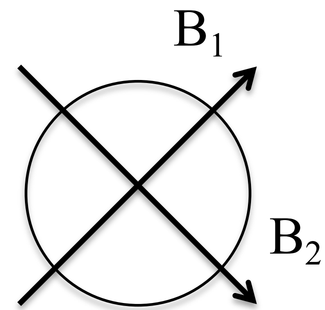

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Every measurement output defines a perfect random bit.

Do POVMs help for certified randomness generation?

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Optimal randomness certification from one entangled bit

A. Acín, S. Pironio, T. Vértesi, P. Wittek
Phys. Rev. A 93, 040102 (2016)

POVMs and Bell-certified randomness

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Question: is this bound attainable?

Answer: Yes. POVM's help for randomness certification.

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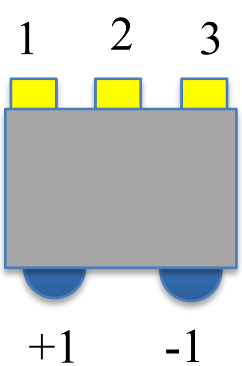

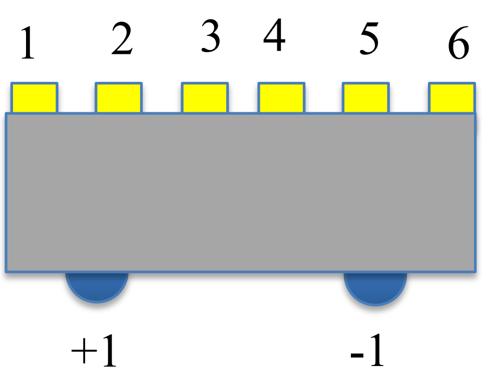

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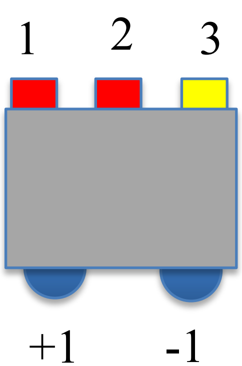

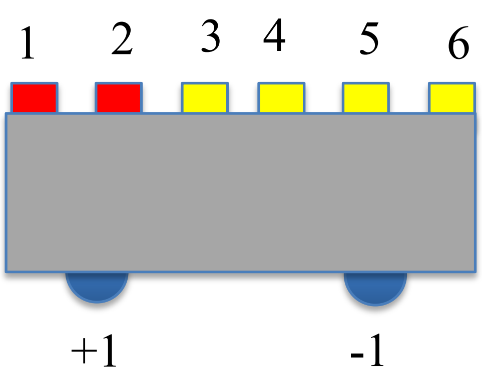

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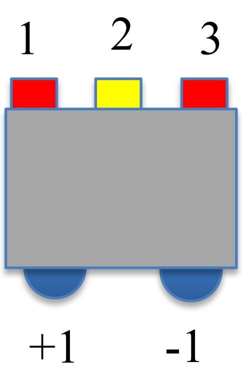

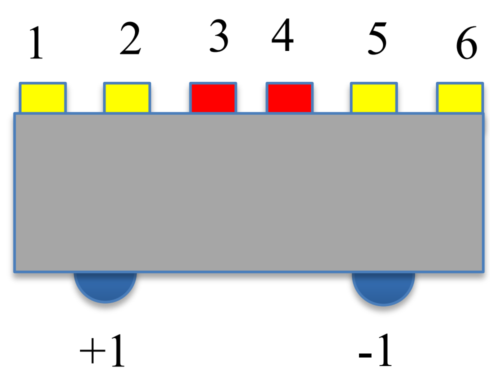

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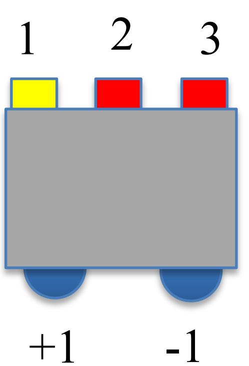

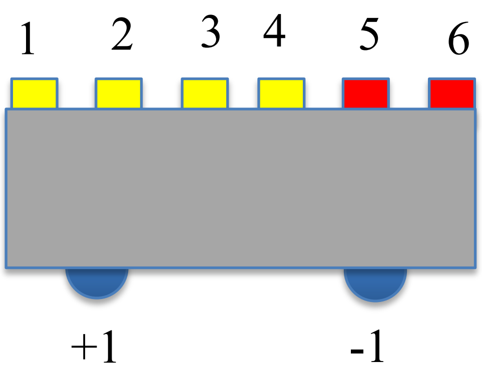

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The maximal violation of this inequality implies that: (i) Alice and Bob are measuring a singlet and (ii) Alice is measuring in the *x, y* and *z* direction.

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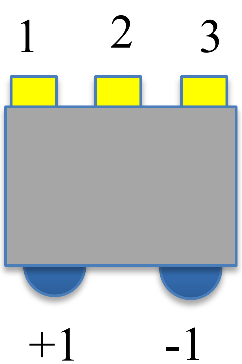
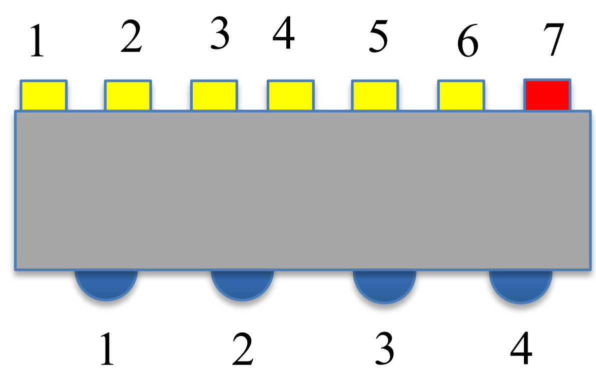
We have one more measurement, of 4 outcomes, on Bob's side.





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Then, the correlations for the 7th measurement by Bob read:

$$p\left(a,b\middle|x,y=7\right)=\mathrm{tr}\left(\begin{matrix}|\pm x\rangle\langle\pm x|\\ |\pm y\rangle\langle\pm y|\\ |\pm z\rangle\langle\pm z|\end{matrix}\otimes M_{b|y=7}|\Phi\rangle\langle\Phi|\right)=\begin{matrix}\langle\pm x|M_{b|y=7}|\pm x\rangle\\ \langle\pm y|M_{b|y=7}|\pm y\rangle\\ \langle\pm z|M_{b|y=7}|\pm z\rangle\end{matrix}$$

POVMs and Bell-certified randomness

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The correlations can be used to reconstruct the POVM elements. If we put the tetrahedron POVM, the certify it ➔ 2 random bits when acting on half of a singlet.

POVMs and Bell-certified randomness

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This is the intuition behind the construction, although in the end the analytical proof is a bit different and follows from extremality. In fact, 2 random bits can be certified from any extremal 4-outcome POVM such that:

$$\mathrm{tr}\left(M_i\right)=1/2$$

POVMs and Bell-certified randomness

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Elegant Bell inequality (Gisin):

$$I_{\mathrm{el}} = \left(A_1 + A_2 + A_3\right)B_1 + \left(A_1 - A_2 - A_3\right)B_2 + \left(-A_1 + A_2 - A_3\right)B_3 + \left(-A_1 - A_2 + A_3\right)B_4 \underset{C}{\leq} 6$$

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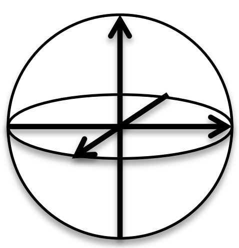
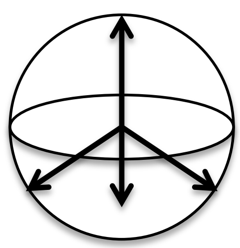
We proved that the known violation was optimal. The known way of getting it is by measuring a singlet in the (x,y,z) bases for Alice and measurements pointing into the directions of a tetrahedron on Bob.

Alice



Bob

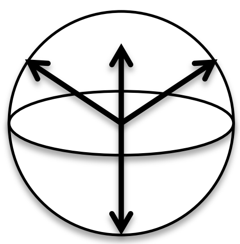


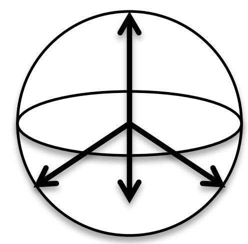
POVMs and Bell-certified randomness

$$I_{\mathrm{el}} - \sum_{i=1}^{4} p\left(A_4 = i, B_i = +1\right) \underset{Q}{\leq} 4\sqrt{3}$$

Idea: The only way of saturating the bound is if the POVM elements of A_4 on Alice's side are anti-aligned with the directions defining the 4 measurements on Bob's side. They define the tetrahedron POVM.

Alice

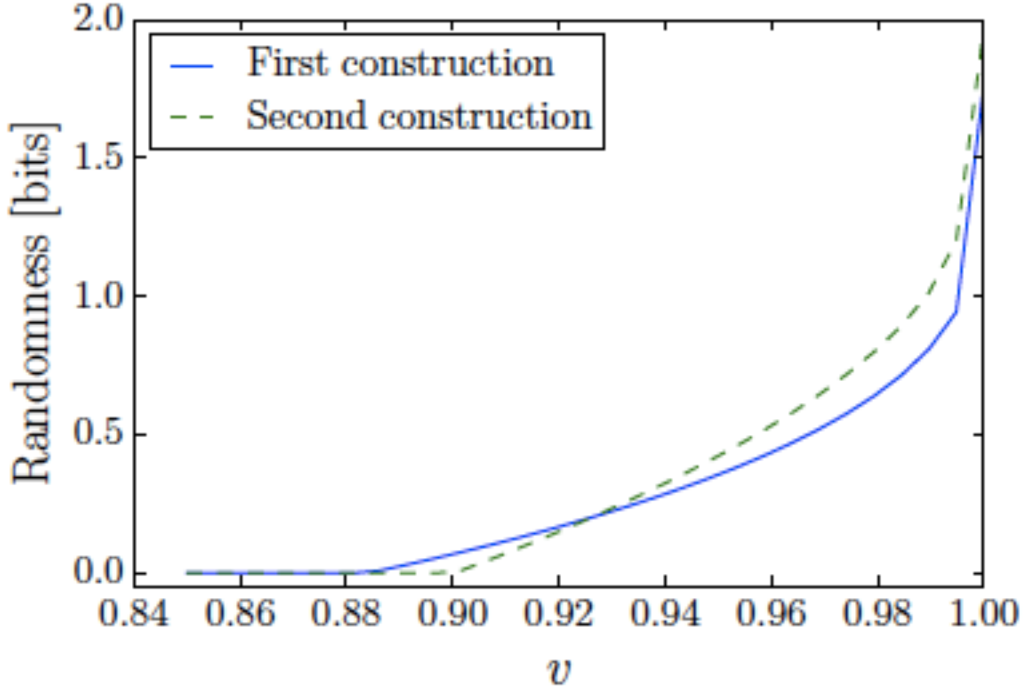




Bob

We confirm this intuition and proved the certification of 1.9999998947470 bits using the NPA hierarchy.

POVMs and Bell-certified randomness



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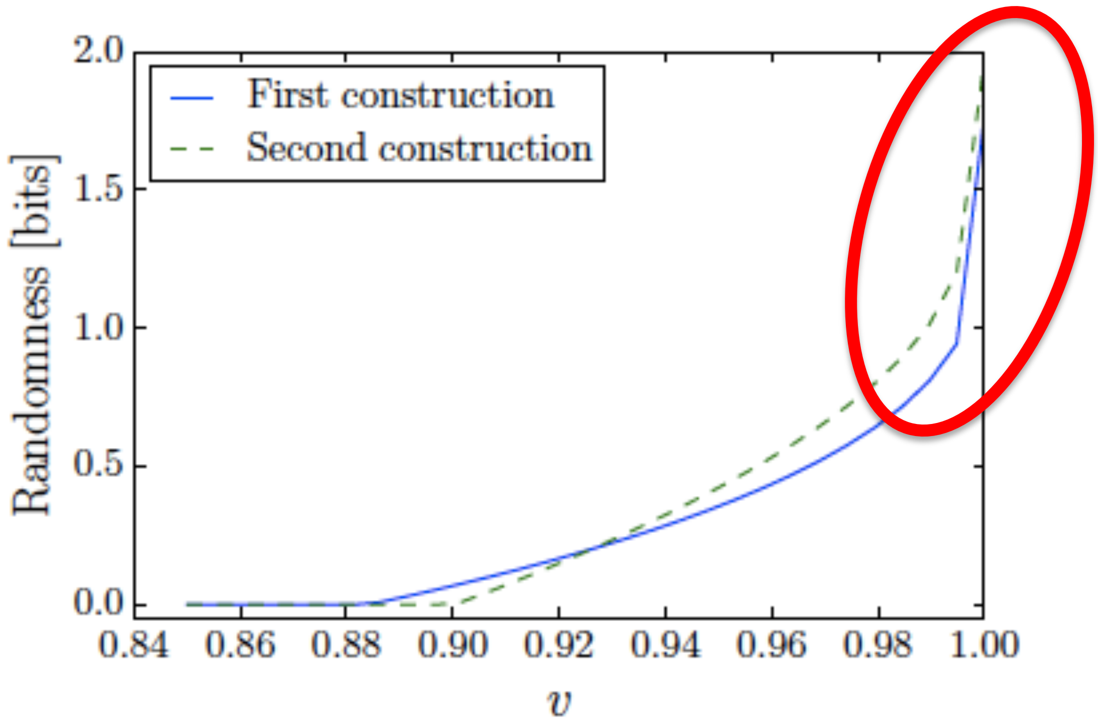
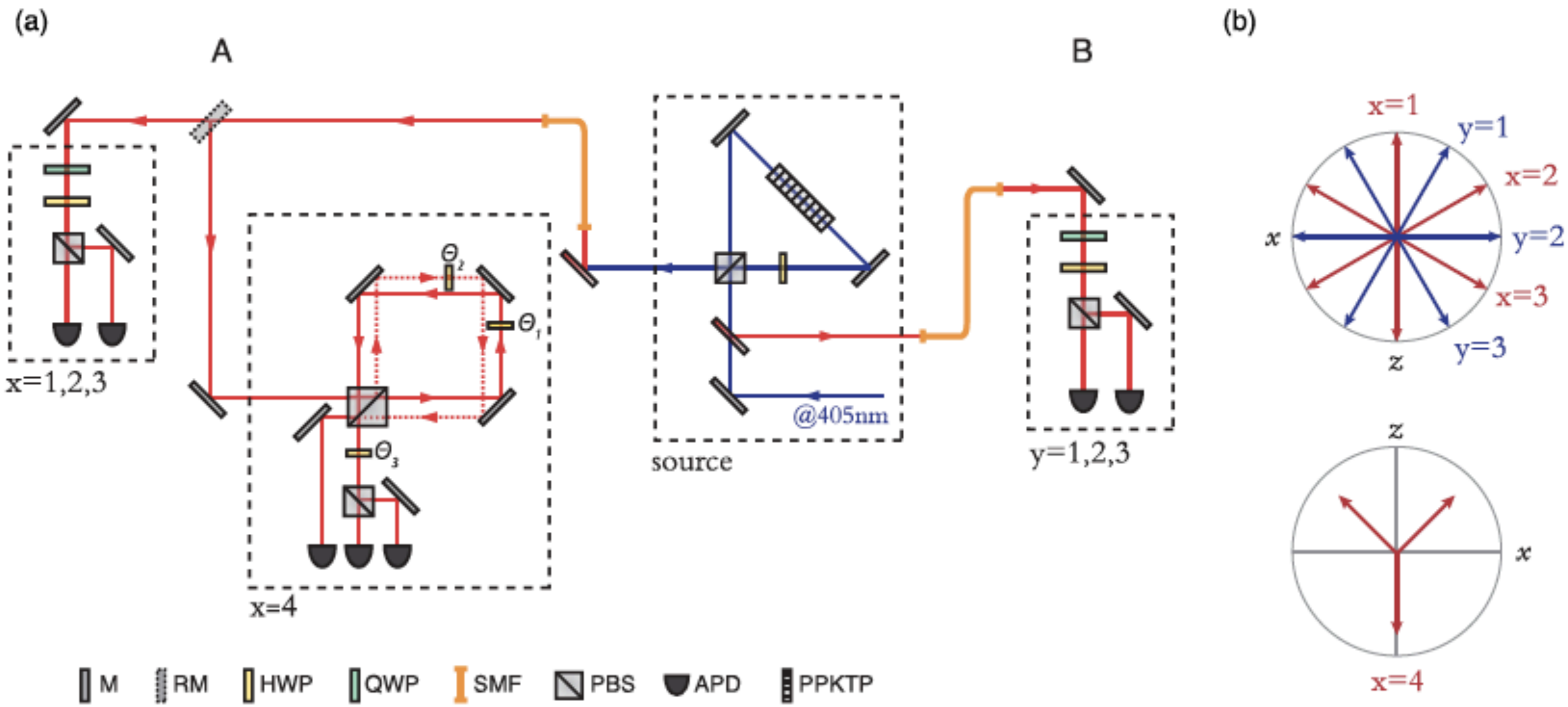


The constructions are demanding: high visibilities are necessary for >1 bit.

Experimental nonlocality-based randomness generation with nonprojective measurements

S. Gómez, A. Mattar, E. S. Gómez, D. Cavalcanti, O. J. Farías, A. Acín, G. Lima
Phys. Rev. A 97, 040102 (2018)

Experimental setup



Visibility = $(99.7 \pm 0.2)\%$

Randomness = (1.18 ± 0.08) bits

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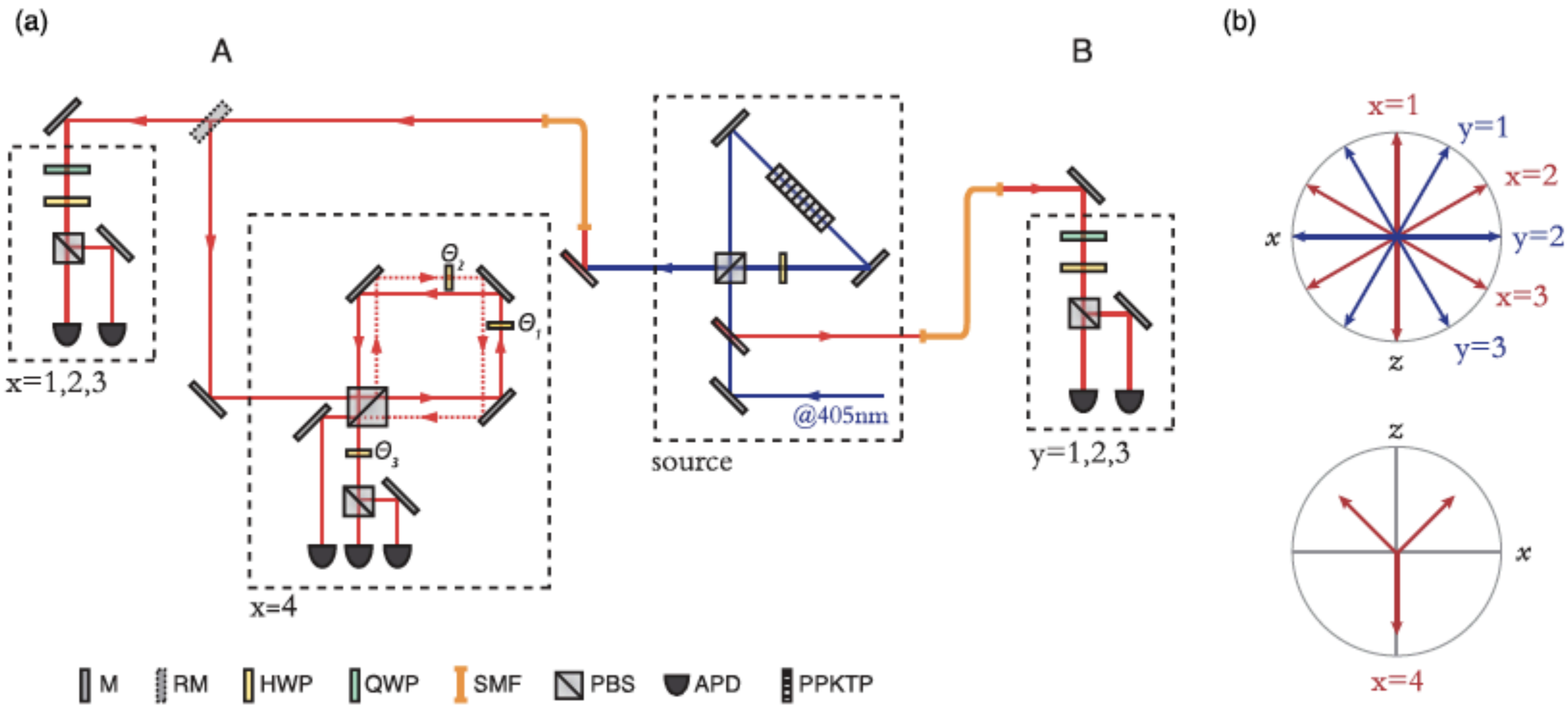

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Four-outcome POVM: Smania *et al.*, arXiv:1811.12851.

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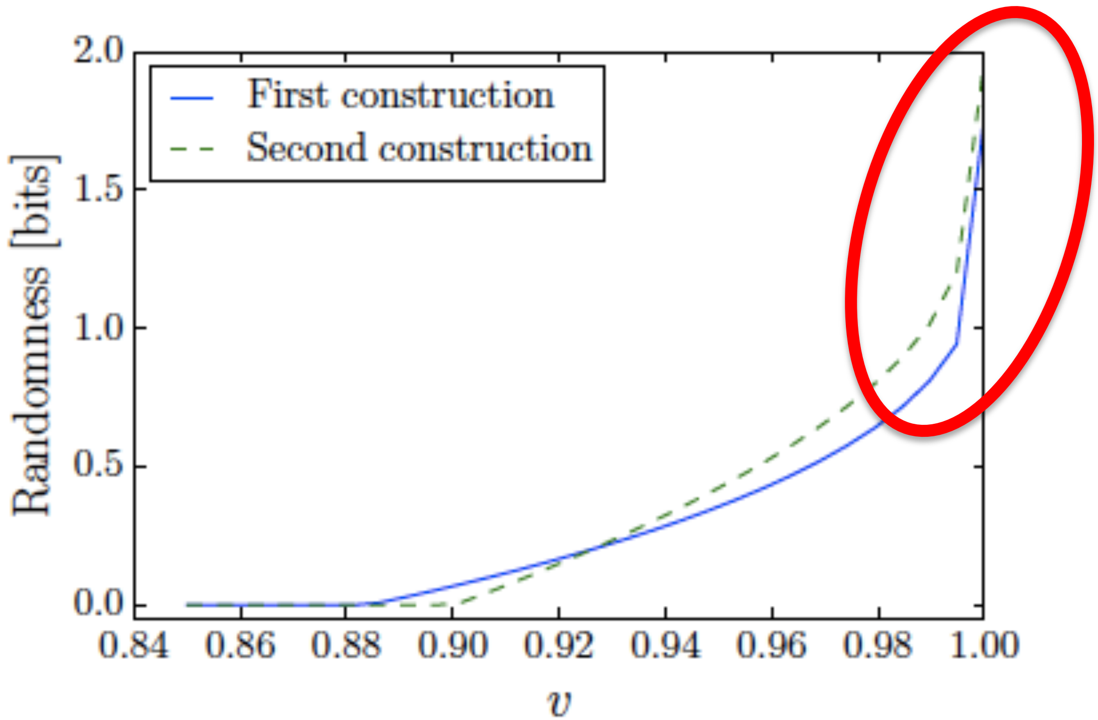

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When do POVMs provide an advantage over projective measurements?

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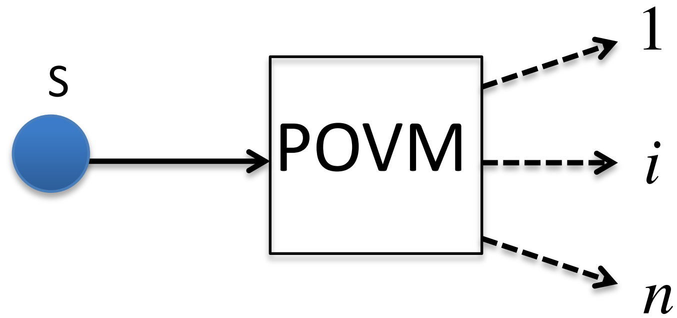
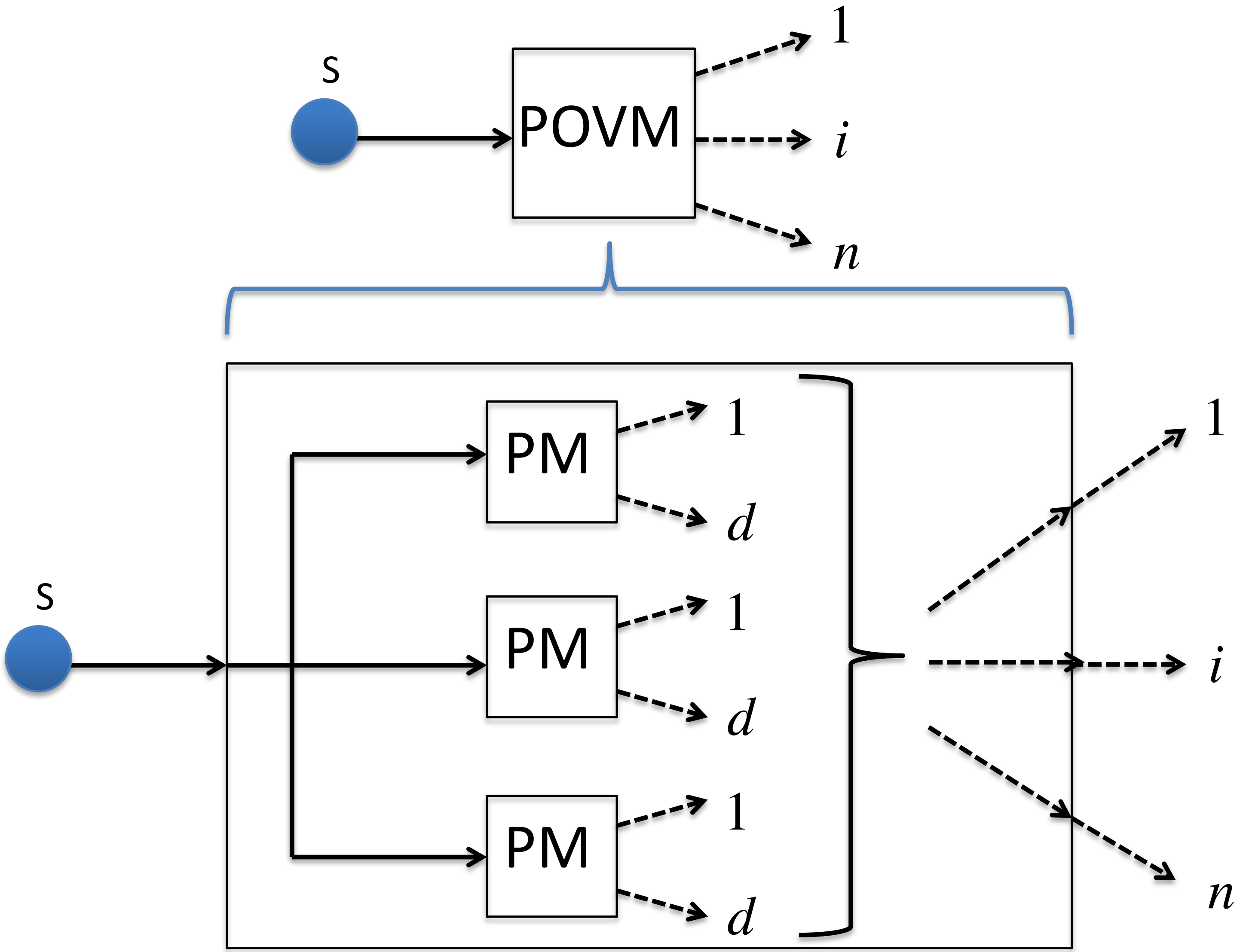
Simulating positive-operator-valued measures with projective measurements

M. Oszmaniec, L. Guerini, P. Wittek, A. Acín
Phys. Rev. Lett. 119, 190501 (2017)

POVM simulation



POVM simulation



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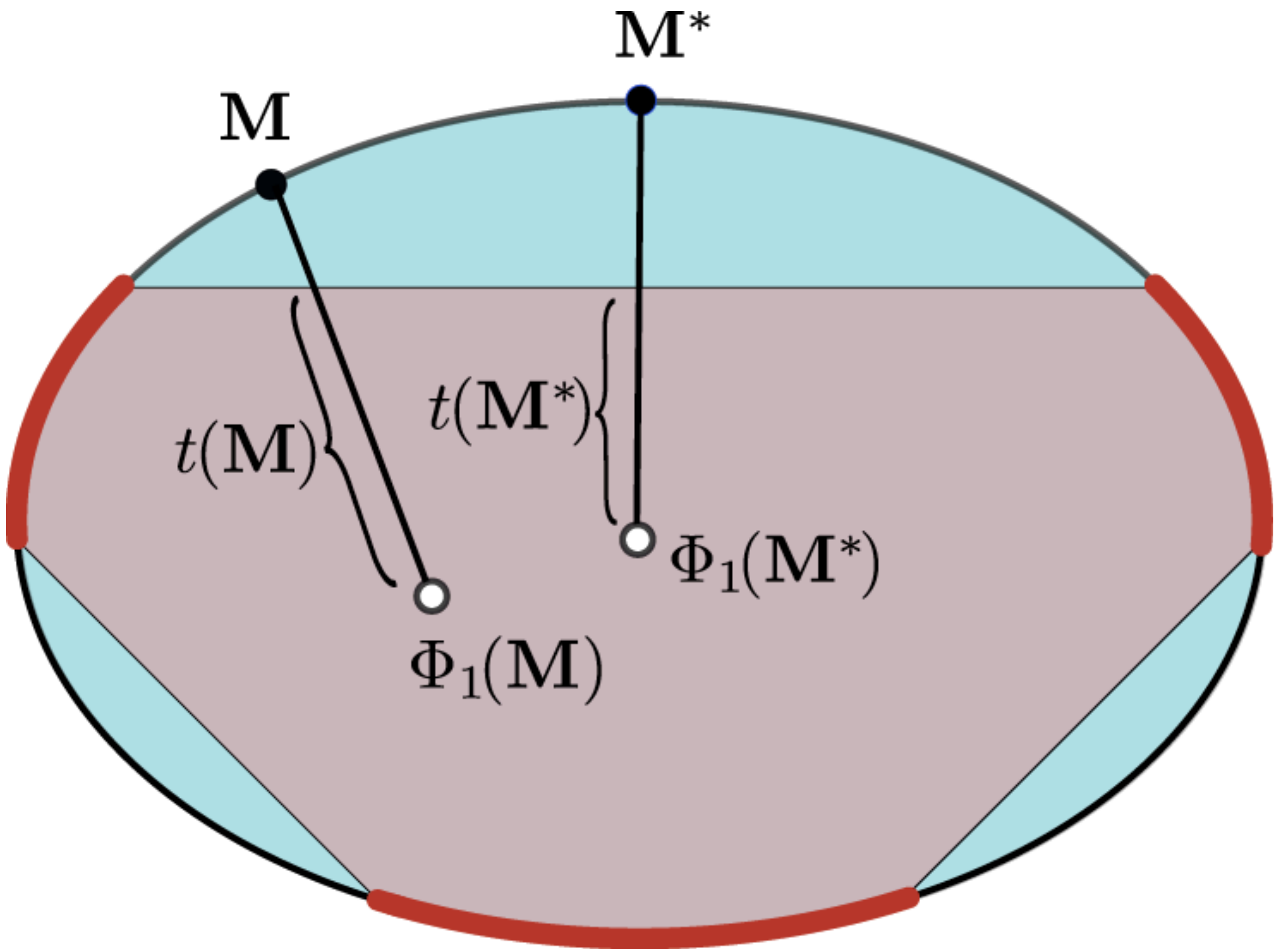
- For a given dimension *d, t*(d) denotes the minimum of $t(\mathbf{M})$ over all measurements. The solution to this problem defines the most non-projective POVM.

POVM simulation: geometry



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- For qubits, a numerical search indicates that the tetrahedron is the most robust POVM, with visibility

$$t_{\text{tetra}} = \sqrt{\frac{2}{3}} \approx 0{,}8165$$

Note: in fact, we could prove that all measurements are PS simulable whenever $t \leq 0.8143$, but we now know that t_{tetra} is the actual value of $t(2)$, see Hirsch *et al.*, Quantum'17*.*

POVM simulation: applications

Correlations obtained when applying noisy measurements on a state are the same as applying noise-less measurements on a noisier state.

Bowles *et al.*, PRL15

$$\mathrm{tr}\Big(\Phi_t\big(M_{a|x}\big)\otimes\Phi_t\big(M_{b|y}\big)\rho\Big)=\mathrm{tr}\Big(M_{a|x}\otimes M_{b|y}\big(\Phi_t\otimes\Phi_t(\rho)\big)\Big)$$

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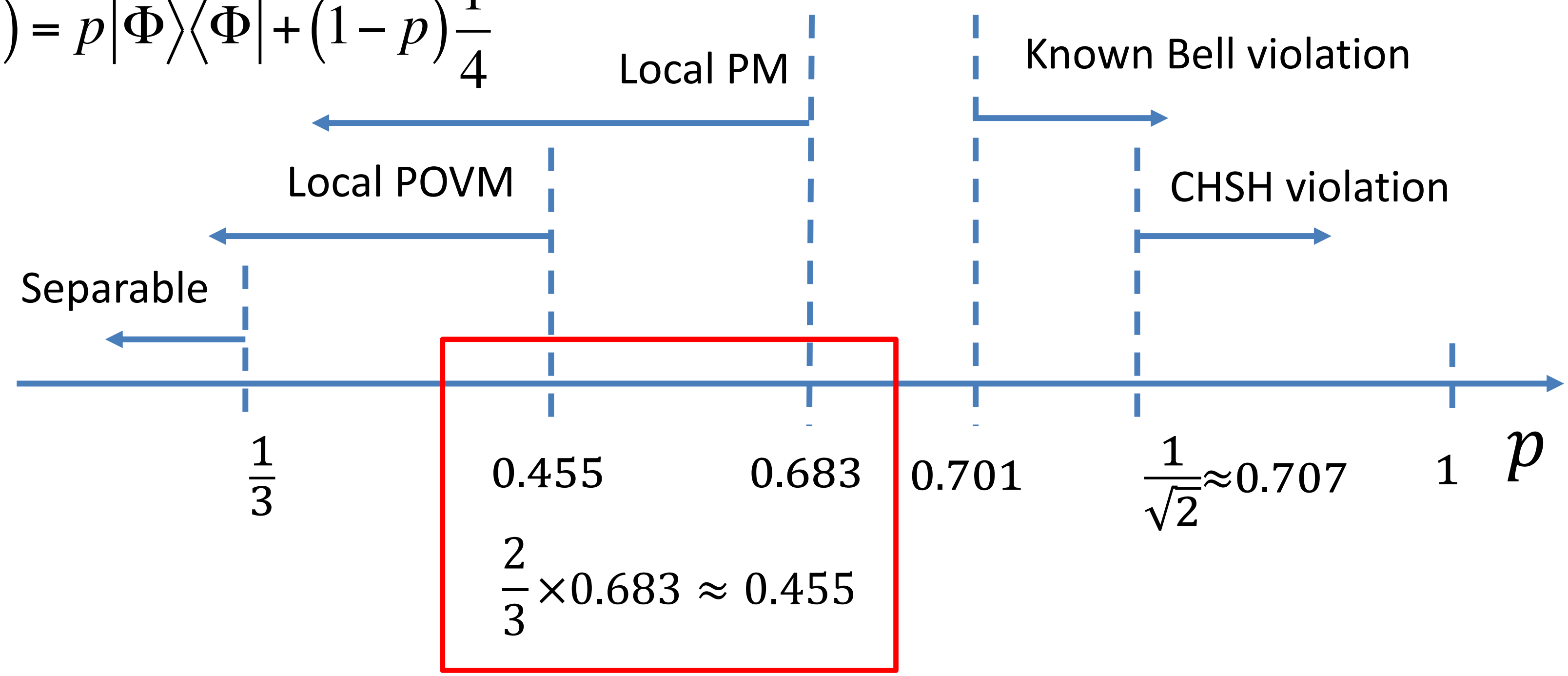


POVM simulation: open questions

- What's the value of critical visibility for larger dimensions?

- Is it always attained for a symetrically informationally complete (SIC) POVM?

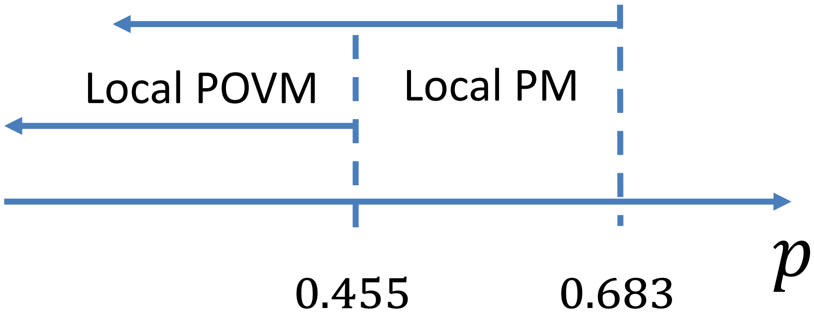
- What's the scaling of t(d)? Even simpler: does it tend to zero when d tends to infinity? If yes, how?

- Resource theory for POVMness?

Concluding remarks

$$\rho(p) = p|\Phi\rangle\langle\Phi| + (1-p)\frac{1}{4}$$

- General measurements provide an advantage for many quantum information tasks. Also for non-locality detection?



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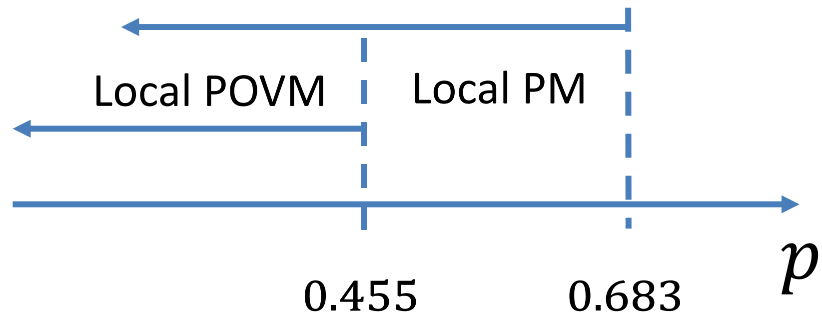

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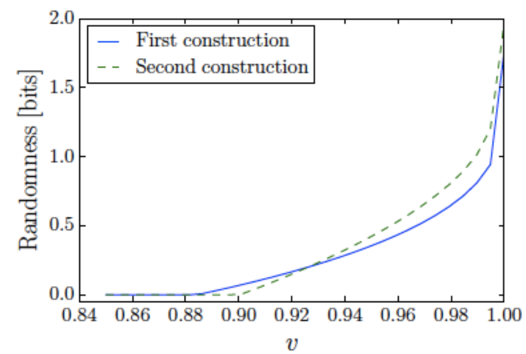

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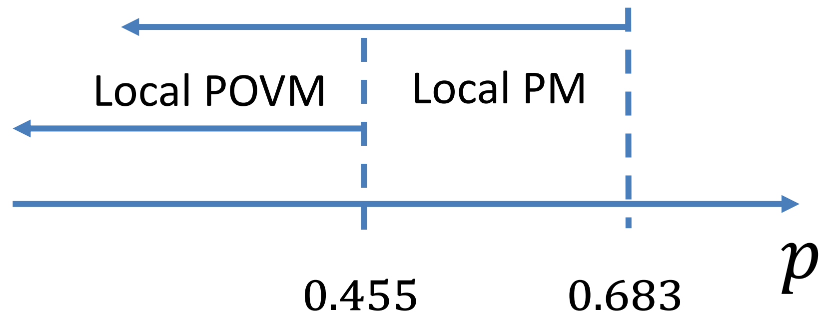

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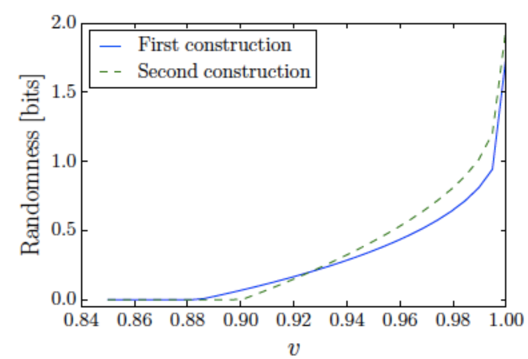

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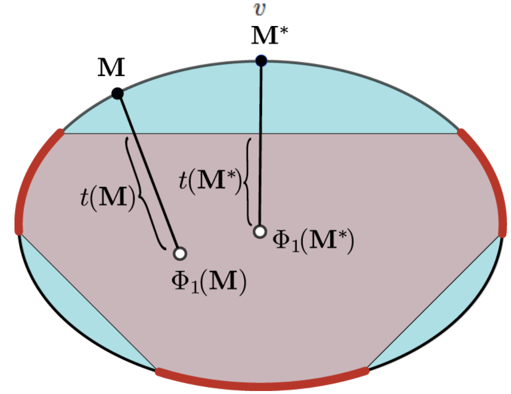

- Noisy general measurements can be projective measurements, hence any advantage is lost.