Nonlinear phase coupling functions: a numerical study

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Contents of the talk

1. An introduction: - phase of a limit cycle oscillator
   - models of phase dynamics
   - linear and nonlinear coupling functions

2. Nonlinear coupling functions: a numerical approach

3. A simple case: forced Stuart-Landau oscillator

4. A less simple case: - forced Rayleigh oscillator
   - forced Rössler oscillator

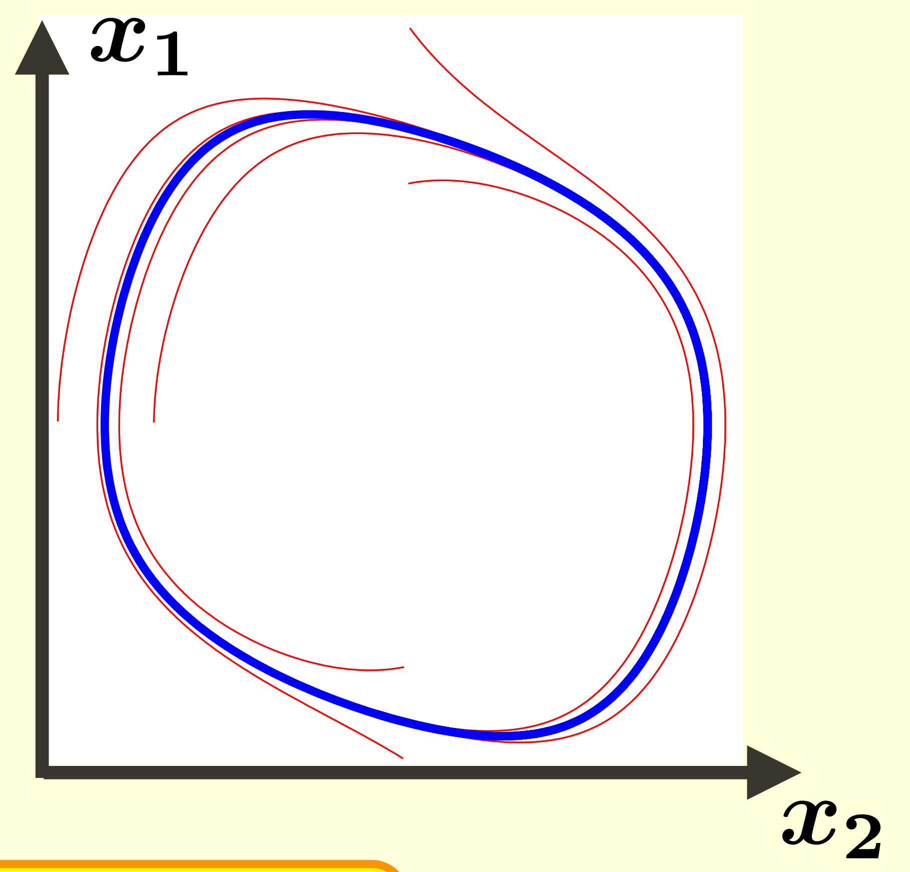
5. Conclusions

Phase dynamics: brief summary

Consider general N-dimensional self-sustained oscillator

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) \;, \mathbf{x} = (x_1, x_2, \ldots, x_N)$$

with a **stable limit cycle** $\mathbf{x}_T$



Phase is defined from the condition $\dot{\varphi} = \omega = 2\pi/T$

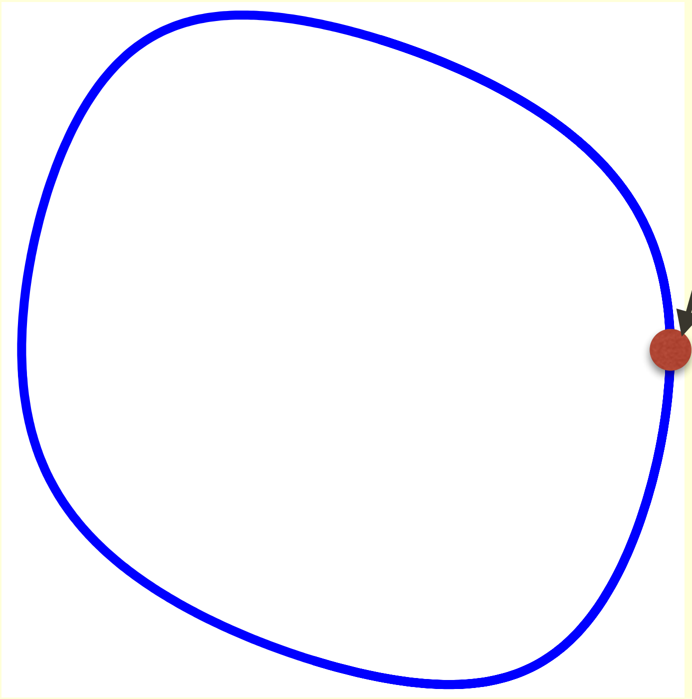
and can be introduced in two steps:

1. phase on the limit cycle

2. phase in the basin of attraction of the limit cycle

Phase on the limit cycle

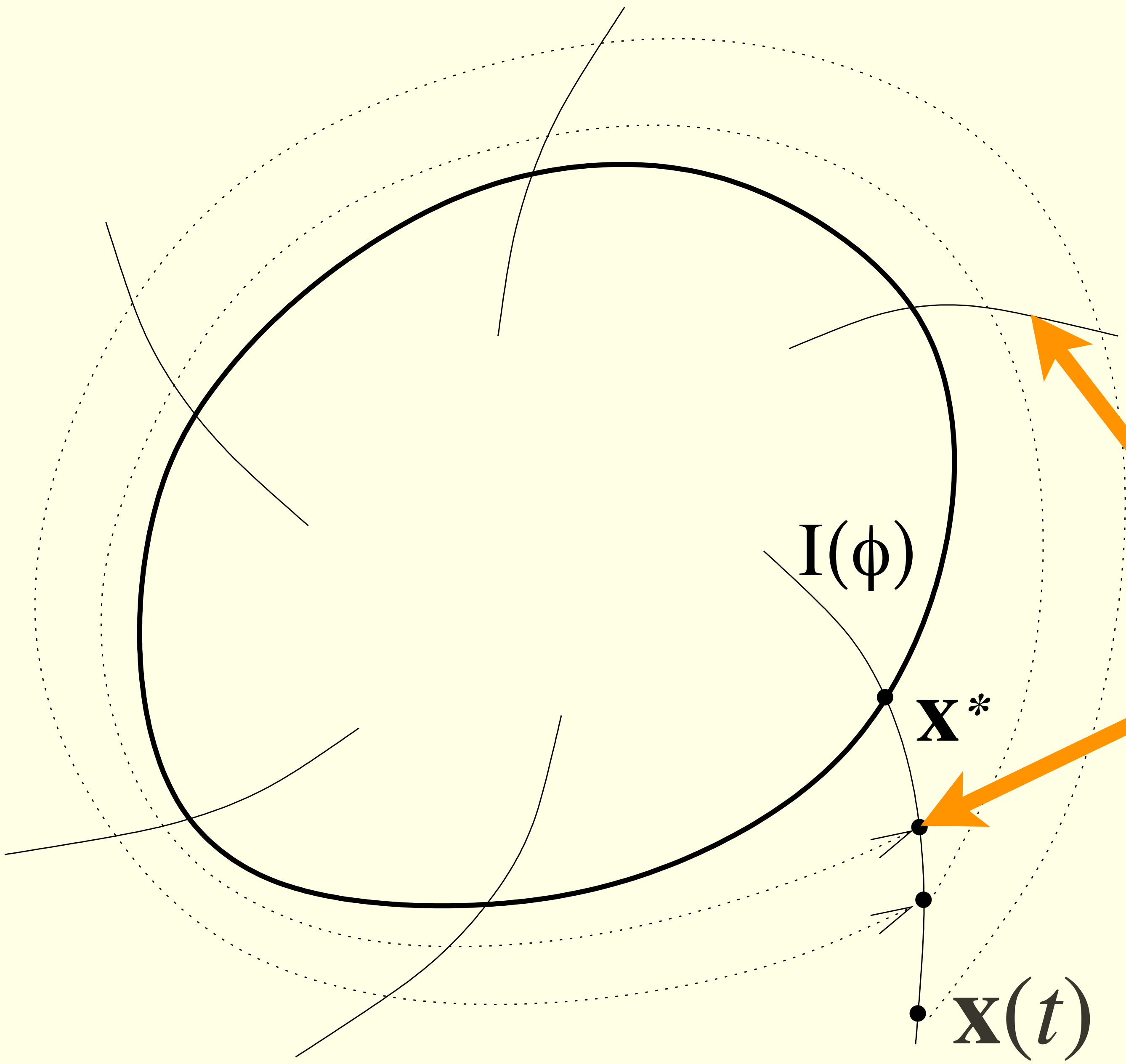
We start with some (arbitrary) zero point, $\mathbf{x}(t_0) \to \varphi(\mathbf{x}(t_0)) = 0$



and define phase as $$\varphi = 2\pi \frac{t - t_0}{T}$$

A remark: phase can be defined either on $[0,2\pi)$ interval or on the real line

Phase in the vicinity of the cycle: Isochrons



Stroboscopic observation with the period $T = 2\pi/\omega$

Isochrons:
Lines of constant phase
(Generally, they are ***N-1*** dimensional hypersurfaces)

$$\varphi\left(\mathbf{x}(t)\right) = \varphi(\mathbf{x}^*) \text{ WHERE } \mathbf{x}^* = \lim_{m\to\infty} \mathbf{x}(t+mT)$$

Thus, we have $\varphi = \varphi(\mathbf{x})$

Phase reduction

Perturbation technique for weak coupling,
Malkin 1956, Kuramoto 1984

The forced system $\dot{\mathbf{x}} = G(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

coupling strength, small parameter

$\varepsilon \ll |\boldsymbol{\lambda}_-|$

Negative Lyapunov exponent
(determines the stability of the limit cycle)

Phase reduction

Perturbation technique for weak coupling,
Malkin 1956, Kuramoto 1984

The forced system $\dot{\mathbf{x}} = G(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

coupling strength, small parameter

→ in the first approximation in ε one writes

$$\dot{\varphi}(\mathbf{x}) = \frac{\partial \varphi}{\partial \mathbf{x}} \dot{\mathbf{x}} = \frac{\partial \varphi}{\partial \mathbf{x}} \left[\mathbf{G}(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t) \right]$$

$$= \omega + \frac{\partial \varphi}{\partial \mathbf{x}} \varepsilon \mathbf{p}(\mathbf{x}, t) \approx \omega + \left. \frac{\partial \varphi}{\partial \mathbf{x}} \right|_{\mathbf{x}_T} \varepsilon \mathbf{p}(\mathbf{x}_T, t)$$

where we
1. use the phase definition for the **unperturbed system**
2. we compute the r.h.s. **on the cycle**

Phase reduction

The forced system $\dot{\mathbf{x}} = G(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

$$\dot{\varphi}(\mathbf{x}) = \omega + \left. \frac{\partial \varphi}{\partial \mathbf{x}} \right|_{\mathbf{x}_T} \varepsilon \mathbf{p}(\mathbf{x}_T, t)$$

Let force be periodic → we characterise it by its phase $\psi, \dot{\psi} = \nu$

Then $\mathbf{p}(\mathbf{x}_T, t) \to \mathbf{p}(\psi, t)$

Points on the limit cycle are in a one-to-one correspondence to φ, i.e. $\mathbf{x}_T = \mathbf{x}_T(\varphi)$ → we obtain a closed equation for the phase:

$$\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi)$$

Phase reduction: the coupling function

The forced system $\dot{\mathbf{x}} = G(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

coupling function

Phase equation $\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi)$

where $Q(\varphi, \psi) = \left.\frac{\partial \varphi}{\partial \mathbf{x}}\right|_{\mathbf{x}_T} \mathbf{p}(\mathbf{x}_T(\varphi), \psi)$

Phase reduction: the Winfree form

The forced system $\dot{\mathbf{x}} = G(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

coupling function

Phase equation $\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi)$

where $Q(\varphi, \psi) = \left.\dfrac{\partial \varphi}{\partial \mathbf{x}}\right|_{\mathbf{x}_T} \mathbf{p}(\mathbf{x}_T(\varphi), \psi)$

Notice: if the forcing is scalar, $\mathbf{p}(\mathbf{x}, \psi) = p(\psi)$ then

$$Q(\varphi, \psi) = Z(\varphi) p(\psi)$$

Phase Sensitivity Curve, or
Phase Response Curve (PRC)

Thus, we have $\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi) = \omega + \varepsilon Z(\varphi) p(\psi)$

the Winfree form

Phase reduction: the Kuramoto-Daido form

The forced system $\dot{\mathbf{x}} = G(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

coupling function

Phase equation $\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi)$

If norm $\| \varepsilon Q \| \ll \omega$ the phase equation can be averaged, keeping the resonance terms

If $\omega / \nu \approx m/n$ then averaging yields

$$\dot{\varphi} = \omega + \varepsilon h(n\varphi - m\psi)$$

the Kuramoto-Daido form

Phase reduction: beyond first approximation

Thus, for weak coupling, i.e. in the **first approximation** one obtains

$$\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi)$$

Let us denote this explicitly:

$$\dot{\varphi} = \omega + \varepsilon Q_1(\varphi, \psi)$$

order of approximation

Phase reduction: beyond first approximation

Thus, for weak coupling, i.e. in the **first approximation** we obtained

$$\dot{\varphi} = \omega + \varepsilon Q(\varphi, \psi)$$

Let us denote this explicitly:

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order of approximation

Generally, one expects

$$\dot{\varphi} = \omega + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \ldots = \omega + Q(\varphi, \psi)$$

… but it is unknown, how to compute $Q_2, Q_3, \ldots$

Phase reduction: problems

1. Even computation of Q_1 is difficult if the isochrons are not known analytically

2. Power series representation remains a conjecture; there no algorithms for computation of $Q_2, Q_3, \dots$

Phase reduction: problems

1. Even computation of Q_1 is a problem if the isochrons are not known analytically
2. Power series representation remains a conjecture; there no algorithms for computation of $Q_2, Q_3, \ldots$

… and approaches

1. Extension to the case of strong coupling with account of deviations from the limit cycle: a number of attempts, see e.g. recent review B. Monga et al, Biol. Cybern., 2019
2. We suggest a **numerical approach**

The simplest model: the Stuart-Landau system

$$\dot{A} = (\mu + i\eta)A - (1 + i\alpha)|A|^2 A + \varepsilon p(\psi), \quad \psi = \nu t$$

In polar coordinates, with $A = Re^{i\theta}$:

$$\dot{R} = \mu R - R^3 + \varepsilon p(\psi) \cdot \cos\theta$$

$$\dot{\theta} = \eta - \alpha R^2 - \varepsilon p(\psi) \cdot \sin\theta / R$$

(*)

Isochrons are known analytically:

$$\varphi = \theta - \alpha \ln(R/R_0) \text{ WITH } R_0 = \sqrt{\mu}$$

Derivation with account of (*) yields

$$\dot{\varphi} = \omega - \frac{\alpha\cos\theta + \sin\theta}{R}\varepsilon p(\psi) \text{ WITH } \omega = \eta - \alpha\mu$$

The Stuart-Landau system: PRC and coupling functions

$$\dot{\varphi} = \omega - \frac{\alpha \cos\theta + \sin\theta}{R}\varepsilon p(\psi) \text{ WITH } \omega = \eta - \alpha\mu$$

For weak force $R \approx R_0 = \sqrt{\mu}, \theta \approx \varphi$ → **well-known results**

PRC: $Z(\varphi) = -\mu^{-1/2}(\alpha\cos\varphi + \sin\varphi)$

Linear coupling function for harmonic forcing $p(\psi) = \cos\psi$:

$$Q_1(\varphi, \psi) = -\mu^{-1/2}(\alpha\cos\varphi + \sin\varphi)\cos\psi$$

Averaging Q_1 for $\nu \approx \omega$ yields the Kuramoto-Daido function

$$h(\varphi - \psi) = -0.5\mu^{-1/2}[\alpha\cos(\varphi - \psi) + \sin(\varphi - \psi)]$$

Computing nonlinear coupling functions

$$\dot{\varphi} = \omega + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \ldots = \omega + Q(\varphi, \psi)$$

$$Q(\varphi, \psi) = \varepsilon Q_1 + Q_{nlin}$$

known from the theory → εQ_1; **shall be obtained numerically** → Q_{nlin}

We simulate the forced Stuart-Landau system to obtain $\varphi(t), \dot{\varphi}(t)$

and fit the rest term $\dot{\varphi}_r = \dot{\varphi} - \omega - \varepsilon Q_1$ by a function of φ, ψ

Practically: we use kernel density estimation on an 100x100 grid

Fitting the coupling function

→ we **fit** the equation $\dot{\varphi}_r = Q_{nlin}(\varphi, \psi)$

We use kernel density estimation on an $n \times n$ grid

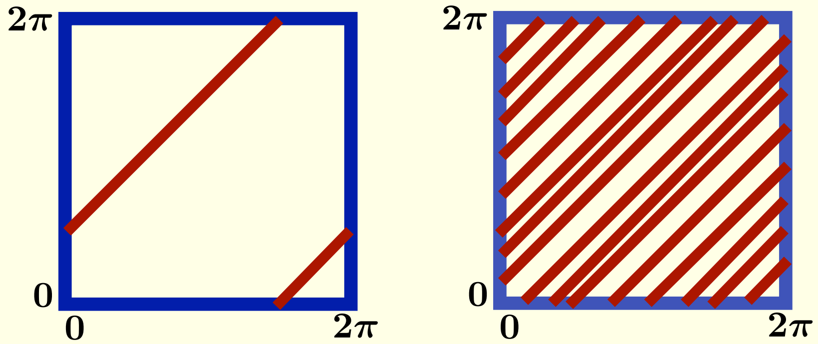
with kernel $K(x, y) = \exp\left[\frac{n}{2\pi}(\cos x + \sin y)\right]$

We start with time series $\varphi_{r,k}, \varphi_k, \psi_k$

and for each point φ, ψ on the equidistant grid compute

$$Q(\varphi, \psi) = \frac{\sum_k \dot{\varphi}_{r,k} K(\varphi - \varphi_k, \psi - \psi_k)}{\sum_k K(\varphi - \varphi_k, \psi - \psi_k)}$$

Notice: the fitting works in the absence of locking!



Computing nonlinear coupling functions

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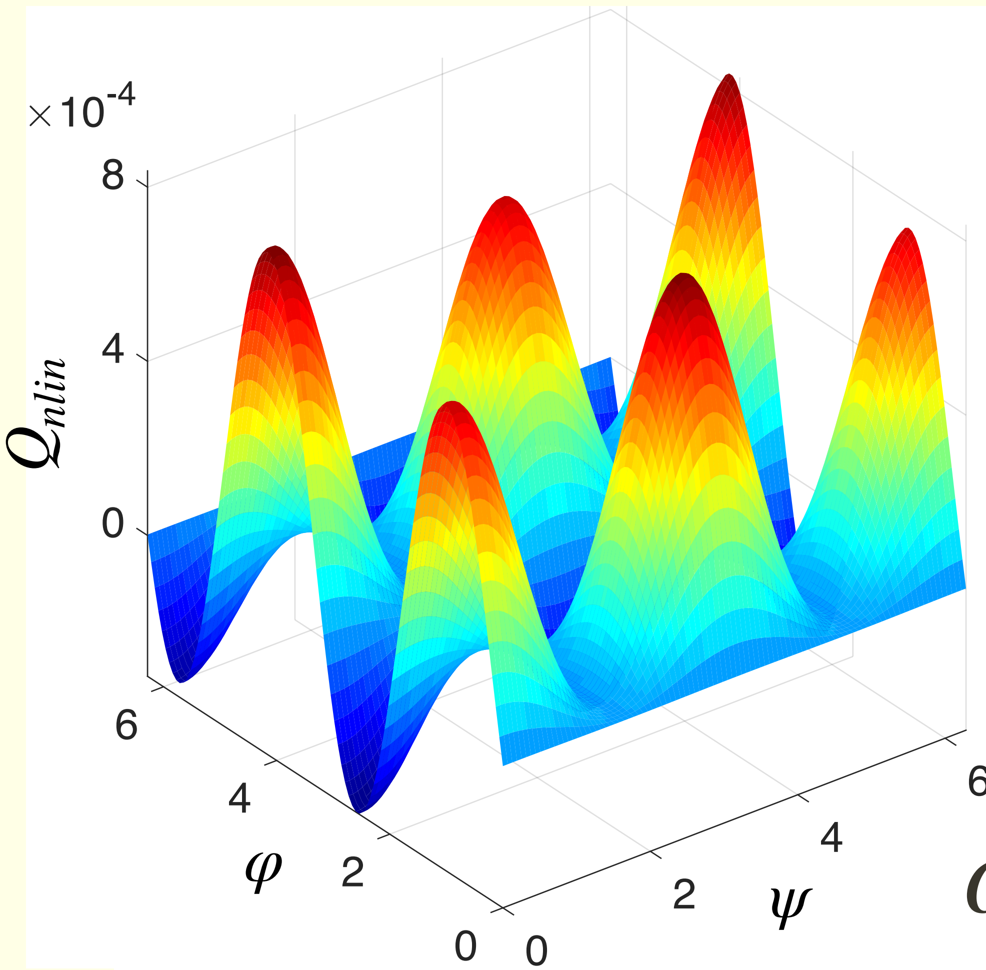
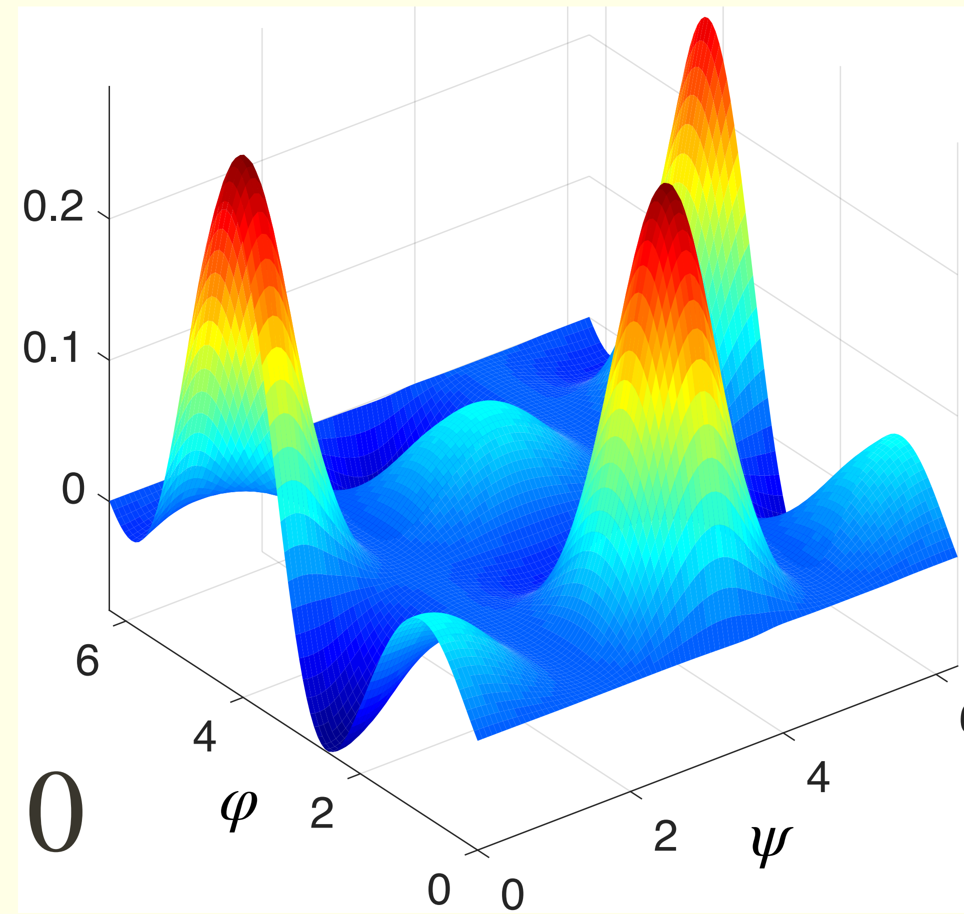
Practically: we use kernel density estimation on an 100x100 grid

We compute Q_{nlin} for $\nu = \text{const}$ and different values of ε and obtain $Q_{2,3,4}$ performing a polynomial fit in ε

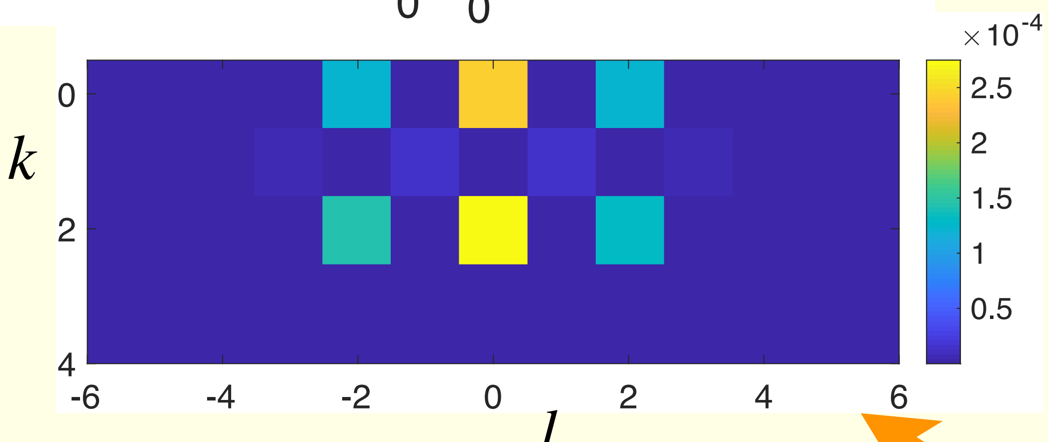
Stuart-Landau oscillator: nonlinear coupling functions

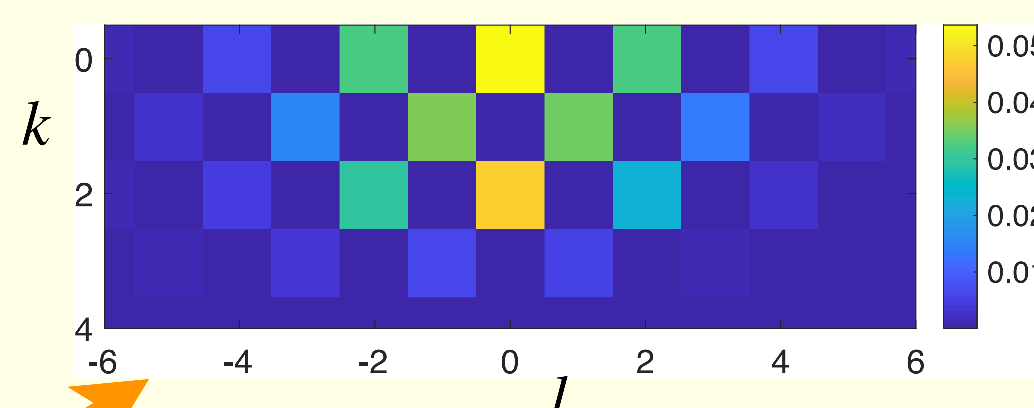
$\varepsilon = 0.05$ $\nu = 0.3$ $\varepsilon = 0.55$





$\alpha = 0$

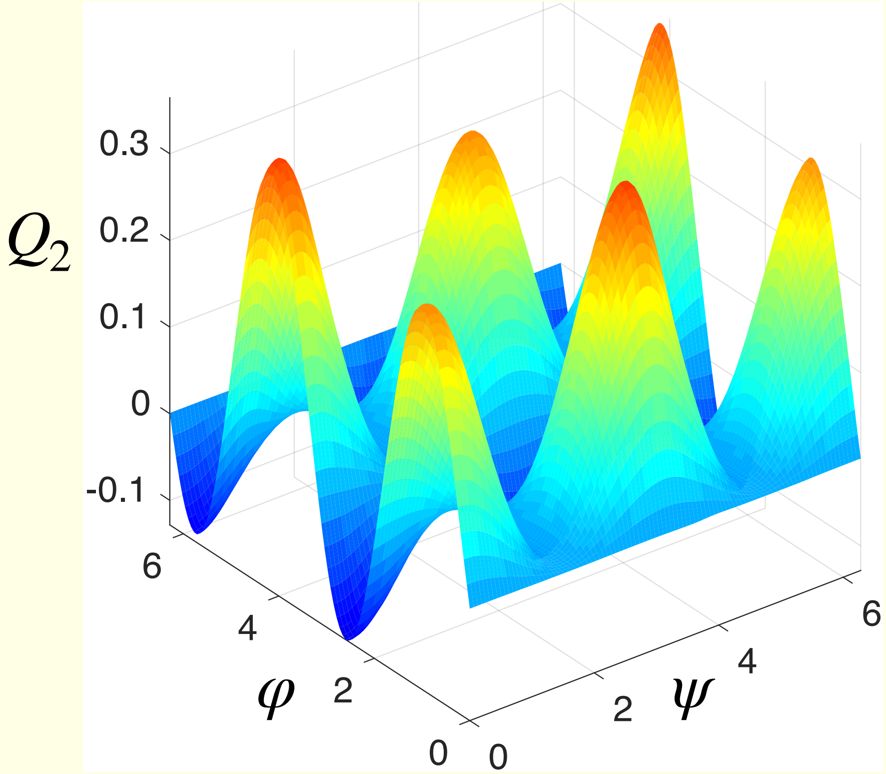


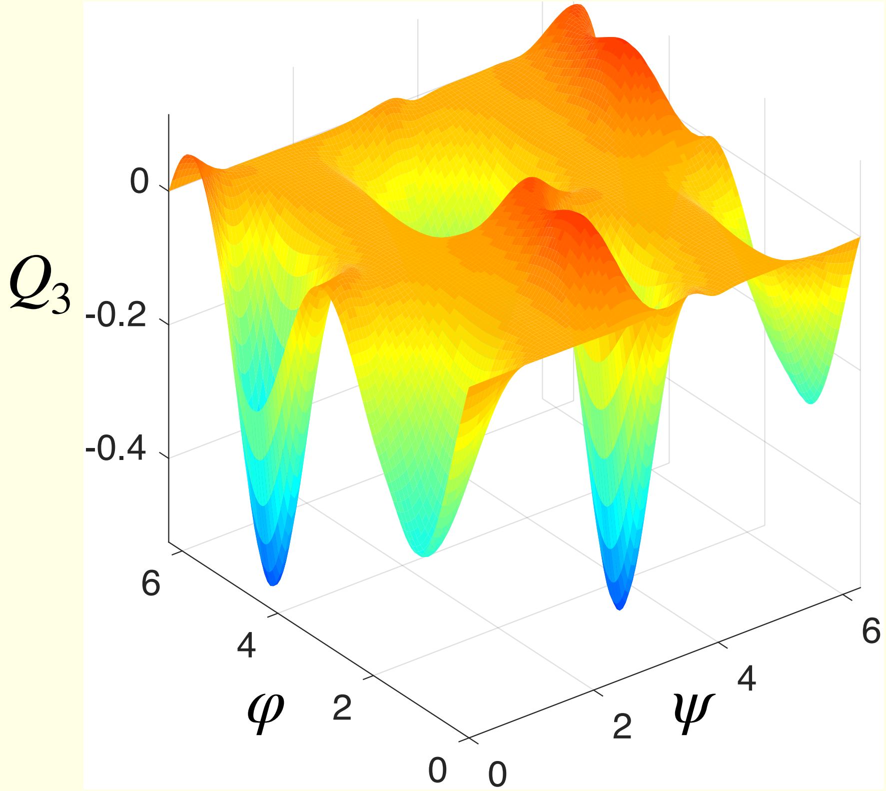


amplitudes of the Fourier modes

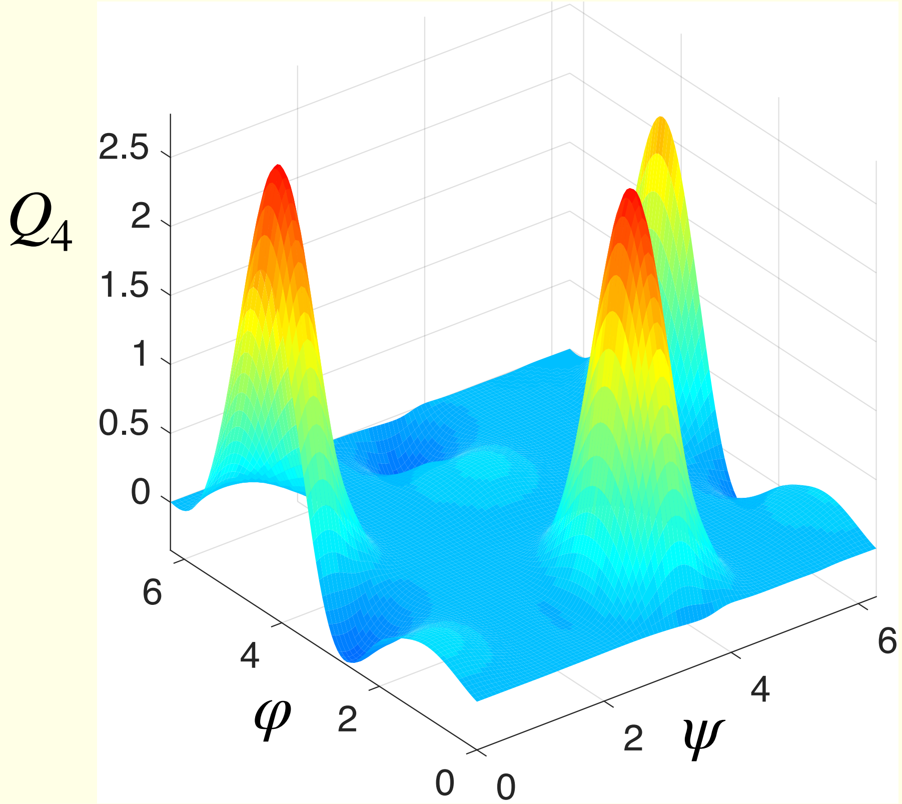
Stuart-Landau oscillator: nonlinear coupling functions

$\nu = 0.3, \alpha = 0$ $\qquad$ $Q_{nlin} \approx \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^4 Q_4$





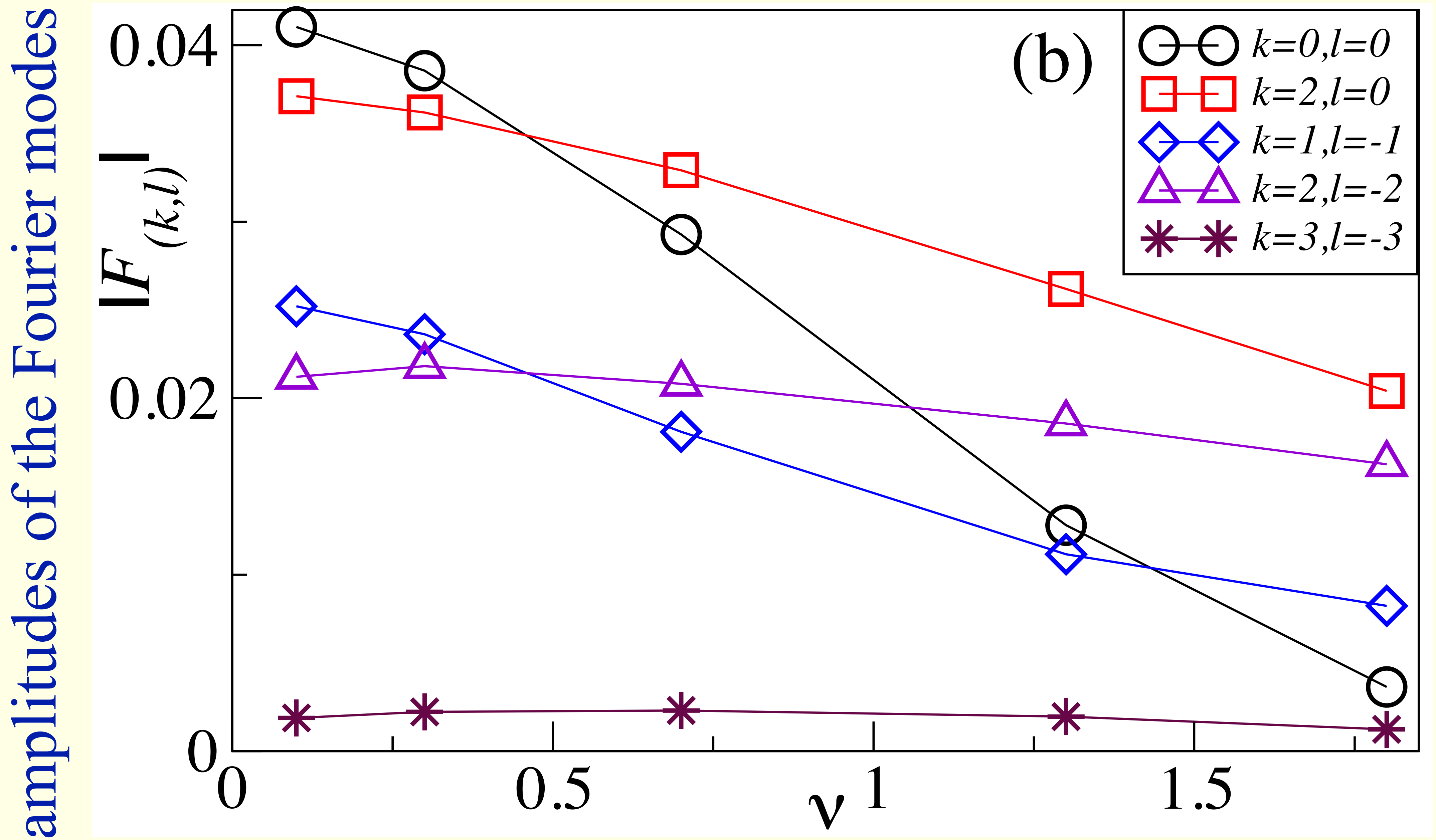
Description via nonlinear function is valid for coupling as strong as $\varepsilon = 0.55$!



Nonlinear coupling functions: frequency dependence

Q_1 does not depend on the frequency of the force

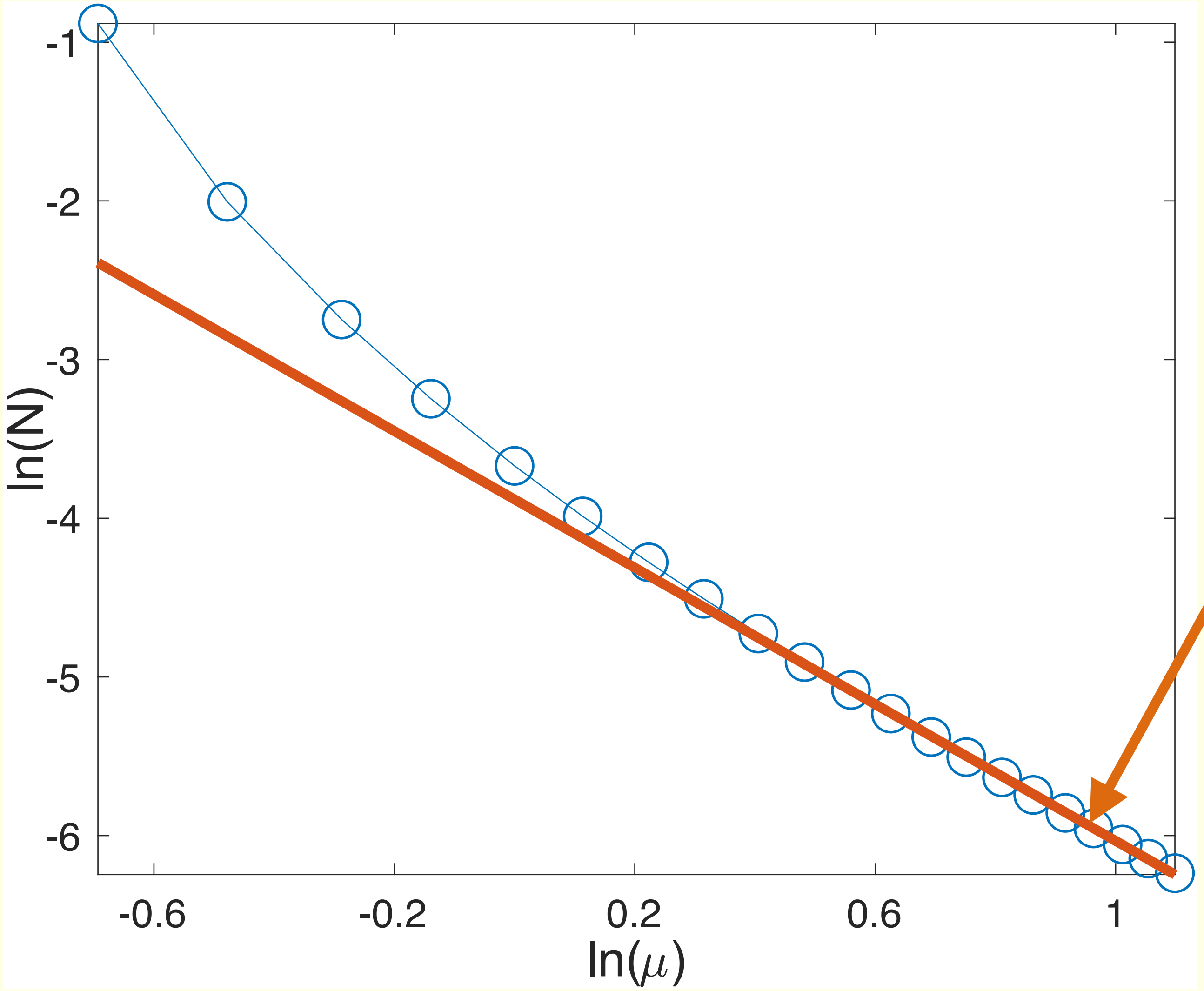
Q_{nlin} depends on the frequency of the force



frequency of the force

Nonlinear coupling functions: μ-scaling

Parameter μ determines stability of the limit cycle



slope ≈ 2.15

Norm of Q_{nlin} scales as $\mu^{-2.15}$

Norm of Q_1 scales as $\mu^{-0.5}$

Nonlinear effects are less visible for $\mu \to \infty$

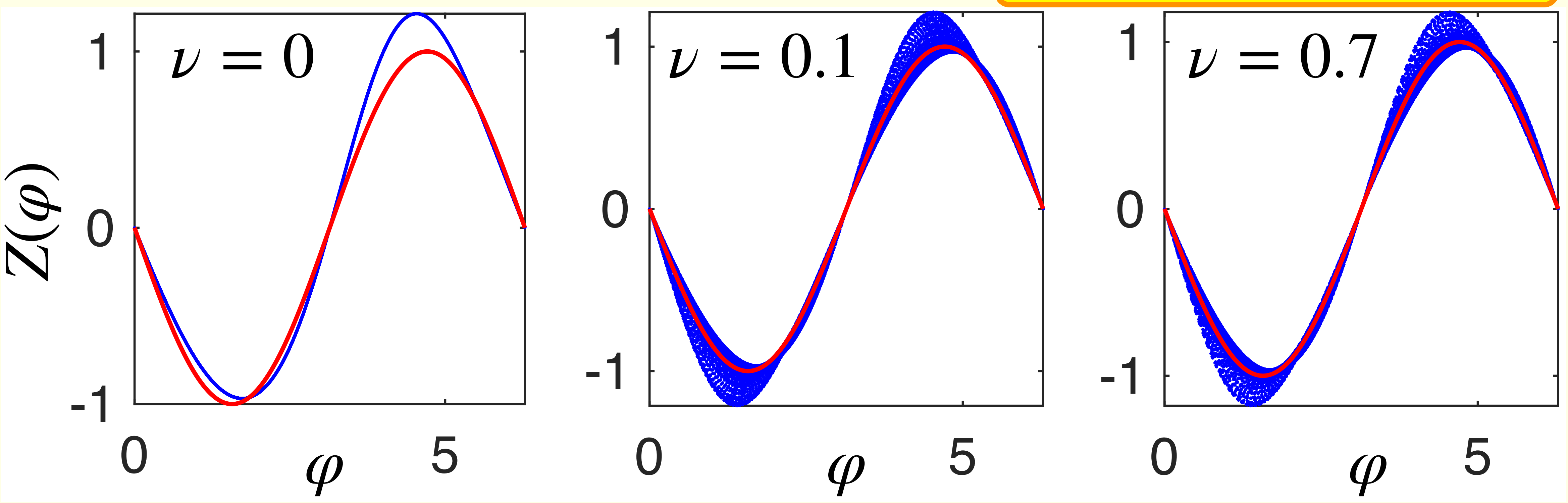
The Winfree form for strong forcing

In the first approximation: $\dot{\varphi} = \omega + \varepsilon Z(\varphi)\cos(\nu t)$

For large ε we obtain "effective" $Z(\varphi)$ by plotting

$$\frac{\dot{\varphi} - \omega}{\varepsilon \cos(\nu t)} \text{ VS. } \varphi \text{ FOR } \varepsilon\cos(\nu t) > 10^{-5}$$

Red curve is linear PRC



Generally, the nonlinear coupling function cannot be represented as a product!

Predicting synchronization regions with nonlinear coupling functions

First order approximation for the Stuart-Landau system, for

$$\alpha = 0, \mu = 1, \nu \approx \omega$$

$$\dot{\varphi} = \omega - \varepsilon \sin\varphi \cos\psi$$

$$= \omega - \frac{\varepsilon}{2}\sin(\varphi + \psi) + \frac{\varepsilon}{2}\sin(\psi - \varphi)$$

Averaged equation $\dot{\varphi} = \omega - \frac{\varepsilon}{2}\sin(\psi - \varphi)$

This term determines 1:1 locking

Other locked state do not appear in the averaged equation!

Beyond the linear approximation: many Fourier terms,

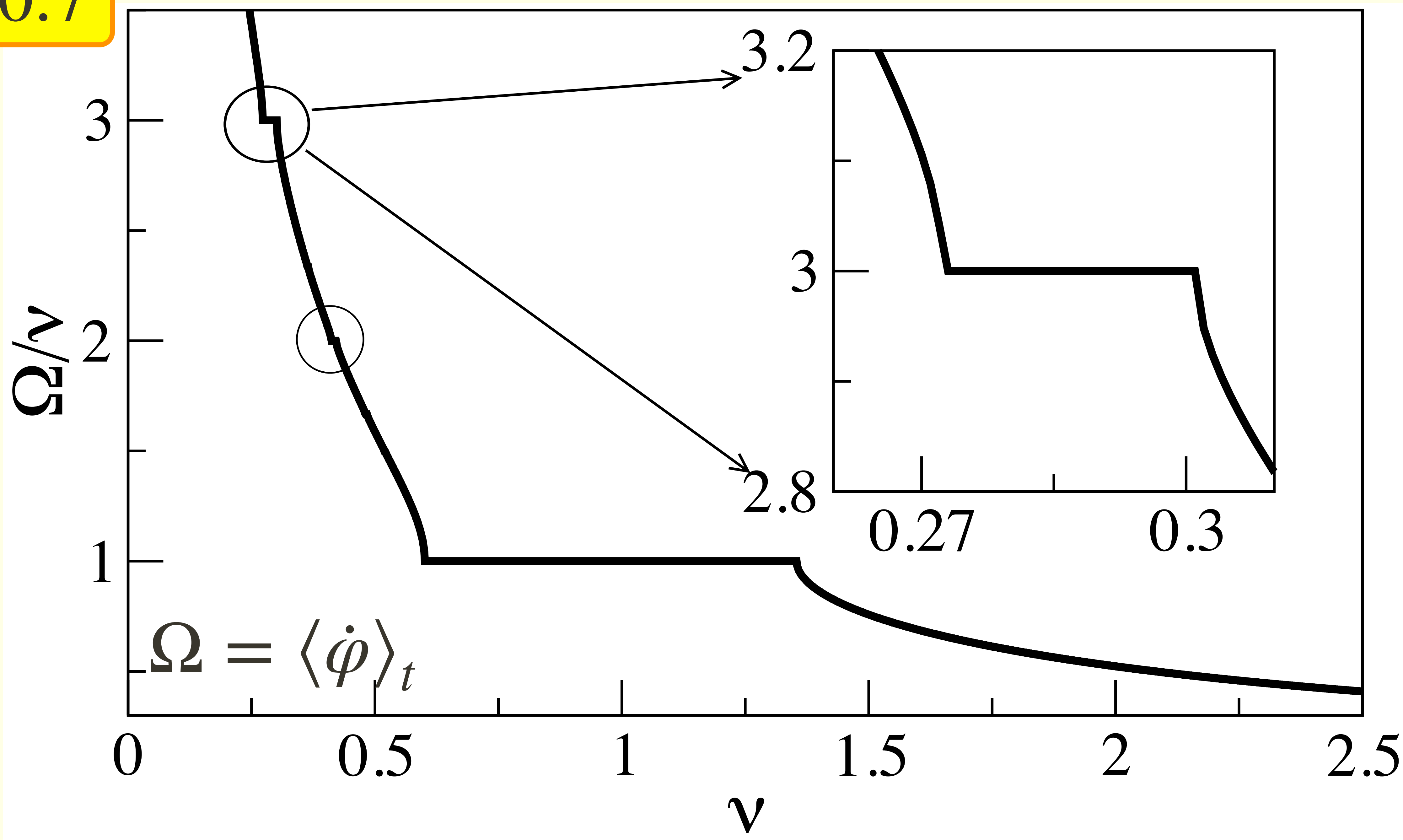
many locking regions!

Stuart-Landau oscillator: synchronization domains

Beyond the linear approximation: many Fourier terms,

many locking regions!

$\varepsilon = 0.7$



Can a nonlinear phase model describe high-order locking?

Stuart-Landau oscillator: synchronization domains

Can a nonlinear phase model describe high-order locking?

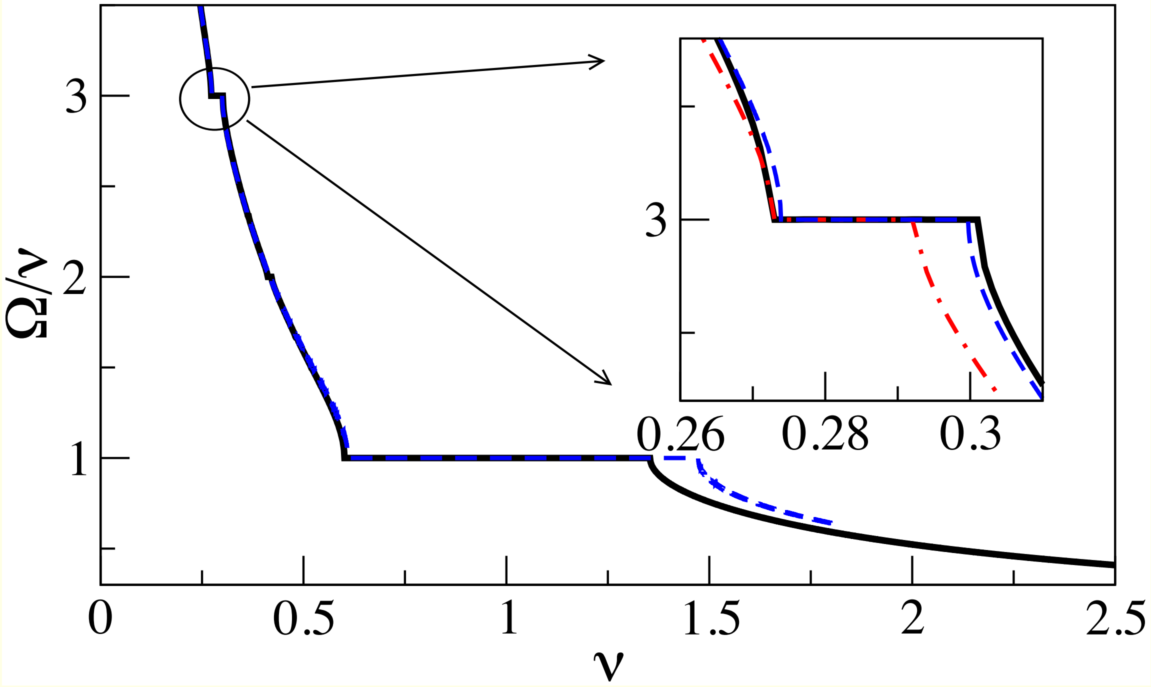
We use a model reconstructed for $\nu = 0.3$, $\varepsilon \leq 0.55$
to make a prediction for $\varepsilon = 0.7$
(for $\varepsilon > 0.55$ model reconstruction fails because of synchrony)

$$\dot{\varphi} = \omega + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^4 Q_4 \quad \text{4th-order fit}$$
$$\dot{\psi} = \nu$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\psi} = (\omega + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \varepsilon^4 Q_4)/\nu$$

We solve this equation numerically

for **different** ν and $\varepsilon = 0.7$



Stuart-Landau oscillator: synchronization domains

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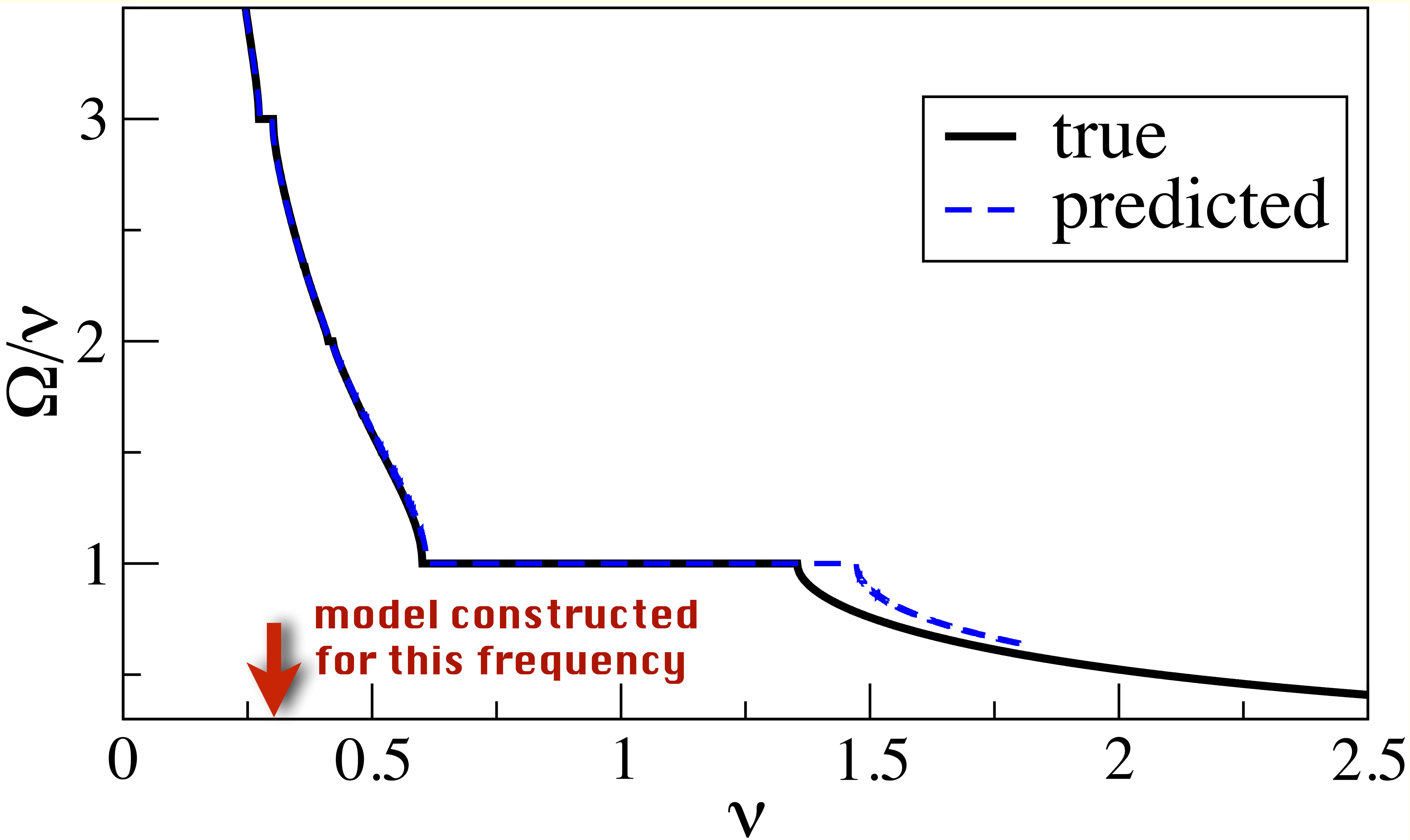
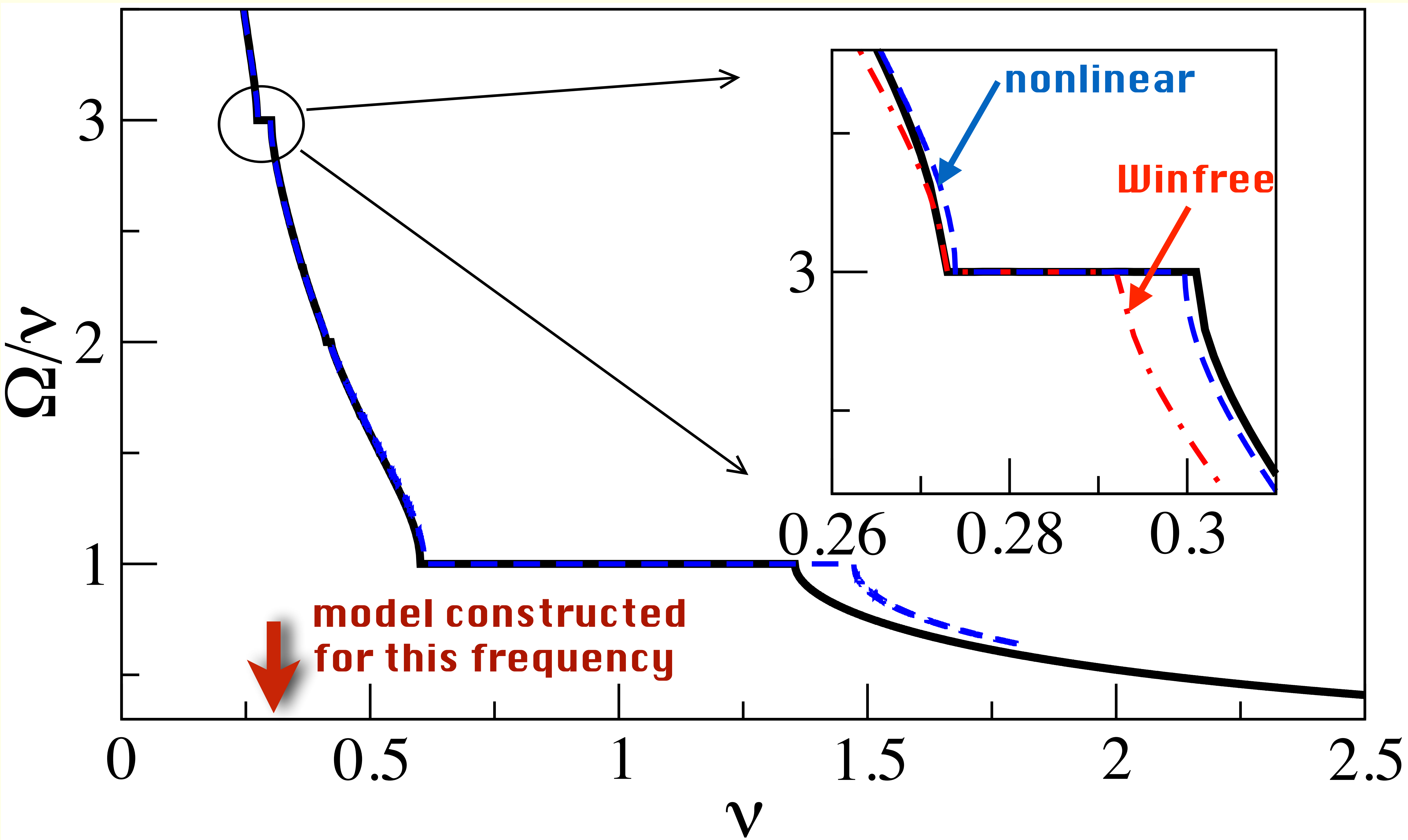


Stuart-Landau oscillator: synchronization domains

Nonlinear phase model describes high-order locking better than integration of the first-order Winfree approximation



How good is the Kuramoto-Daido model?

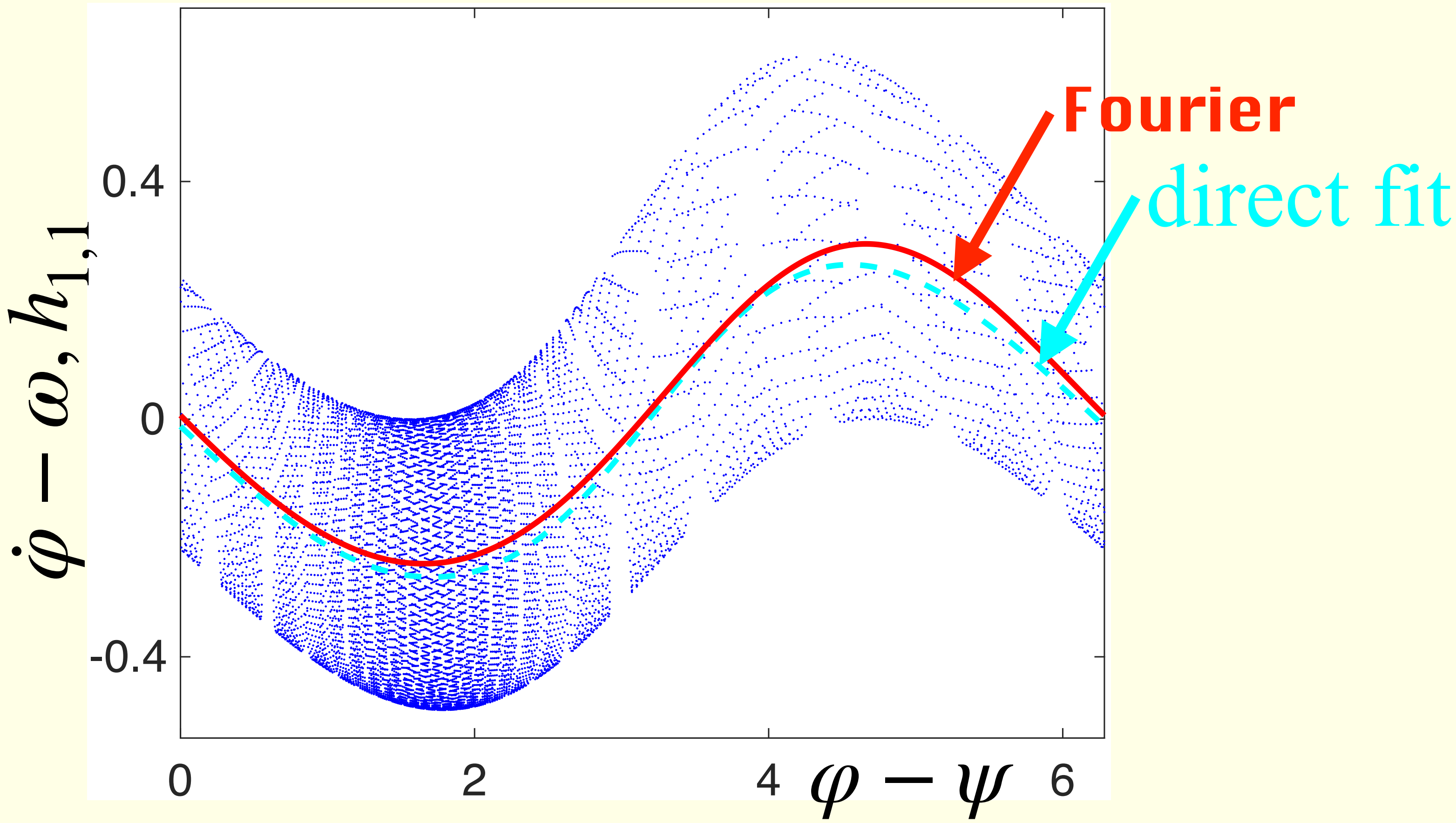
This model is obtained by averaging for weak forcing, but we can formally exploit it for large amplitudes as well

$$\dot{\varphi} = \omega + h_{n,m}(n\varphi - m\psi)$$

We construct the model either via Fourier modes of the full model,

$$h_{n,m}(n\varphi - m\psi) = \sum_k F_{(kn,-km)} \exp(ikn\varphi - ikm\psi)$$

or by a direct fit of $\dot{\varphi} - \omega$ vs. $n\varphi - m\psi$ MOD 2π



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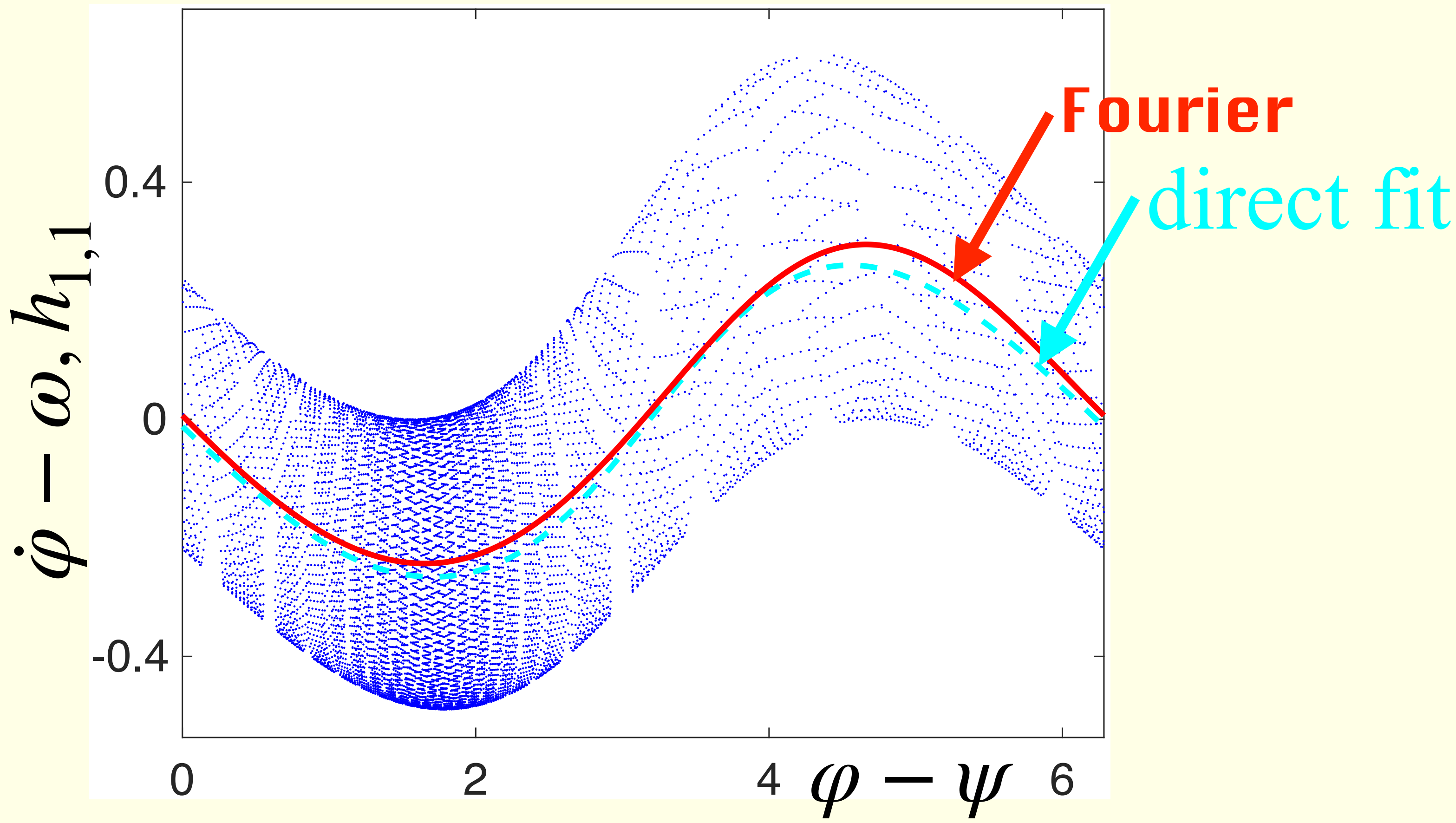


The model $h_{1,1}$ yields a good prediction of the 1:1 locking domain

Prediction by the $h_{1,3}$ model is bad

The Stuart-Landau oscillator is a good model: here the isochrons are known analytically

What to do in a general case?

Computing true phases on the fly

Consider a forced system: $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

Let us solve it numerically to obtain $\mathbf{x}_k = \mathbf{x}(t_k) = \mathbf{x}(k\Delta t)$

Recall that the perturbation approach operates with phases ***defined for the autonomous, unperturbed system***

→ to obtain phase for each $\mathbf{x}_k$ we integrate a copy of the autonomous system, $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$, with initial condition $\mathbf{y}(0) = \mathbf{x}_k$ and integration time $NT, N \in \mathbb{N}$

→ for sufficiently large N, $\mathbf{y}(NT)$ is ***on the limit cycle***

Hence, we can easily compute phase of $\mathbf{y}(NT)$ and therefore phase φ_k of $\mathbf{x}_k$

(N is the only parameter of the algorithm)

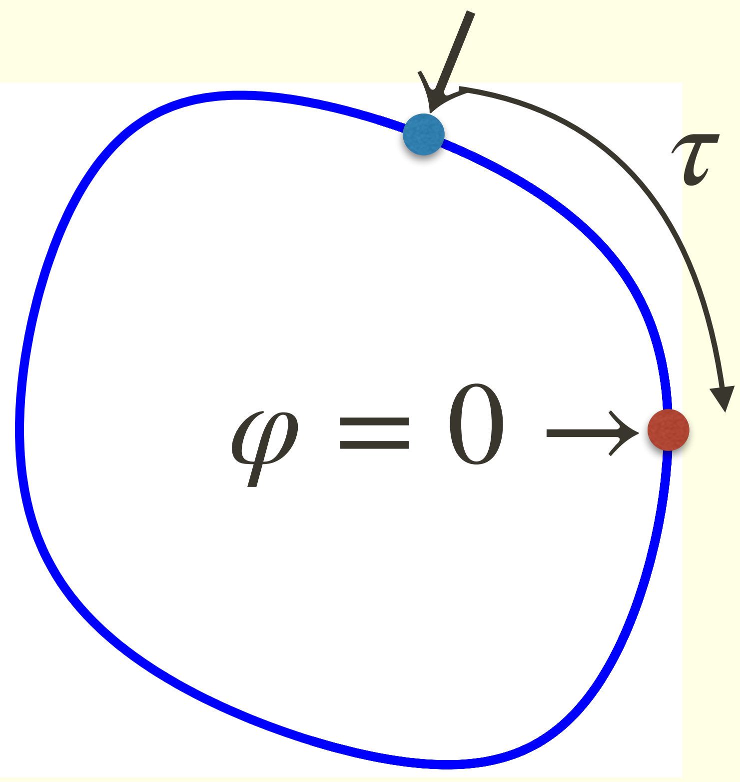
Computing true phases on the fly II

for sufficiently large N, $\mathbf{y}(NT)$ is ***on the limit cycle***

Hence, we can easily compute phase of $\mathbf{y}(NT)$

and therefore phase φ_k of $\mathbf{x}_k$

$$\varphi_k = \varphi(\mathbf{x}_k) = 2\pi \frac{T-\tau}{T}$$



Numerical phase reduction

Thus, for a forced system: $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) + \varepsilon \mathbf{p}(\mathbf{x}, t)$

we obtain φ_k for each $\mathbf{x}_k$, and numerically $\dot{\varphi}_k$

Suppose we know the phase of the force, ψ_k

Natural frequency ω is also known

→ we can **fit** the equation $\dot{\varphi} = \omega + Q(\varphi, \psi)$

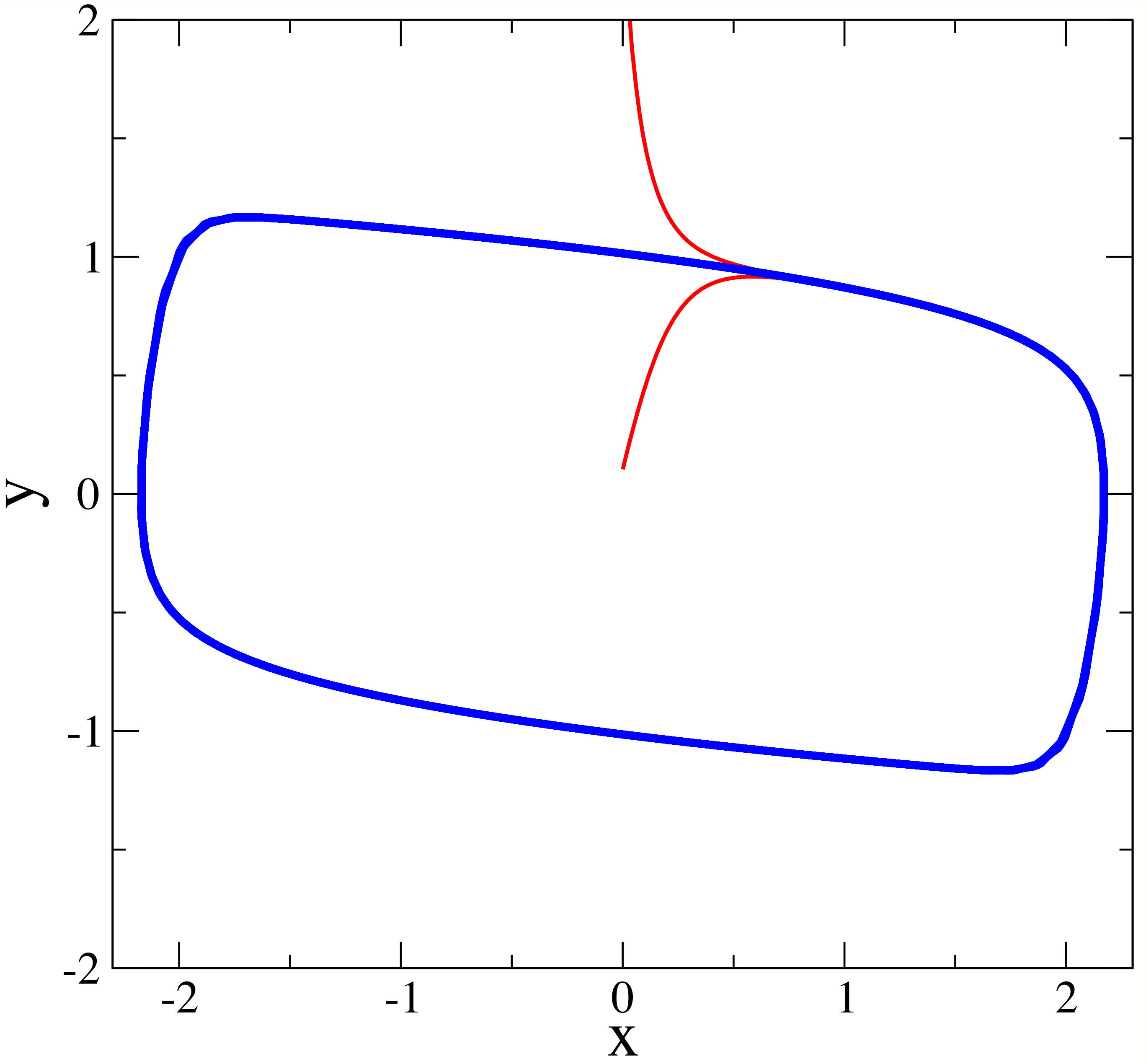
Practically: we use Savitzky-Golay smoothing filter for derivation

we use kernel density estimation on an $n \times n$ grid

$$K(x, y) = \exp\left[\frac{n}{2\pi}(\cos x + \sin y)\right]$$

Numerical phase reduction: Example I

Rayleigh oscillator $\ddot{x} - 4(1 - \dot{x}^2)\dot{x} + x = \varepsilon \cos(\nu t)$



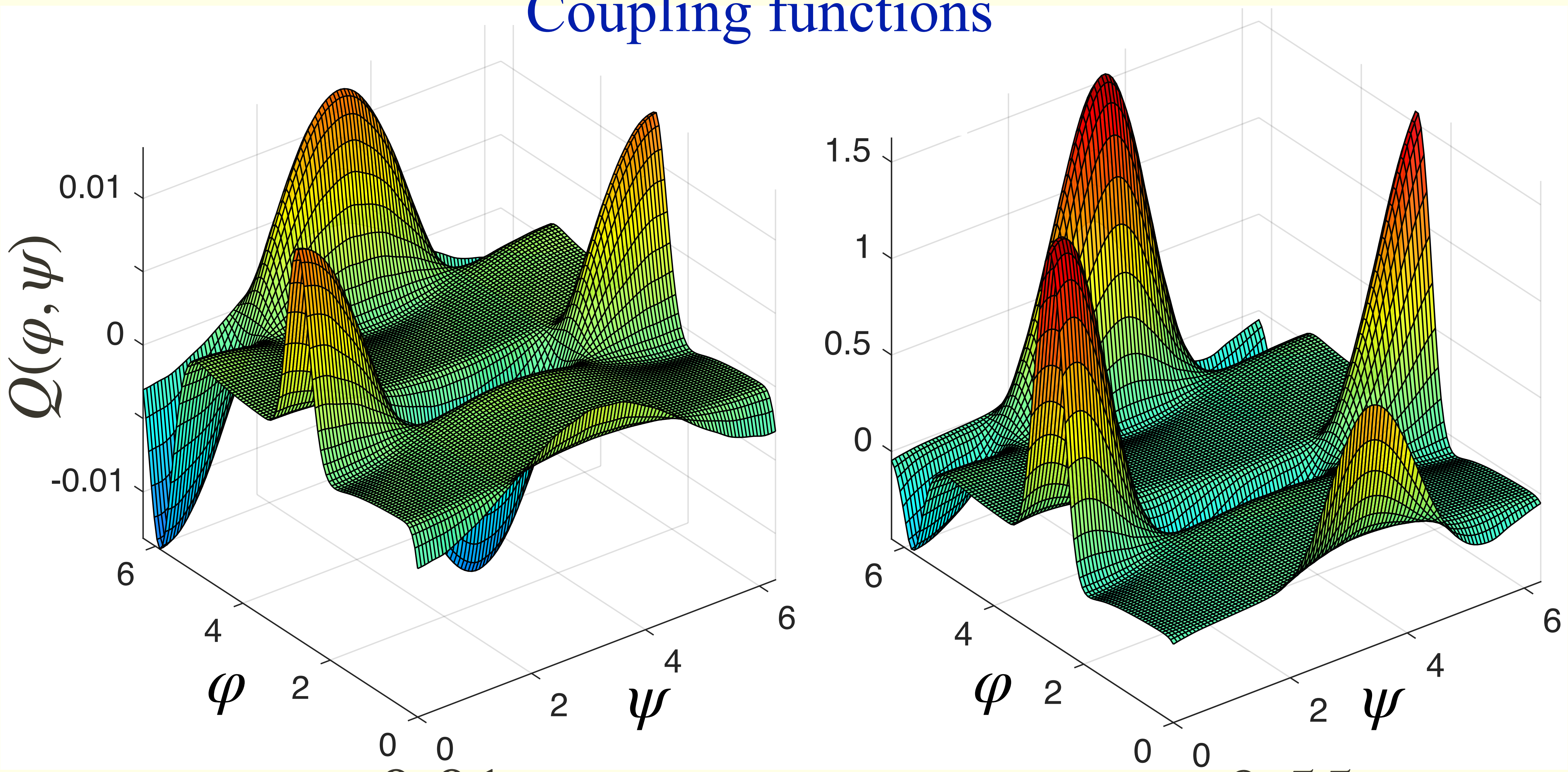
Strong stability of the limit cycle: phase approximation shall work

Numerical phase reduction: Example I

Rayleigh oscillator $\ddot{x} - 4(1 - \dot{x}^2)\dot{x} + x = \varepsilon \cos(\nu t)$

Fixed $\nu = 0.8$, varied $\varepsilon = 0.01, \ldots, 0.55$

Coupling functions



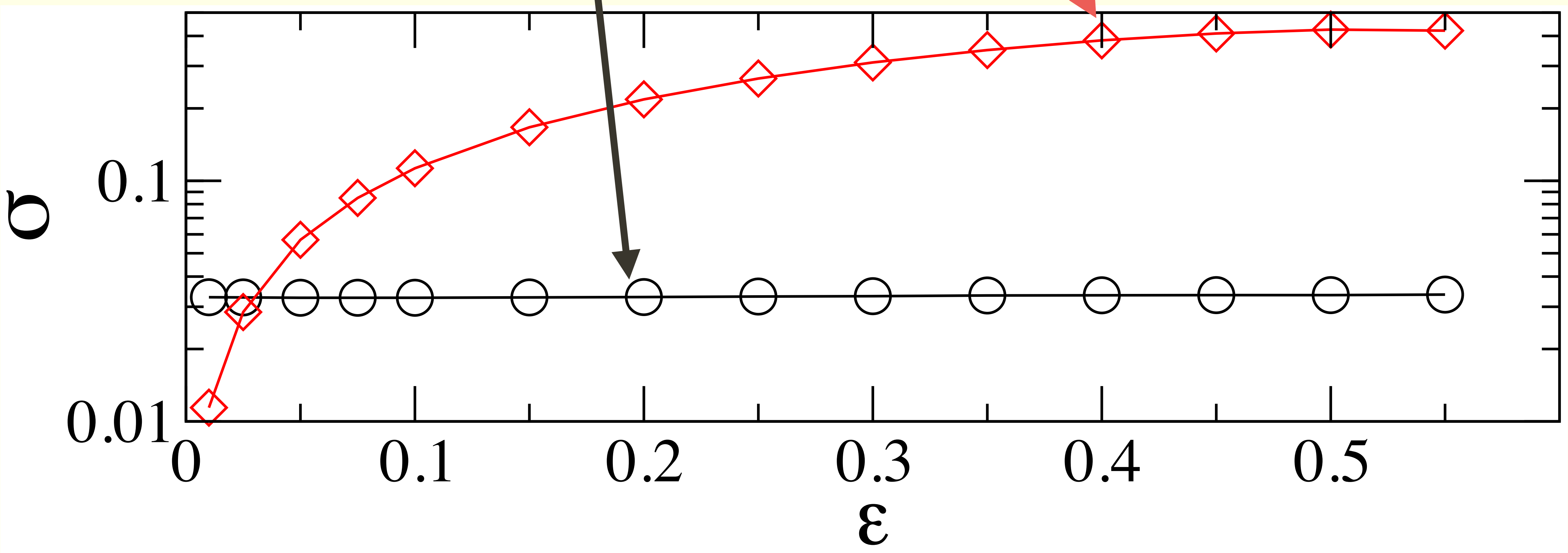
$\varepsilon = 0.01$ $\varepsilon = 0.55$

Example I: how good is the model?

Rest term $\xi_k = \dot{\varphi}_k - \omega - Q(\varphi_k, \psi_k)$

Quality of the model $\sigma = \mathrm{STD}(\xi_k)/\mathrm{STD}(\dot{\varphi}_k)$

Error of the Winfree-form representation



Phase equation works very well even for strong coupling!

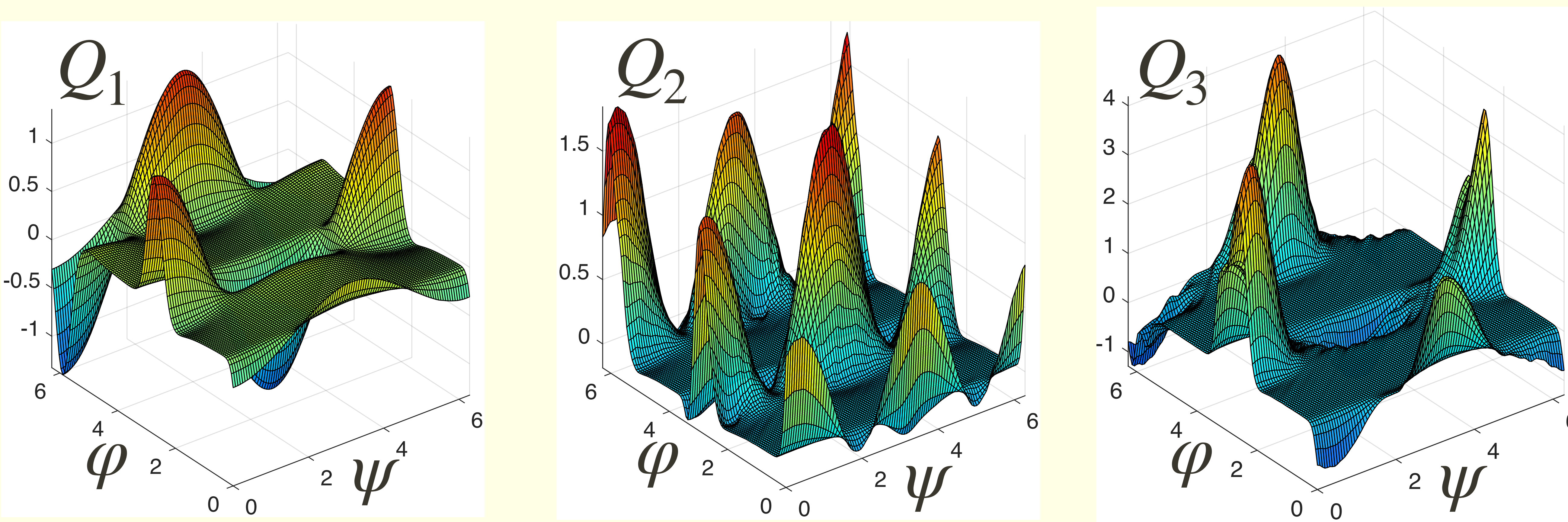
Coupling function: power series representation

Generally, one expects

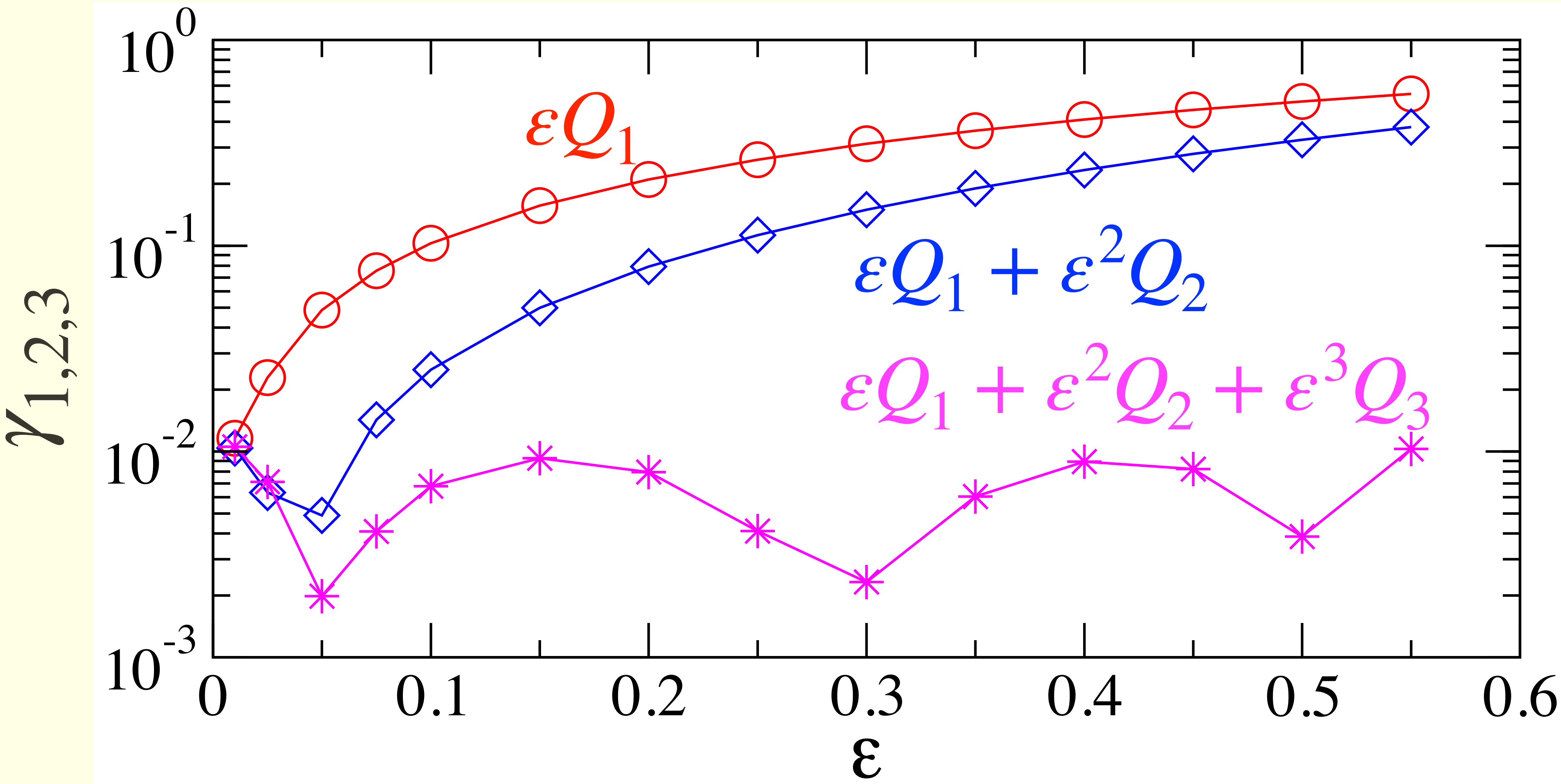
$$\dot{\varphi} = \omega + \varepsilon Q_1 + \varepsilon^2 Q_2 + \varepsilon^3 Q_3 + \ldots = \omega + Q(\varphi, \psi)$$

Since we have computed $Q(\varphi, \psi, \varepsilon)$ for many different ε

we can fit $Q(\varphi, \psi, \varepsilon)$ by a polynomial in ε



Quality of the power series representation

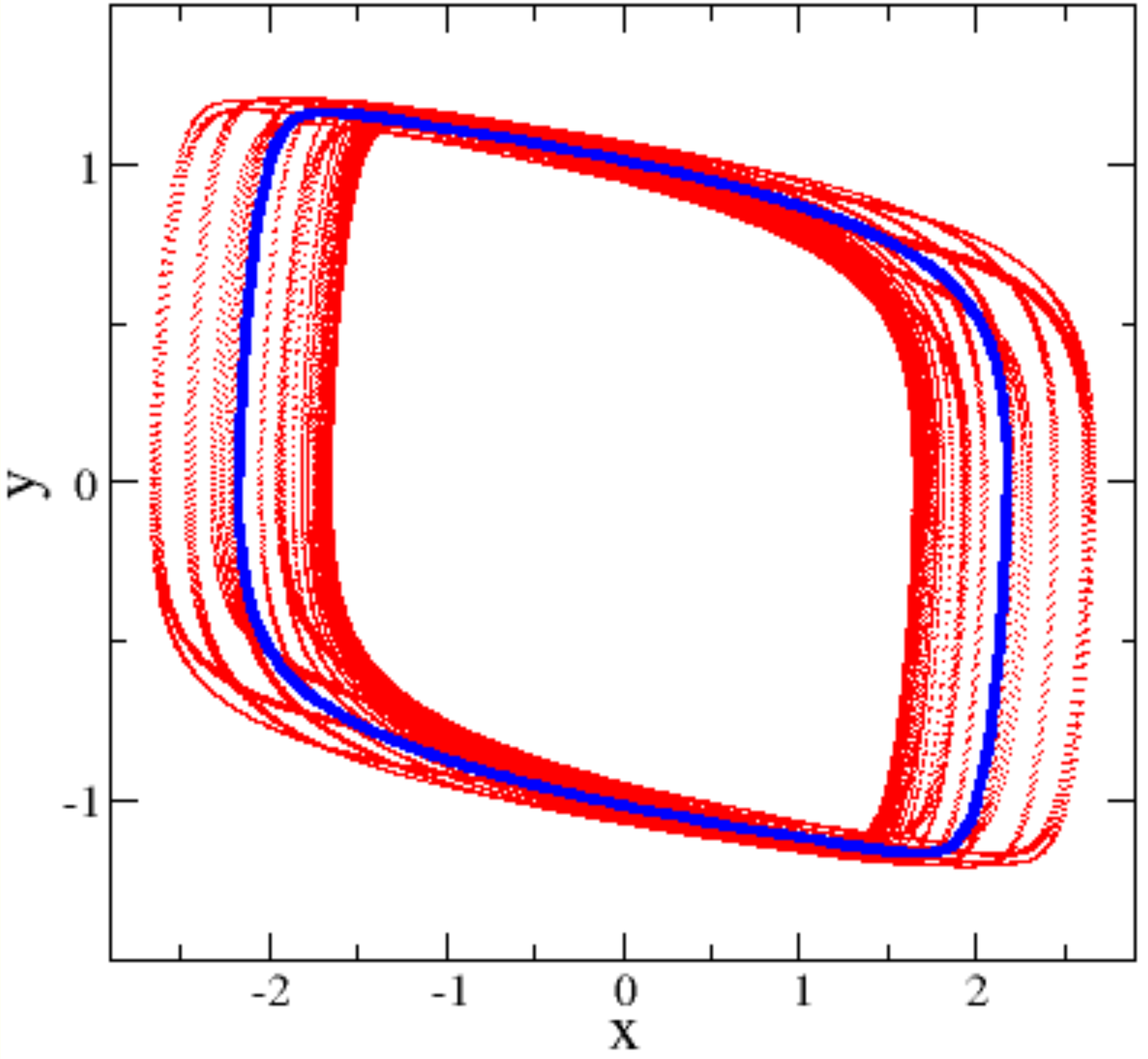


For 3rd-order approximation the error is below 1%

$$\gamma_n(\varepsilon) = \mathrm{STD}\left(Q(\varepsilon) - \sum_{i=1}^{m} \varepsilon^i Q_i \right) / \mathrm{STD}[Q(\varepsilon)]$$

Intermediate summary

Phase approximation works well even for quite strong coupling when the deviation from the limit cycle is large

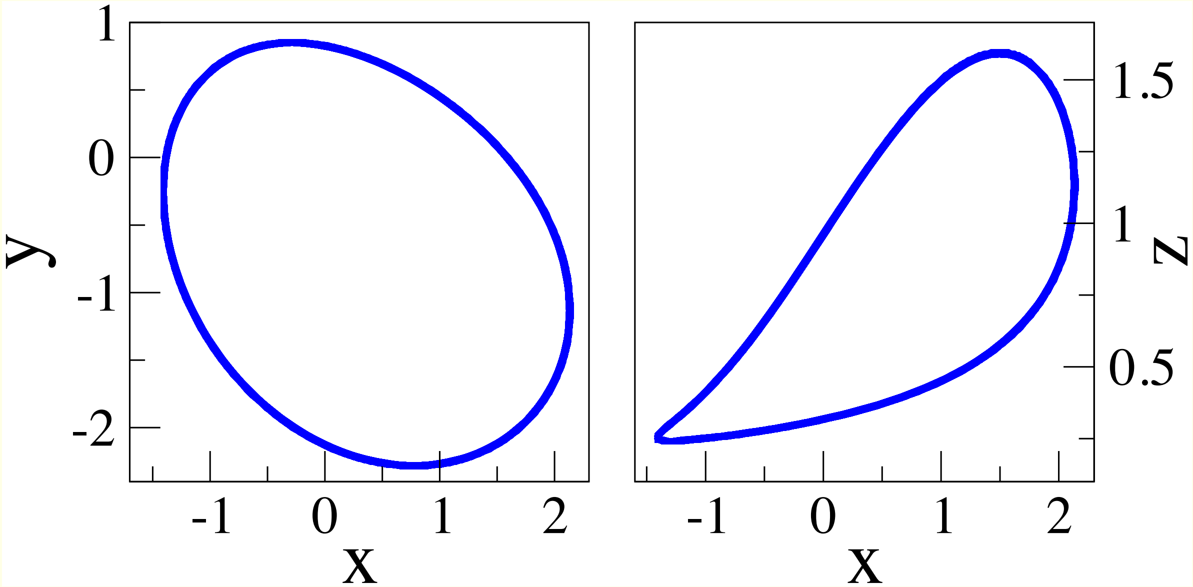


$\varepsilon = 0.55$

Numerical phase reduction: Example II

Rössler oscillator in a periodic state:

$$\dot{x} = -y - z$$
$$\dot{y} = x + 0.34y$$
$$\dot{z} = 0.8 + z(x-2)$$

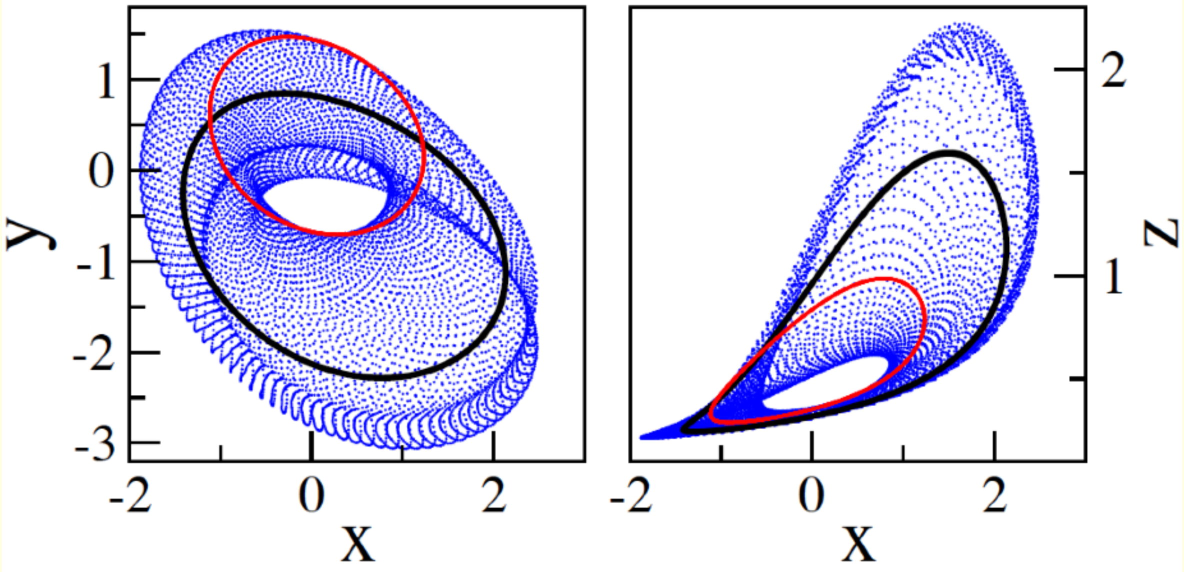


Weak stability of the limit cycle:

complex multiplicators $\mu = (-8.7 \pm 12.4i) \cdot 10^{-3}$

Forced system:

$$\dot{x} = -y - z + \varepsilon\cos(\nu t)$$
$$\dot{y} = x + 0.34y$$
$$\dot{z} = 0.8 + z(x-2)$$



with $\nu = 0.5$, $\varepsilon = 0.05,\ldots,0.7$

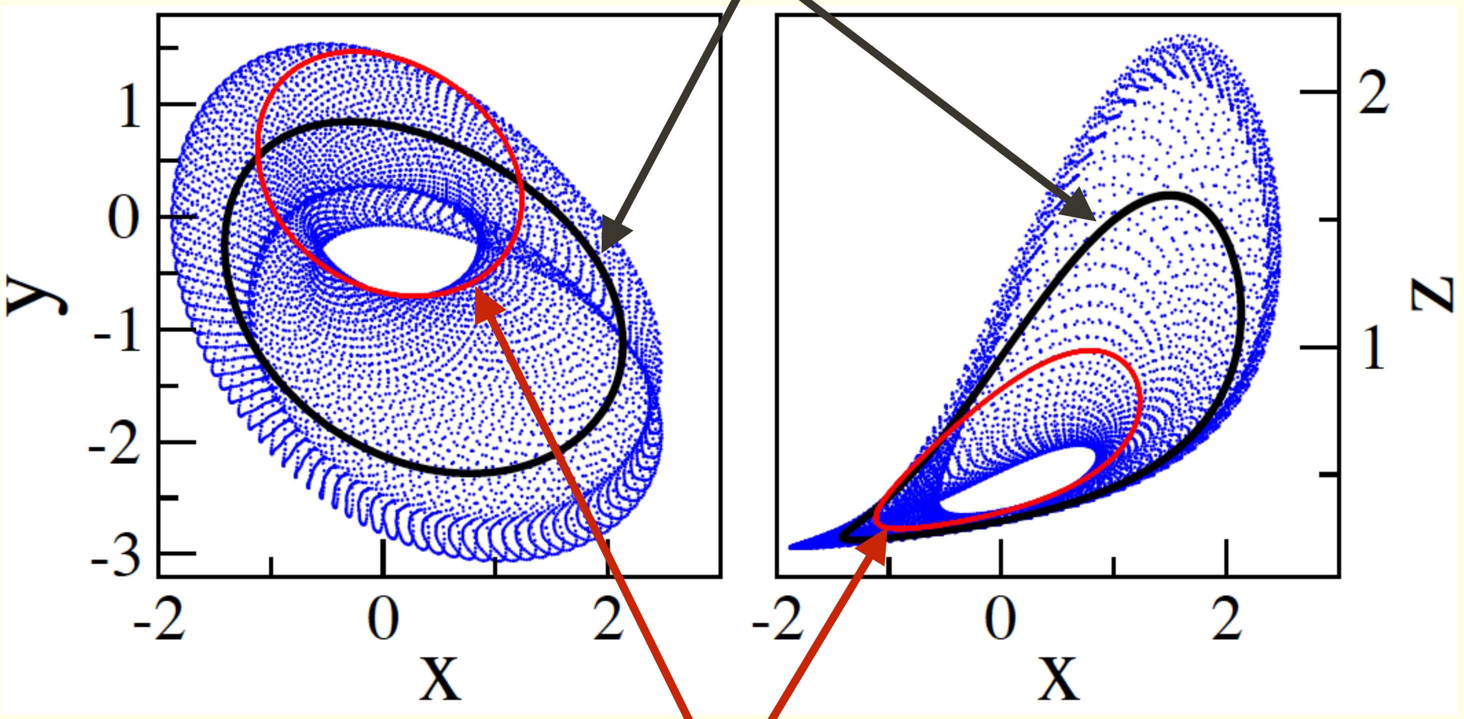
Numerical phase reduction: Example II

Forced system:

$$\dot{x} = -y - z + \varepsilon\cos(\nu t)$$
$$\dot{y} = x + 0.34y$$
$$\dot{z} = 0.8 + z(x-2)$$

with $\nu = 0.5\,,\varepsilon = 0.55$

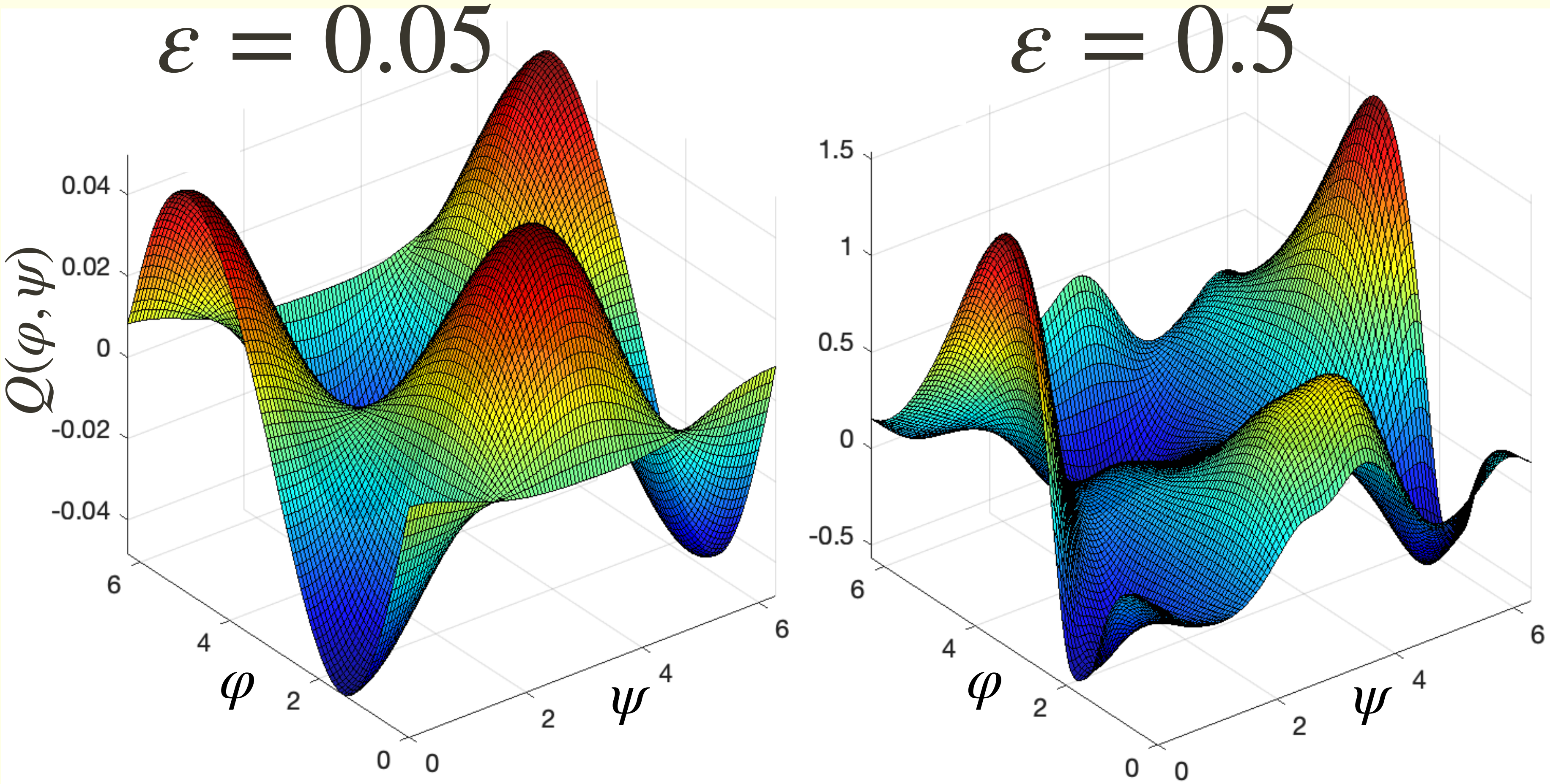
unperturbed limit cycle



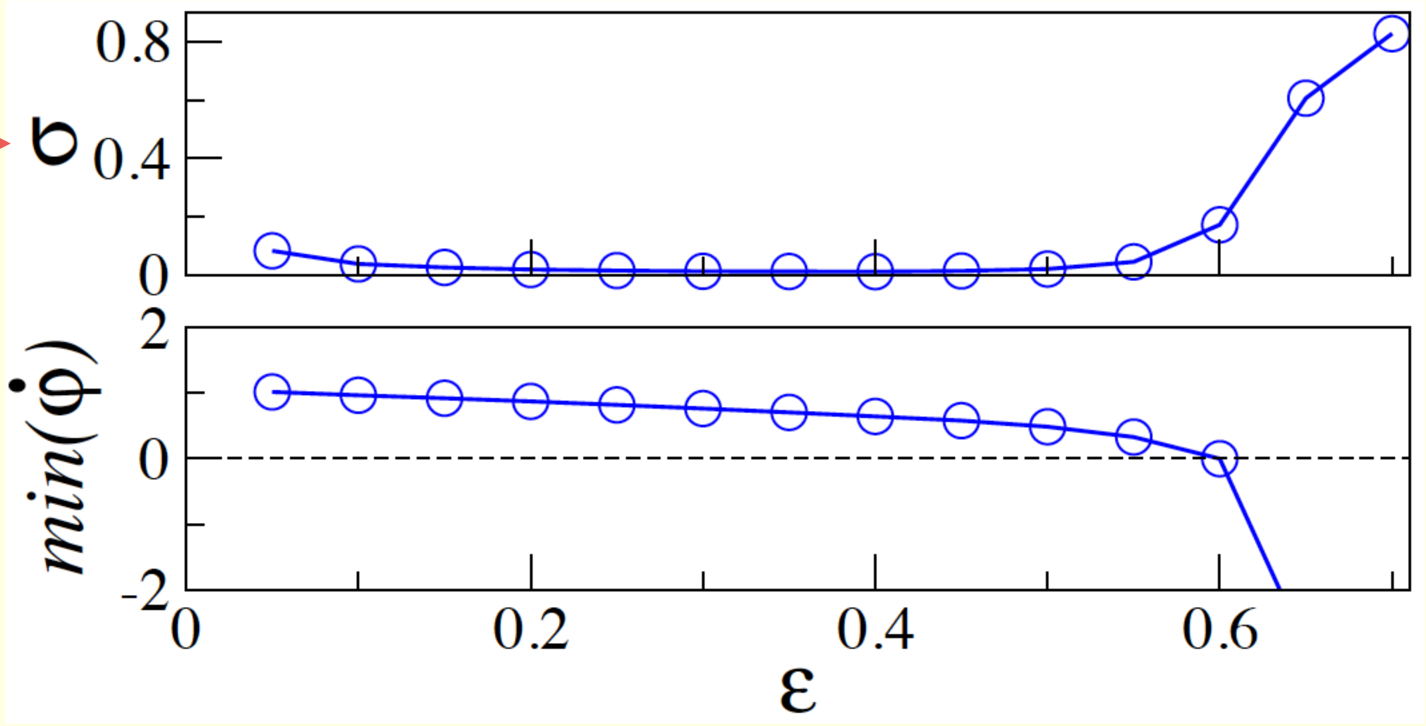
Poincare section

$\nu t \bmod 2\pi = \text{const}$

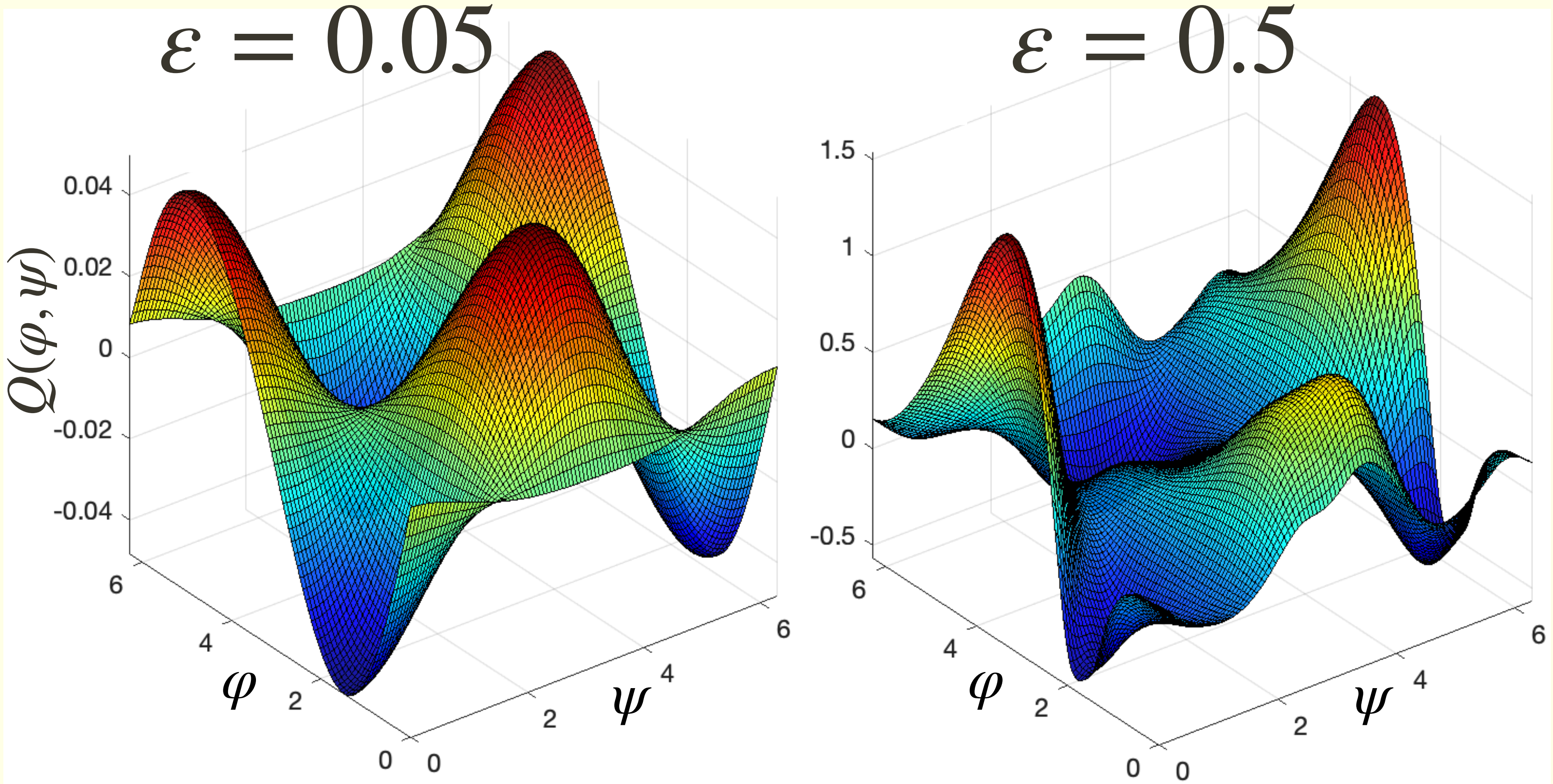
Example II: coupling functions



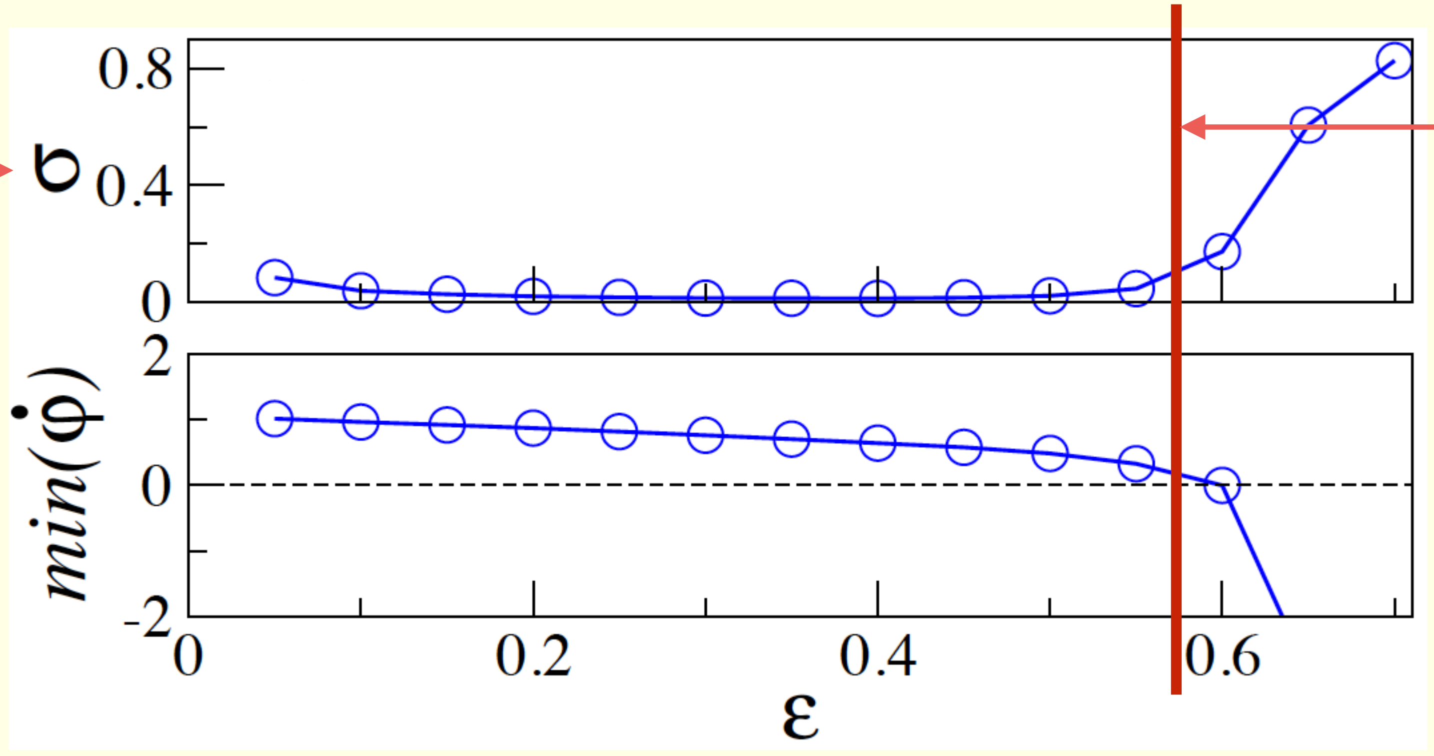
Error of fit



Example II: coupling functions



Error of fit

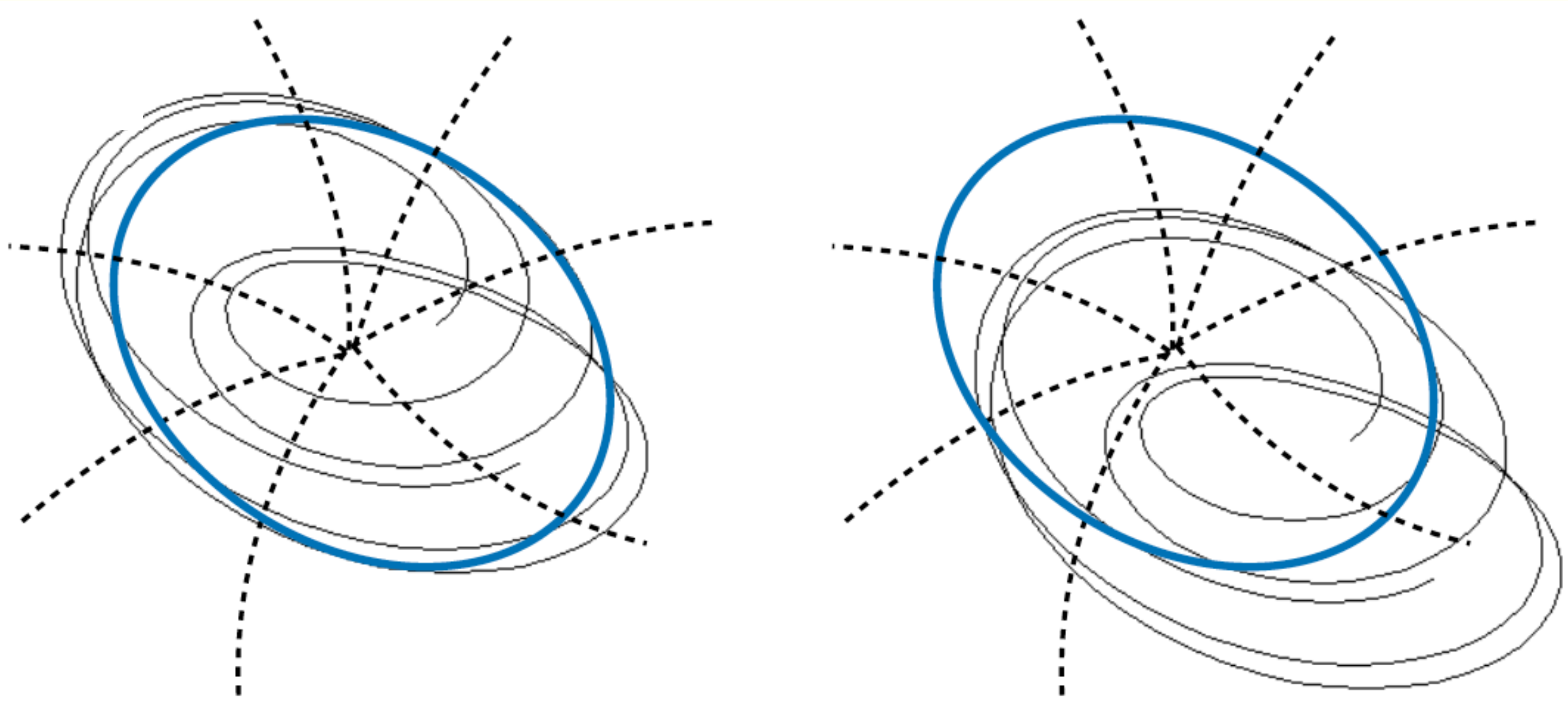


here the method fails

When the technique fails?

1. If coupling is so strong that the system gets locked to the force; the inference via fit does not work anymore (this happened in the first example)

2. If the torus becomes too "thick" and trajectories start to cross "wrong isochrones (second example)



3. If strong forcing destroys smooth attractive torus (see Afraimovich, Shilnikov 1983)

Conclusions

- Phase description works for quite strong coupling, but
  - coupling function is amplitude- and frequency dependent
  - description in terms of phase response curve generally fails

- Phase dynamics equations can be inferred numerically

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Numerical phase reduction beyond the first order approximation

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Nonlinear phase coupling functions: a numerical study

M. Rosenblum[1,2] and A. Pikovsky[1,2]