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How to observe and quantify quantum discord in electronic systems

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Overview

- Quantum correlations in bipartite systems: entanglement and beyond
- Discord: quantumness of separable states related to conditional von Neumann entropy
- Detecting and quantifying discord via interference correlations

Bipartite system: pure state

Pure state of a composite system: $\rho = |\psi\rangle\langle\psi| \equiv \hat{\Pi}_\psi$

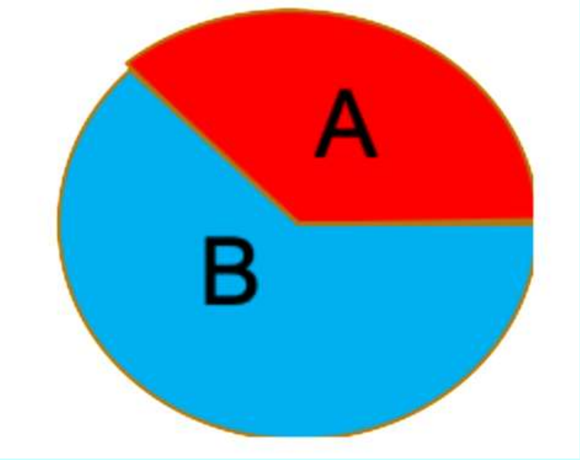
Von Neumann entropy $S = -\operatorname{Tr} \rho \ln \rho = 0$ - no uncertainty

Quantumness (or its absence) reveals in partitioning the system

Schmidt decomposition:

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$$

with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_1 \geqslant \lambda_2 \cdots \geqslant 0$



Entanglement of pure state

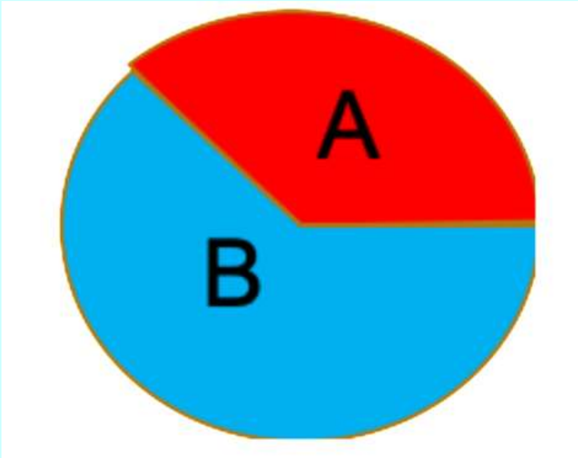
Subsystem A:
marginal (reduced) density matrix $\rho^A = \mathrm{Tr}_B\, \rho$

Marginal (entanglement) entropy:

$$S^A = -\,\mathrm{Tr}_A\, \rho^A \ln \rho^A = -\sum_i \lambda_i \ln \lambda_i$$

$S^A = 0$ iff $\lambda_1 = 1$:
classical product state
– tracing out B makes no impact

$S^A > 0$ when $\lambda_1 < 1$:
quantum entangled states:
– tracing out B creates uncertainty



Bipartite system: mixed state

Is AB entangled or not? – not necessarily obvious, e.g. Werner state:

$$\rho = \frac{1}{4}(1-z)\mathbb{1} + z\,|\psi\rangle\langle\psi| \quad \text{with} \quad \psi \equiv \frac{1}{\sqrt{2}}\big[|00\rangle + |11\rangle\big]$$

which is entangled only for $z < 1/3$.

Generically, a mixed state of a bipartite system is not entangled iff
(Werner, '89)

$$\rho^{AB} = \sum_{\nu=1} w_\nu \rho_\nu^A \otimes \rho_\nu^B, \qquad \sum_\nu w_\nu = 1\,, \qquad \rho_\nu^X = |x_\nu\rangle\langle x_\nu|$$

However, such a separable mixed state can still have quantumness exemplified by quantum discord.

(Ollivier and Zurek, '01; Henderson and Vedral, '01)

Classical mutual information

The concepts of quantum discord comes from comparing quantum and classical conditional entropies in bipartite systems.

Shannon entropy of system AB with joint probability distribution $p(a,b)$:

total: $$H(AB) = -\sum_{a,b} p(a,b) \ln p(a,b)$$

marginal: $$H(A) = -\sum_{a} p(a) \ln p(a) \text{ with } p(a) \equiv \sum_{b} p(a,b)$$

conditional: $$H(A|b) = -\sum_{a} p(a|b) \ln p(a|b) \text{ with } p(a|b) = p(a,b)/p(b)$$

$$H(A|B) = \sum_{b} p(b) H(A|b)$$

Mutual information:

$$I(A:B) = H(A) + H(B) - H(AB)$$

or

$$J(A:B) = H(A) - H(A|B)$$

$I(A:B) = J(A:B)$ as $H(A|B) = H(AB) - H(B)$

Quantum mutual information

Quantum analogue: I,J(A:B) → $\mathcal{I},\mathcal{J}$(AB) with $H \to S$:

$$\mathcal{I}(\hat{\rho}^{AB}) = S(\hat{\rho}^{A}) + S(\hat{\rho}^{B}) - S(\hat{\rho}^{AB})$$

$$\mathcal{J}_A(\hat{\rho}^{AB}) = S(\hat{\rho}^{A}) - S(\hat{\rho}^{A|B}), \quad \text{where}$$

$$S(\hat{\rho}^{A|B}) = \sum_\mu p_\mu \operatorname{Tr}\left[\rho^{A|b} \ln \rho^{A|b}\right], \text{ and}$$

$$p_\mu = \operatorname{Tr}\left[\hat{\Pi}^b_\mu \hat{\rho}\, \hat{\Pi}^b_\mu\right], \quad \hat{\rho}^{A|b} = p_\mu^{-1} \hat{\Pi}^b_\mu \hat{\rho}\, \hat{\Pi}^b_\mu$$

However, $\mathcal{J}_A(\rho^{AB})$ is more tricky as a basis-independent definition of conditional entropy requires optimization over all possible measurements over 'passive' subsystem B. So the more precise definition is

$$\mathcal{J}_A(\hat{\rho}^{AB}) = S(\hat{\rho}^{A}) - \max_{\{\Pi^B_\mu\}} S(\hat{\rho}^{A|B})$$

minimizing ignorance about A, i.e. picking the best measurement basis

Quantum discord

In general, $\mathcal{I}(\hat{\rho}^{AB}) \neq \mathcal{J}(\hat{\rho}^{AB})$. Hence - quantum discord:

$$\boxed{\mathcal{D}_A(\hat{\rho}^{AB}) \equiv \mathcal{I}(\hat{\rho}^{AB}) - \mathcal{J}_A(\hat{\rho}^{AB}) \geqslant 0}$$

(Ollivier and Zurek, '01; Henderson and Vedral, '01)

If $\mathcal{D}_A(\hat{\rho}^{AB}) > 0$, the composite system is A-discorded.

In general, $\mathcal{D}_A(\hat{\rho}^{AB}) \neq \mathcal{D}_B(\hat{\rho}^{AB})$:

A-discorded system is not necessarily B-discorded.

Quantum discord

Alternative expression for quantum discord

$$\mathcal{D} = \max_{\{\Pi_\mu^B\}} S(\hat{\rho}^{A|B}) - \left[S(\hat{\rho}^{AB}) - S(\hat{\rho}^B)\right]$$

Pure state: discord ≡ entanglement

For a pure state (Schmidt decomposition),

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\,|a_i\rangle \otimes |b_i\rangle \text{ and } S(\hat{\rho}^{AB}) = 0\,,$$

but post-measurement state is also pure:

$$\hat{\rho}^{A|B} = |a\rangle\langle a| \text{ with } |a\rangle \equiv \textstyle\sum_b \sqrt{\lambda_i(b)}\,|a_i\rangle \quad \Rightarrow \quad S(\hat{\rho}^{A|B}) = 0$$

$$\mathcal{D} = \max_{\{\Pi^B_\mu\}} S(\hat{\rho}^{A|B}) - \left[S(\hat{\rho}^{AB}) - S(\hat{\rho}^B)\right]$$

Hence, discord $\mathcal{D} = S(\hat{\rho}^B) =$ entanglement entropy.

If a mixed state is entangled, it is always discorded – $\mathcal{D}$ adds little.

Hence, our main interest is in discord of separable – unentangled – states.

Discord of mixed separable state

$\hat{\rho}^{AB} = \sum_{\nu=0}^{n} w_\nu \hat{\rho}_\nu^A \otimes \hat{\rho}_\nu^B$ – generic unentangled state with

$\rho_\nu^A = |a_\nu\rangle\langle a_\nu|$ and $\rho_\nu^B = |b_\nu\rangle\langle b_\nu|$

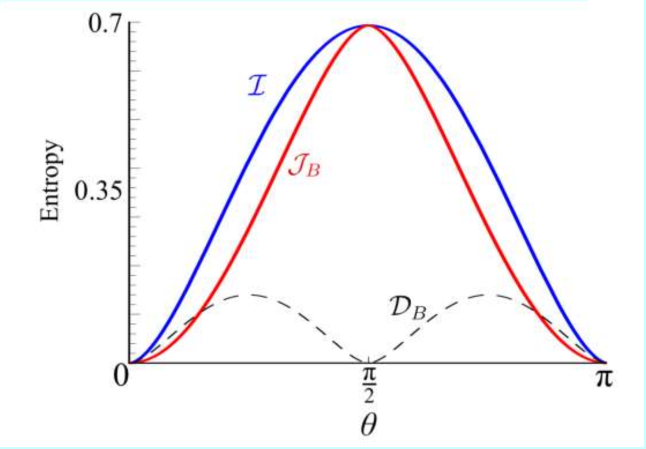
Let us choose for a two-qubit system with $w_0 = w_1$ and

$|a_0\rangle = |0\rangle\,,\ |a_1\rangle = |1\rangle$ and $|b_0\rangle = |0\rangle\,,\ |b_1\rangle \equiv |\theta\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$

i.e. $\hat{\rho}^{AB} = \frac{1}{2}\Big[\,|00\rangle\,\langle 00| + |1\theta\rangle\,\langle 1\theta|\,\Big].$

Here $\mathcal{D}_A$=0 for any θ.

For $\theta=0,\pi$ the subsystems are totally uncorrelated and $\mathcal{D}_B$=0
For $\theta=\pi/2$, the classical mutual information is maximal but it is entirely classical and $\mathcal{D}_B$=0 again.



Discord of mixed separable state

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$\rho_\nu^A = |a_\nu\rangle\langle a_\nu|$ and $\rho_\nu^B = |b_\nu\rangle\langle b_\nu|$

This state is A-non-discorded in either a trivial case – all $|a_\nu>$ coincide, or when all $|a_\nu>$ are orthogonal.

Otherwise, they are discorded independently of B

Our aim: to find ***linear*** in ρ characteristics of a bipartite system that detect and quantify discord

Entanglement witness

A (non-optimized) witness: Bell–CHSH correlator for a bipartite system

$$C(\rho) = \mathrm{Tr}\,\hat{\rho}^{AB}\left[\hat{\boldsymbol{S}}_A \otimes \left(\hat{\boldsymbol{S}}_B + \hat{\boldsymbol{S}}'_B\right) + \hat{\boldsymbol{S}}'_A \otimes \left(\hat{\boldsymbol{S}}_B - \hat{\boldsymbol{S}}'_B\right)\right]$$

For a (Schmidt-decomposed) pure state of a bipartite system,

$$\max C(\rho) = 2\left(1 + \sum_{i\neq j}\sqrt{\lambda_i\lambda_j}\right) \quad \begin{cases} = 2 & \text{if } \lambda_1 = 1 \text{ -- classical} \\ > 2 & \text{if } \lambda_1 < 1 \text{ -- entangled} \end{cases}$$

For the generic separable state, $\max C(\rho) = 2\sum_\nu w_\nu = 2$ – unentangled.

Q. How to detect and quantify the remaining quantumness – discord – linearly, as $\mathrm{Tr}\,(\rho\ \ldots)$, without full or partial quantum tomography.

Theorem: no linear witness of discord (R. Rahimi and A. SaiToh, 2010)

Way around: **repeated** measurements of certain correlations.

Discord via correlations

(1) Prepare the system in a mixed state: $\hat{\rho}^{AB} = \sum_{\nu=0}^{n} w_\nu \hat{\rho}_\nu^A \otimes \hat{\rho}_\nu^B$

(2) Let the system to evolve: $\hat{\rho}^{AB} \to \mathsf{S}\hat{\rho}^{AB}\mathsf{S}^\dagger$ with $\mathsf{S} = \mathsf{S}^A \otimes \mathsf{S}^B$

(3) Test post-evolution A-basis rotation: $\mathsf{S}^A \to \mathsf{S}_d(\phi_d)\mathsf{S}^A$

(4) Make correlated projective measurements on both subsystems,

$$K(\phi_d) = \mathrm{Tr}\left[\Pi_A\,\Pi_B\,\mathsf{S}\,\rho^{AB}\,\mathsf{S}^\dagger\right] = \mathrm{Tr}_A\left[\mathsf{S}_d\,\widetilde{\rho}^{A|B}\,\mathsf{S}_d^\dagger\Pi_A\right]$$

(5) Repeat with changing ϕ_d to get interference pattern

$$K(\phi_d) = \mathcal{C} + \left(\mathcal{A}e^{i\phi_d} + \mathrm{c.c.}\right).$$

(6) Measure interference visibility,

$$\mathcal{V} = |\mathcal{A}/\mathcal{C}| = \frac{\max[K(\phi_d)] - \min[K(\phi_d)]}{\max[K(\phi_d)] - \min[K(\phi_d)]}$$

Visibility as discord witness

Visibility vanishes when post-evolution $\widetilde{\rho}^{A|B} = \mathsf{S}_A\,\rho^{A|B}\,\mathsf{S}_A^{\dagger}$ is diagonal:

$$K(\phi_d) = \mathrm{Tr}_A\left[\mathsf{S}_d\,\widetilde{\rho}^{A|B}\,\mathsf{S}_d^{\dagger}\Pi_A\right] \text{ is } \phi\text{-independent}$$

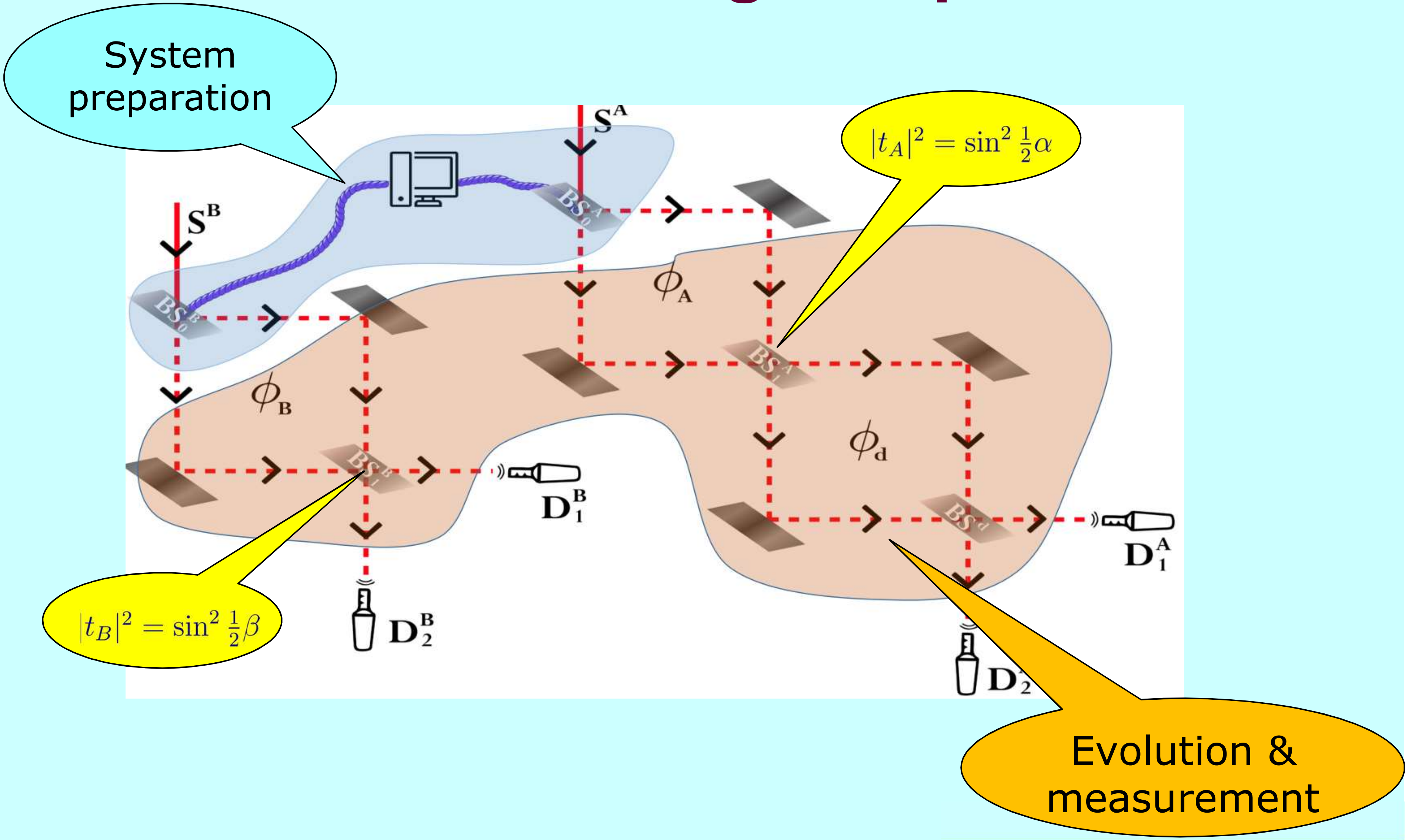
Changing S_A one always find S_A^0 that reduces $\mathcal{V}$ to 0.

Interference pattern as will always have lines of zero visibility..

Crucial:

Zero- $\mathcal{V}$ lines are B independent iff $\mathcal{D}_A = 0$.

Measuring setup



Input state

$\hat{\rho}^{AB} = \sum_{\nu=0}^{2} w_\nu \hat{\rho}_\nu^A \otimes \hat{\rho}_\nu^B$ with $\rho_\nu^X = |x_\nu\rangle\langle x_\nu|$

and $|x_\nu\rangle = \cos\theta_\nu^x |\uparrow\rangle + \sin\theta_\nu^x |\downarrow\rangle$

Two simple examples with $|\pm\rangle \equiv |\uparrow\rangle \pm |\downarrow\rangle$:

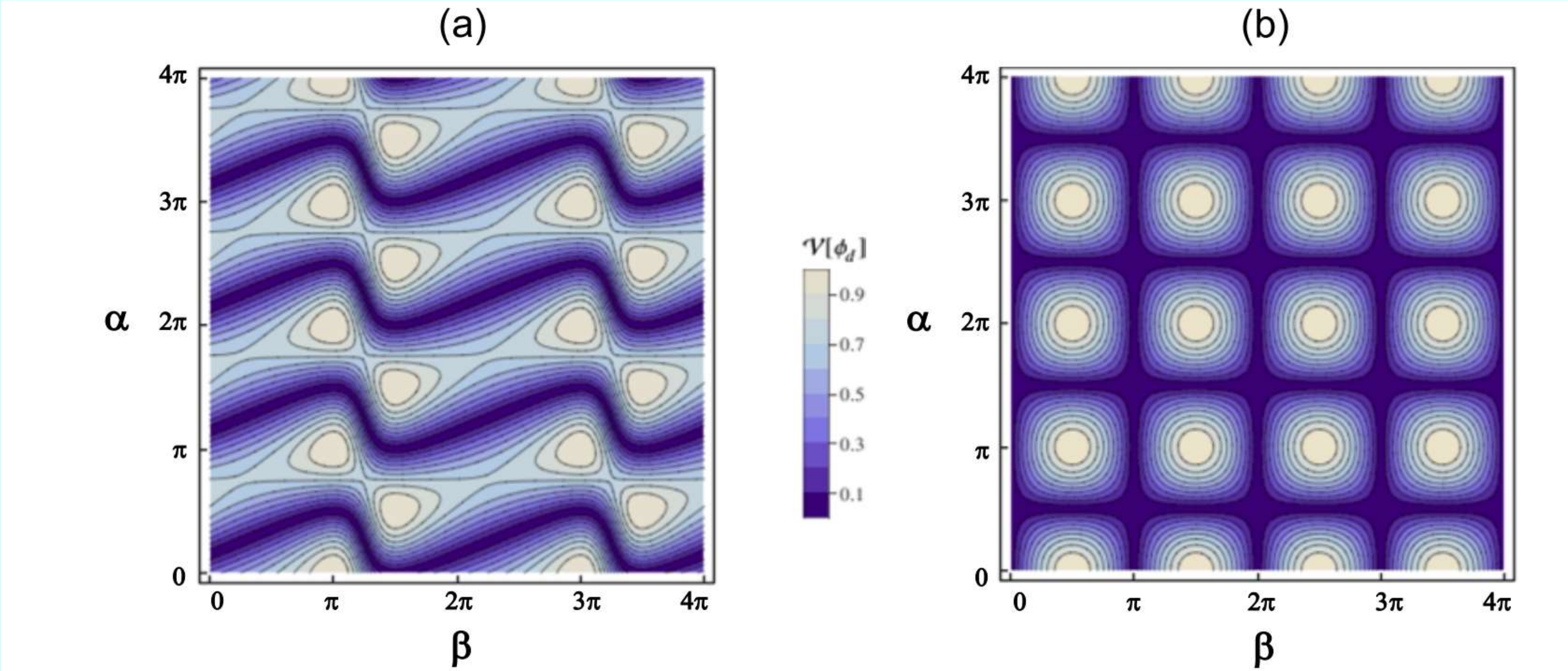
(a) $\rho^{AB} = \frac{1}{2}\left[|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|++\rangle\langle++|\right]$

discorded

(b) $\rho^{AB} = \frac{1}{2}\left[|++\rangle\langle++| + \frac{1}{2}|--\rangle\langle--|\right]$

nondiscorded

Visibility pattern

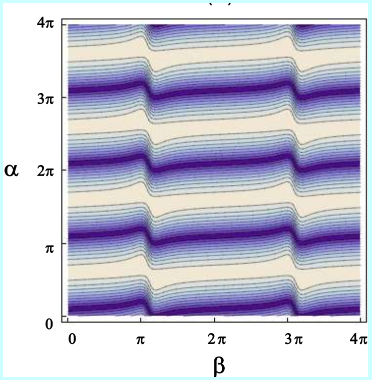


(a) $\rho^{AB} = \frac{1}{2}\left[\left|\uparrow\uparrow\right\rangle\left\langle\uparrow\uparrow\right| + \frac{1}{2}\left|++\right\rangle\left\langle++\right|\right]$

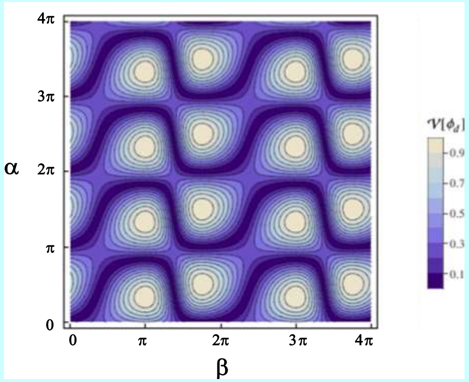
(b) $\rho^{AB} = \frac{1}{2}\left[\left|++\right\rangle\left\langle++\right| + \frac{1}{2}\left|--\right\rangle\left\langle--\right|\right]$

Discord witnesses

The visibility plots for $\rho^{AB} = \frac{1}{2}\,|\uparrow\uparrow\rangle\,|\uparrow\uparrow\rangle + \frac{1}{2}\,|\theta\theta\rangle\langle\theta\theta|$

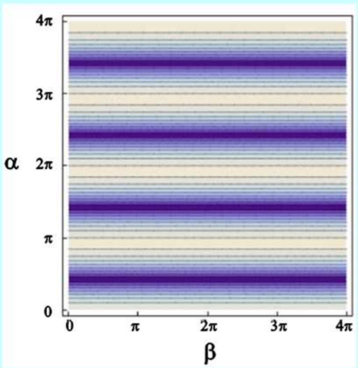


$\theta = 5\pi/6$

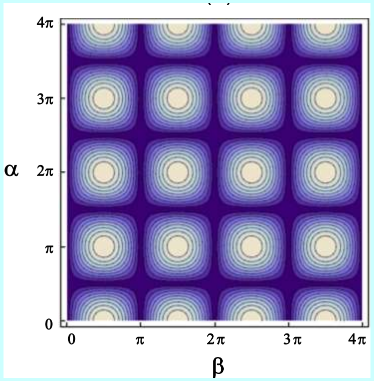


$\theta = \pi/6$

discorded



$\theta = 0$

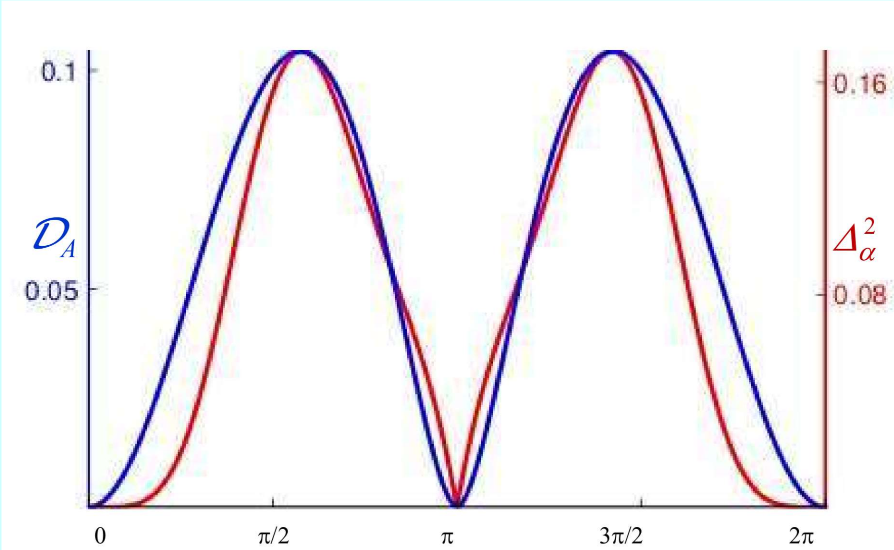


$\theta = \pi/2$

nondiscorded

Quantifying discord

From zero-visivility line $\alpha_0(\beta)$:
get $f_\alpha(\beta) \equiv \cos^2[\alpha_0(\beta)]$;
build its deviation from the mean



$$\Delta_\alpha^2 = \int_0^{2\pi} \frac{d\beta}{2\pi} \left[f_\alpha(\beta) - \overline{f}_\alpha\right]^2,$$

$$\overline{f}_\alpha = \int_0^{2\pi} \frac{d\beta}{2\pi} f_\alpha(\beta).$$

Summary

- Discord is hard to measure. Alternatives (geometric discord) are based on full or partial quantum tomography – hardly extendable to condensed matter systems
- The proposed discord witness – the visibility in (linear in ρ) interference pattern
- The proposed quantifier gives results similar to the original.